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PREFACE.

The report published in this volume of "Verslagen en Verhandelingen" ("Reports and Transactions") of the Nationaal Luchtvaartlaboratorium (National Aeronautical Research Institute) served the author as a thesis for the degree of doctor in the Technical Sciences at the Technical University of Delft.

Due to particular circumstances this Vol. XXII appears before Vol. XXI has been printed. The issue of Vol. XXI is, however, envisaged before long.

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Boundary value problems in lifting surface theory

by

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Summary.

A lifting surface of circular planform in steady and unsteady incompressible flow with any downwash distribution is considered. The problems are formulated as boundary value problems for the Laplace equation. In order to solve these problems appropriate orthogonal coordinates are introduced. Suitable solutions of the Laplace equation are found by separation of variables. By means of these solutions Green's function of the second kind can be constructed. A comparison of the regular velocity potential and the regular acceleration potential leads to a general expression for singular solutions of the Laplace equation. The complete solution of each of the physical problems can be written as the sum of the regular acceleration potential and a singular solution, which is singular only along the leading edge of the wing. This singular solution contains an unknown weight-function, which must be determined by requiring that the normal velocity at the wing surface corresponding to the complete acceleration potential coincides with the given normal velocity. The resulting integral equation is replaced by an infinite system of linear algebraic equations. In order to arrive at numerical results the infinite set of equations is broken off. The fast convergence of the process is illustrated by several examples.

Introduction.

In 1937 Kinner (ref.9) solved the problem of the determination of the pressure distribution, forces and moments on an aerofoil of circular planform in steady incompressible flow. In 1940 Schade (ref.19) extended Kinner's theory to the problem of the oscillating circular wing in incompressible flow. Numerical results for the oscillating circular aerofoil were published by Krienes and Schade in 1942 (ref.10). Although these papers greatly contributed to the lifting surface theory at that time, they do not well fit in with the present state of the lifting surface theory.

One of the main objections against Kinner's and Schade's solutions is that they fulfil the Kutta condition only in a finite number of points of the trailing edge of the wing. The number of these points depends on the number of linear equations by which a certain infinite system of algebraic equations, occurring in their theories, is approximated. Between the points of the trailing edge mentioned the pressure distribution becomes infinite.

Neither Kinner nor Schade give any information about the convergence of their solutions, when the number of linear equations increases. Schade treats his problem of the oscillating circular wing for the six downwash distributions up to the second degree in x and y . An extension to downwash distributions of higher degree in x and y requires such a lot of analysis that it becomes inconvenient for practical use.

In the last decade several approximation methods for the calculation of the pressure distribution and lift and moments on three-dimensional wings of arbitrary planform in steady incompressible flow have been developed. One of the most important methods is that of Multhopp's (ref.16). Application of Multhopp's method yields more reliable results as the aspect ratio of the wing becomes larger. At the N.L.L. an approximate lifting surface theory has been developed which is a slight modification of Multhopp's method with the advantage, however, of being especially adapted to electronic computing; moreover it does not have the restriction of two pivotal points in chordwise direction (ref.23). In order to check the accuracy of such an approximation, it would be most useful if an exact solution for a three-dimensional wing of not too large an aspect ratio should exist. Such an exact solution can be found for the wing of circular planform. However, the numerical results of Kinner's theory are so scarce that a further elaboration of this theory would be necessary.

For the non-stationary theory the situation is somewhat different. In the last years some approximate theories for oscillating wings of arbitrary planform were developed. In fact, most of these theories are extensions of known steady methods. In particular mention can be made of Garner's extension of Multhopp's steady lifting surface theory to pitching oscillations of low frequency (ref.3). Garner applies his method among other things to the circular wing and compares the values of some aerodynamic derivatives with those found by Krienes and Schade. He remarks that for the damping derivatives these comparisons cast doubt on the results of Krienes and Schade and emphasize the desirability of checking the complicated analysis developed in their method.

Recently two other methods were developed which claim a larger range of validity as to the frequency (refs. 4 and 18). However, some insight into the reliability of these theories does not exist. Therefore it would be important if there existed an exact solution for an oscillating three-dimensional wing of moderate aspect ratio.

Since 1954 Küssner has published a series of papers, dealing with the problem of the wing of elliptic planform in steady and unsteady flow (refs. 11, 12, 13, 14, 15). In some of the papers mentioned numerical results are given for the elliptic lifting surface of infinite aspect ratio and the circular wing in steady incompressible flow. These values differ considerably from those of Kinner's and from the values found in this report. The reason of these discrepancies can be ascribed to several errors in Küssner's theory.

In this paper a new method is developed for solving the problem of the circular wing in steady and unsteady incompressible flow. In fact the method used is an extension of a method applied by Timman for solving a two-dimensional boundary value problem for the wave equation. In his thesis (ref.24) Timman solved the problem of the harmonically oscillating aerofoil in subsonic compressible flow. Due to criticism on the numerical results of his theory he re-examined his method of solution in 1954, which resulted in a new, and more straight-forward theory (ref.25). In the latter paper it was recognized that singular solutions of the boundary value problem for the two-dimensional wave equation can be expressed in terms of Green's function of the boundary value problem concerned. In particular the generalization of this idea has opened the possibility of attacking the problem of the circular wing. It may be remarked that the same method is applicable to a wing of elliptic planform, but the analysis required for the evaluation of the ultimate pressure distribution is much more cumbersome due to the very complicated nature of the Lamé functions which have to be used instead of the associated Legendre functions corresponding to the circular wing.

In the present paper the two main problems treated, are firstly that of the circular wing in steady incompressible flow, and secondly that of the circular wing harmonically oscillating in an incompressible flow.

Chapter I contains the derivation of the basic equations of linearized incompressible aerofoil theory.

Chapter II deals with the method of solution and its underlying theory. However, the method is applied to the circular wing in steady flow and several examples of prescribed downwash distributions on the wing surface are evaluated, while the numerical results are compared with results of Kinner and with values calculated by an approximate theory developed at the N.L.L.

In chapter III the theory is used to derive the equations required for the harmonically oscillating circular aerofoil. Because of the complexity of the formulas to be used, only the simpler case of low frequency is elaborated. Several cases of special modes of vibration of the wing are treated. It appears that the numerical results differ very much from those of Krienes' and Schade's, which is mainly caused by some errors in Schade's theory.

In the appendix a closed expression is derived for the function of Green for the boundary value problem considered, by means of a method given by Sommerfeld.

Chapter I. The basic equations of linearized incompressible aerofoil theory.

1 The velocity potential.

The fluid considered in this paper is a non-viscous incompressible gas. With regard to the mechanical properties of the gas this implies that the force acting across any interior or bounding surface is normal to that surface. This force is called pressure. The flow of a gas with the mentioned properties can be described by specifying as functions of time t and position (x, y, z) in rectangular Cartesian coordinates, the components (u, v, w) of the velocity vector $\vec{q}$. These dependent variables are related by the law of mass conservation, which is expressed by

$$\operatorname{div} \vec{q} = 0 \tag{1,1,1}$$

and the equations of motion (Euler's equations)

$$\frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \operatorname{grad})\, \vec{q} = -\frac{1}{\rho} \operatorname{grad} p \; . \tag{1,1,2}$$

The quantities p and ρ denote the pressure and the density respectively In the last equation gravity and other external forces have been omitted, as is customary in aerodynamics.

In aerofoil theory an essential role is played by a certain function which reduced the number of dependent variables, notably the velocity potential $\varphi(x,y,z,t)$. The existence of this function depends on the condition of irrotationality of the flow, which means physically that all fluid elements have zero angular velocity. This condition is expressed mathematically by the disappearance of the curl of the vector $\vec{q}$, or in component form

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \;, \quad \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \;, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \; . \tag{1,1,3}$$

It is well known that the vanishing of the curl in a vector field is a necessary and sufficient condition to assure that the vector is the gradient of some scalar function. In the present case this function is the velocity potential $\varphi(x,y,z,t)$. This fact is expressed by the relation

$$\vec{q} = \operatorname{grad} \varphi$$

or in component form

$$u = \frac{\partial \varphi}{\partial x} \;, \quad v = \frac{\partial \varphi}{\partial y} \;, \quad w = \frac{\partial \varphi}{\partial z} \tag{1,1,4}$$

The velocity potential is related to the pressure, which is a quantity of more direct physical significance, by means of the equations of motion and a first integral of them.
In irrotational flow equation (1,1,2) is equivalent to

$$\frac{\partial}{\partial t} \operatorname{grad} \varphi + \operatorname{grad}\left(\frac{q^2}{2}\right) = -\frac{1}{\rho} \operatorname{grad} p \; . \tag{1,1,5}$$

Because of the incompressibility of the fluid this equation can be written as

$$\operatorname{grad}\left[\frac{\partial \varphi}{\partial t} + \frac{q^2}{2} + \frac{p}{\rho}\right] = 0 \tag{1,1,6}$$

Integration of this equation shows that the sum of the three quantities in parentheses is a constant throughout the field of flow at any particular instant of time, so this sum can at most equal some function of time:

$$\frac{\partial\varphi}{\partial t}+\frac{q^2}{2}+\frac{p}{\rho}=F(t) \qquad (1,1,7)$$

The expression is known as the unsteady Bernoulli's equation. The function $F(t)$ can be eliminated by a redefinition of the velocity potential. Thus φ may be replaced by $\varphi^x+\int F(t)dt$ without altering the velocity field. When the fluid motion at infinity consists of parallel streamlines with velocity U the following relation holds

$$F(t)=\frac{1}{2}U^2 \quad . \qquad (1,1,8)$$

A partial differential equation for the velocity potential is found by inserting the eq. (1,1,4) into the continuity equation (1,1,1). Using the fact that the divergence of the gradient is equal to Laplace's operator

$$\Delta = \mathrm{div.\ grad} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

eq. (1,1,1) can be rewritten into the following concise form

$$\Delta\varphi=0 \qquad (1,1,9)$$

We may conclude that in the case of an incompressible fluid Laplace's equation is the exact unsteady flow equation to be satisfied by the velocity potential.

2 Linearization of the boundary conditions.

In order to get a complete mathematical formulation of flow problems we still have to consider the boundary conditions. These conditions can be divided into two types: condition at infinity and conditions arising from the presence of the aerofoil.

The former depend on the nature of the partial differential equation governing φ . As this is Laplace's equation, they require only that the fluid be at rest or has some specified uniform motion at remote points.

The condition at the surface of a body states simply that the velocity of the fluid relative to the surface of the body is everywhere tangential to its surface.

If the equation of the surface is given by

$$F(x,y,z,t)=0 \qquad (1,2,1)$$

the boundary condition reads

$$\frac{\partial F}{\partial t}+\vec{q}\cdot \mathrm{grad}\ F=0 \qquad (1,2,2)$$

on the surface F.

The proof of this relation can be given as follows. Suppose that the velocity of motion of the surface $F=0$ is indicated by the components u', v' and w'.

From eq. (1,2,1) it follows

$$\frac{\partial F}{\partial t}+u'\frac{\partial F}{\partial x}+v'\frac{\partial F}{\partial y}+w'\frac{\partial F}{\partial z}=0 \ . \qquad (1,2,3)$$

If ℓ , m and n are the direction-cosines of the normal to the surface $F=0$ at the point (x,y,z) , one has at any point of the surface

$$\ell u' + m v' + n w' = \ell u + m v + n w \tag{1,2,4}$$

Since

$$\ell = \frac{\frac{\partial F}{\partial x}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}, \quad m = \frac{\frac{\partial F}{\partial y}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}},$$

$$n = \frac{\frac{\partial F}{\partial z}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

it follows from eq. (1,2,4) that

$$u' \frac{\partial F}{\partial x} + v' \frac{\partial F}{\partial y} + w' \frac{\partial F}{\partial z} = u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} \; . \tag{1,2,5}$$

Inserting this relation into eq. (1,2,3) the desired result is readily derived. It is just the form of boundary conditions (1,2,2) that gives rise to the small disturbance concept underlying the technique for linearizing the aerodynamic problem. We think of the wing, which translates with velocity U in the negative direction of x . The wing is fixed to the coordinate system in such a way that it coincides nearly with the xy -plane.

Supposing that the variable z can be explicitly solved from eq. (1,2,1), this formula can be split in two other equations, one of which defining the upper surface of the wing and the other one defining its lower surface:

$$\begin{aligned} F_u &= z - z_u \;(x,y,t) = 0 \\ F_\ell &= z - z_\ell \;(x,y,t) = 0 \end{aligned} \tag{1,2,6}$$

Since $\frac{\partial F}{\partial z}=1$, we are able to solve eq. (1,2,2) for the values of the vertical velocity w over the wing surface:

$$\begin{aligned} w &= \frac{\partial z_u}{\partial t} + u \frac{\partial z_u}{\partial x} + v \frac{\partial z_u}{\partial y} \quad \text{for } z = z_u ,\ (x,y) \text{ in } S \\ w &= \frac{\partial z_\ell}{\partial t} + u \frac{\partial z_\ell}{\partial x} + v \frac{\partial z_\ell}{\partial y} \quad \text{for } z = z_\ell ,\ (x,y) \text{ in } S \end{aligned} \tag{1,2,7}$$

where S is the projection of the wing on the xy -plane.
The eqs. (1,2,7) are exact; however, we shall approximate them by assuming that over almost the entire area of the wing the following suppositions hold:

(1) the slopes $\frac{\partial z_u}{\partial x}$, $\frac{\partial z_u}{\partial y}$, etc., are very small compared with unity, and

(2) the fluid velocity vector $\vec{q}$ differs only slightly in direction and magnitude from the free-stream velocity U .

The second condition points to the desirability of introducing a disturbance velocity potential $\bar{\varphi}$ by isolating the contribution of the uniform flow,

$$\varphi = \bar{\varphi} + Ux \tag{1,2,8}$$

The perturbation velocity components

$$\bar{u} = \frac{\partial \bar{\varphi}}{\partial x} = u - U \; , \quad v = \frac{\partial \bar{\varphi}}{\partial y} \; , \quad w = \frac{\partial \bar{\varphi}}{\partial z} \qquad (1,2,9)$$

are assumed to satisfy the order-of-magnitude requirement

$$\bar{u}, v, w \ll U \; . \qquad (1,2,10)$$

If the conditions (1) and (2) above are fulfilled, the eqs. (1,2,7) can be approximated by

$$\begin{aligned} w &= \frac{\partial z_u}{\partial t} + U \frac{\partial z_u}{\partial x}, \qquad \text{for } z = z_u, \ (x,y) \text{ in } S \\ w &= \frac{\partial z_\ell}{\partial t} + U \frac{\partial z_\ell}{\partial x}, \qquad \text{for } z = z_\ell, \ (x,y) \text{ in } S. \end{aligned} \qquad (1,2,11)$$

Now we can state that condition (1) is equivalent to assuming that z_u and z_ℓ be small compared with the wing chord.

This fact enables us to proceed one step further with the approximation. We expand w in MacLaurin series about its value just above and below the xy-plane:

$$w(x,y,z_u,t) = w(x,y,0_+,t) + z_u \frac{\partial w(x,y,0_+,t)}{\partial z} + \frac{z_u^2}{2!} \frac{\partial^2 w(x,y,0_+,t)}{\partial z^2} + \cdots\cdots$$

$$w(x,y,z_\ell,t) = w(x,y,0_-,t) + z_\ell \frac{\partial w(x,y,0_-,t)}{\partial z} + \frac{z_\ell^2}{2!} \frac{\partial^2 w(x,y,0_-,t)}{\partial z^2} + \cdots\cdots \qquad (1,2,12)$$

If the derivatives $\frac{\partial w}{\partial z}$, etc., are sufficiently well behaved such that their products with the small quantities z_u, etc., may be neglected with respect to w itself, all but the first terms on the right hand side of eqs. (1,2,12) can be neglected. Then the boundary conditions finally take the forms

$$\begin{aligned} w &= \frac{\partial z_u}{\partial t} + U \frac{\partial z_u}{\partial x}; \qquad \text{for } z = 0_+, \ (x,y) \text{ in } S \\ w &= \frac{\partial z_\ell}{\partial t} + U \frac{\partial z_\ell}{\partial x}; \qquad \text{for } z = 0_-, \ (x,y) \text{ in } S. \end{aligned} \qquad (1,2,13)$$

It is important to remark that the use of eqs. (1,2,13) points to the admissibility of schematizing the actual wing by a mathematical plane surface across which appropriate discontinuities exist.

Concerning the linearization of the differential equation for the velocity potential it must be remarked that the terms in Laplace's equation are already linear and there is no reason to expect anyone of them to be much larger or smaller than the others. Hence in linearized incompressible theory the governing equation for the velocity potential remains Laplace's equation.

3 The acceleration potential.

It has been experienced that in solving boundary value problems there are sometimes great advantages in using the concept of the acceleration potential instead of the velocity potential.

The existence of the acceleration potential is assured by the vector equation of motion for incompressible fluids

$$\frac{d\vec{q}}{dt} = -\frac{1}{\rho}\,\mathrm{grad}\; p = -\,\mathrm{grad}\,\frac{p}{\rho} \quad . \tag{1,3,1}$$

This equation shows that the acceleration vector is the gradient of a scalar function, which we designate by $\psi(x,y,z,t)$, such that

$$\frac{d\vec{q}}{dt} = \mathrm{grad}\;\psi \tag{1,3,2}$$

with the components

$$\frac{du}{dt} = \frac{\partial\psi}{\partial x} \;,\quad \frac{dv}{dt} = \frac{\partial\psi}{\partial y} \;,\quad \frac{dw}{dt} = \frac{\partial\psi}{\partial z} \;.$$

From eqs. (1,3,1) and (1,3,2) it follows

$$\mathrm{grad}\;\psi + \mathrm{grad}\;\frac{p}{\rho} = 0 \tag{1,3,3}$$

so that these quantities differ at most by a function of time,

$$\psi = -\frac{p}{\rho} + G(t) \quad . \tag{1,3,4}$$

$G(t)$ has the same properties as $F(t)$ in eq. (1,1,7) and therefore can be eliminated in some way or another. In particular when ψ is assigned the value zero at infinity where $p = p_\infty$, eq. (1,3,4) reads

$$\psi = \frac{p_\infty - p}{\rho} \tag{1,3,5}$$

Here ψ differs only by a constant factor from the disturbance pressure $p - p_\infty$.

The relation between the acceleration potential ψ and the velocity potential φ is found by inserting eq. (1,3,3) into eq. (1,1,6). When the flow is uniform with velocity U at infinity, the resulting equation reads

$$\psi = \frac{\partial\varphi}{\partial t} + \frac{1}{2}\left(q^2 - U^2\right) \tag{1,3,6}$$

If the small-perturbation conditions (1,2,10) are fulfilled, this equation can be linearized to

$$\psi = \frac{\partial\bar{\varphi}}{\partial t} + U\bar{u} = \frac{\partial\bar{\varphi}}{\partial t} + U\,\frac{\partial\bar{\varphi}}{\partial x} \tag{1,3,7}$$

As the operator $\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$ may be interchanged with the other linear operators in Laplace's equation, it can be concluded that the acceleration potential also satisfies the same differential equation as the velocity potential in the linearized theory.

The relation (1,3,7) can also be considered as a linear partial differential equation of the first order for the unknown velocity potential $\bar{\varphi}$.

In order to get the general solution we write down the characteristic equations:

$$dt = \frac{dx}{U} = \frac{d\bar{\varphi}}{\psi(x,t)} \tag{1,3,8}$$

This system of ordinary differential equations has the solutions

$$x = Ut + C_1$$

$$\bar{\varphi} = \frac{1}{U} \int \psi\ (x,t)\, dx + C_2 = \int \psi\ (Ut + C_1, t)\, dt + C_2 \quad . \tag{1,3,9}$$

The requirement that $\bar{\varphi}$ vanishes at the point $x = -\infty$, enables us to write the solution for $\bar{\varphi}$ into the form

$$\bar{\varphi} = \frac{1}{U} \int_{-\infty}^{x} \psi\left(x_1, y, z, t + \frac{x_1 - x}{U}\right) dx_1 = \int_{-\infty}^{t} \psi\left(x + U(t_1 - t), y, z, t_1\right) dt_1 \tag{1,3,10}$$

Chapter II. The circular wing in steady incompressible flow.

1 Formulation of the boundary value problem in terms of the velocity potential.

Consider an aerofoil of circular planform moving with constant velocity U in an incompressible and non-viscous medium. A right-hand system of rectangular coordinates (x,y,z) is used (see fig.1). The positive direction of the axis of x is taken opposite to the direction of motion of the wing; the axis of y is taken in the spanwise direction. The projection of the aerofoil on the xy-plane is a circle having radius unity with its centre at the origin of the coordinates. The coordinate axes are assumed to be fixed to the wing.

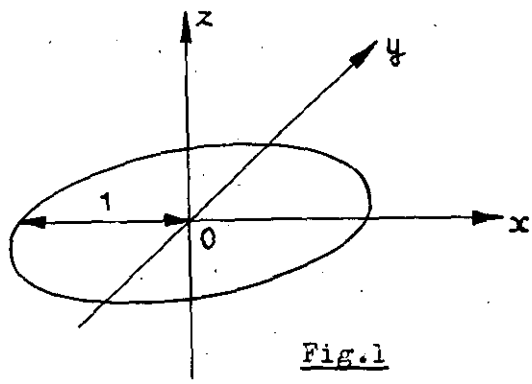

Fig.1

In chapter I it has been proved that the perturbation velocity potential Φ satisfies Laplace's equation:

$$\Delta \Phi \equiv \Phi_{xx}+\Phi_{yy}+\Phi_{zz}=0 \qquad (2,1,1)$$

According to paragraph 2 of chapter I the linearized conditions of tangential flow at the aerofoil surfaces

$$z=z_u(x,y) \quad \text{and} \quad z=z_l(x,y)$$

read as follows:

$$\begin{aligned} w&=U\frac{\partial z_u}{\partial x} \quad \text{for} \quad z=0_+ \;, \quad x^2+y^2\leqq 1 \\ w&=U\frac{\partial z_l}{\partial x} \quad \text{for} \quad z=0_- \;, \quad x^2+y^2\leqq 1 \end{aligned} \qquad (2,1,2)$$

In order to examine the boundary conditions more closely, we split z_u and z_l in an even part z_a and an odd part z_t, as follows

$$z_u=z_a+z_t \;, \quad z_l=z_a-z_t \qquad (2,1,3)$$

where z_t describes a shape that is symmetrical about the xy-plane and defines the distribution of thickness over the wing, whereas z_a gives the angle of attack and camber distribution.

Since the conditions (2,1,2) are linear, the boundary value problems associated with z_t and z_a can be treated separately. The aerofoil properties may be regarded as the superposition of those of a symmetrical aerofoil at zero incidence and a cambered inclined mean plane of zero thickness.

For the symmetrical part of the problem we must solve Laplace's equation subject to the boundary conditions

$$\begin{aligned} w&=U\frac{\partial z_t}{\partial x} \quad \text{for} \quad z=0_+ \;, \quad x^2+y^2\leqq 1 \\ w&=-U\frac{\partial z_t}{\partial x} \quad \text{for} \quad z=0_- \;, \quad x^2+y^2\leqq 1 \end{aligned} \qquad (2,1,4)$$

Because of the wing's symmetry, we expect a flow pattern, which is completely symmetrical with respect to the xy-plane. Thus we can conclude that no pressure-jump exists over the wing surface. In general we can say that the thickness of the aerofoil contributes nothing to the lift, pitching moment and rolling moment experienced by the wing.

The anti-symmetrical part of the problem, associated with the function z_a, is of much more interest to the aerodynamicist. This problem will now be treated extensively for the circular wing in this chapter.

The problem is to find a solution Φ of Laplace's equation, which satisfies the following boundary conditions

$$w = \frac{\partial\Phi}{\partial z} = U\frac{\partial z_a}{\partial x} \quad \text{for} \quad z = 0_+ \quad , \quad x^2+y^2 \leq 1 \tag{2,1,5}$$

$$w = \frac{\partial\Phi}{\partial z} = U\frac{\partial z_a}{\partial x} \quad \text{for} \quad z = 0_- \quad , \quad x^2+y^2 \leq 1 \quad .$$

To supplement these conditions, it is worth-while to emphasize that no discontinuity of the pressure can exist across the xy-plane except over the wing surface. In order to express this condition in terms of the perturbation velocities we apply Bernoulli's equation. Its form may be simplified by observing that the uniform motion at great distance implies

$$F(t) = \frac{1}{2}U^2 + \frac{p_\infty}{\rho} \quad . \tag{2,1,6}$$

The consistent application of the small disturbance assumption calls for the substitution

$$q^2 = (U+u)^2 + v^2 + w^2 \approx U^2 + 2Uu \quad . \tag{2,1,7}$$

With these insertions eq. (1,3,6) becomes in the case of steady flow

$$Uu = \frac{p_\infty - p}{\rho} \tag{2,1,8}$$

As the pressure p possesses no discontinuity across the xy-plane outside the wing planform, it can be concluded that the velocity component u also has no discontinuity across this region.

The conditions (2,1,5) express that the downwash distribution w is a continuous, even function of z. Recalling that integration reverses the evenness or oddness of a function, this means that the velocity potential Φ and the velocity component $\frac{\partial\Phi}{\partial x}$ are odd functions of the variable z. Formulated mathematically this reads

$$\Phi(x,y,z) = -\Phi(x,y,-z)$$
$$\frac{\partial\Phi(x,y,z)}{\partial x} = -\frac{\partial\Phi(x,y,-z)}{\partial x} \quad . \tag{2,1,9}$$

Denoting the pressure jump over the xy-plane by Π, we can write

$$\Pi = 2\rho U u \tag{2,1,10}$$

From this relation we see at once that the velocity component u vanishes at points in the xy-plane outside the aerofoil.

Recapitulating, the boundary value problem can be formulated as follows:

It is required to find a solution of Laplace's equation

$$\Phi_{xx}+\Phi_{yy}+\Phi_{xz}=0$$

which satisfies the conditions

1) $\Phi(x,y,z)=0$ for (x,y,z) at infinity

2) $\frac{\partial\Phi}{\partial z}=w(x,y)$ for $z=0_+$ and $z=0_-$; $x^2+y^2\leqq 1$

3) $\frac{\partial\Phi}{\partial z}=0$ for $z=0$, $x^2+y^2>1$

2 Formal solution of the boundary value problem by means of Green's function.

Considering the two sides of the wing together as forming a closed surface, the problem is an exterior Neumann problem for Laplace's equation. The starting point for the treatment of such problems is a well-known theorem of Green (ref. 8), which reads as follows:

If U and V are continuously differentiable functions in the closed regular region D and if their partial derivatives of the second order are continuous in D , then the following identity holds

$$\iiint\limits_D (U\Delta V-V\Delta U)\,d\tau=\iint\limits_S \left(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\right)d\sigma \tag{2,2,1}$$

S represents the boundary of the region D and n the normal pointing outward from D .

For the exterior problem we consider a region bounded by a surface S and a sphere C with radius R , enclosing completely the boundary S . For this region D we can write

$$\iiint\limits_D (U\Delta V-V\Delta U)\,d\tau=\iint\limits_S\left(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\right)d\sigma+\iint\limits_C\left(U\frac{\partial V}{\partial R}-V\frac{\partial U}{\partial R}\right)d\sigma \tag{2,2,2}$$

The normal n is directed to the interior of the region enclosed by S .

We shall now impose the additional conditions for infinite regions on the functions U and V , that the absolute values of

$$RU,\ R^2\frac{\partial U}{\partial x},\ R^2\frac{\partial U}{\partial y},\ R^2\frac{\partial U}{\partial z}\ ;\ RV,\ R^2\frac{\partial V}{\partial x},\ R^2\frac{\partial V}{\partial y},\ R^2\frac{\partial V}{\partial z} \tag{2,2,3}$$

shall be bounded for all sufficiently large values of R , where R is the distance from any fixed point. Functions, satisfying these conditions, are called regular at infinity.

Under these conditions it can easily be shown that

$$\lim_{R\to\infty}\iint\limits_C\left(U\frac{\partial V}{\partial R}-V\frac{\partial U}{\partial R}\right)d\sigma=0\ , \tag{2,2,4}$$

so that eq. (2,2,2) transforms into

$$\iiint\limits_D (U\Delta V-V\Delta U)\,d\tau=\iint\limits_S\left(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\right)d\sigma \tag{2,2,5}$$

where D now represents the entire region outside the boundary S .

Let $P(x,y,z)$ be any point in D. We take V equal to $1/r$ in the identity (2,2,5), where r is the distance from P to a point Q. In the point P this function does not fulfil the conditions stated in Green's theorem. Therefore we surround P with a small sphere with P as its center and ρ as its radius and remove from D the interior of this sphere. For the resulting region D^{x}, we have, since $1/r$ is harmonic,

$$-\iiint_{D^{x}} \frac{1}{r}\,\Delta U\,d\tau = \iint_{S}\left(U\frac{\partial}{\partial n}\frac{1}{r}-\frac{1}{r}\frac{\partial U}{\partial n}\right)d\sigma + \iint_{\Omega}\left(U\frac{\partial}{\partial n}\frac{1}{r}-\frac{1}{r}\frac{\partial U}{\partial n}\right)d\sigma . \tag{2,2,6}$$

Using the fact that on Ω one has $\frac{\partial}{\partial n}=-\frac{\partial}{\partial \rho}$, it can easily be proved that the second integral on the right-hand-side of eq. (2,2,6) tends to $4\pi U(P)$ as the radius of Ω tends to zero.
Hence we can write

$$-\iiint_{D} \frac{1}{r}\,\Delta U\,d\tau = \iint_{S}\left(U\frac{\partial}{\partial n}\frac{1}{r}-\frac{1}{r}\frac{\partial U}{\partial n}\right)d\sigma + 4\pi U(P) . \tag{2,2,7}$$

If U is harmonic in D, we have

$$U(P) = -\frac{1}{4\pi}\iint_{S}\left(U\frac{\partial}{\partial n}\frac{1}{r}-\frac{1}{r}\frac{\partial U}{\partial n}\right)d\sigma . \tag{2,2,8}$$

If V is also a harmonic function in D we have moreover

$$\iint\left(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\right)d\sigma = 0 \tag{2,2,9}$$

Jointly these eqs. (2,2,8) and (2,2,9) yield

$$U(P) = \frac{1}{4\pi}\iint_{S}\left\{U\frac{\partial}{\partial n}\left(V+\frac{1}{r}\right)-\left(V+\frac{1}{r}\right)\frac{\partial U}{\partial n}\right\}d\sigma \tag{2,2,10}$$

where the normal n is supposed to point to the exterior region D.

In order that $U(P)$ may be expressed only in terms of the boundary values of its normal derivative on S, we must eliminate the first term under the integral sign in (2,2,10). This could be accomplished if we could find a function V harmonic in D, and having a normal derivative which is the opposite of that of $\frac{1}{r}$.

The function $G(Q,P)$ which is defined by

$$G(Q,P) = \frac{1}{r} + V(Q,P) \quad , \tag{2,2,11}$$

where $V(Q,P)$ has the above stated properties, is known as Green's function of the second kind for the region D and the pole P.
In terms of Green's function we can write

$$U(P) = -\frac{1}{4\pi}\iint_{S}\frac{\partial U(Q)}{\partial n}\,G(Q,P)\,d\sigma . \tag{2,2,12}$$

This formula gives $U(P)$ in terms of its normal derivative for the exterior region D.

In our boundary value problem the normal derivative of the velocity potential Φ at the wing surface is prescribed. The problem is thus to find the suitable Green's function. With the aid of Sommerfeld's theory of "Riemann spaces" it is possible to determine a closed expression for

Green's function, but because of the limited usefulness of such analytical formula, we shall postpone the elaboration of this method to the appendix. In the next paragraph we shall derive an expression for Green's function in terms of an infinite series.

3 The regular solution of the boundary value problem.

In this paragraph we shall try to find an infinite series representation for Green's function. For this purpose we apply the method of separation of variables, which is often used in boundary value problems of this kind. We introduce the so-called oblate spheroidal coordinates. These coordinates are formed by rotating confocal elliptic coordinates in the xz -plane about the minor axis of the ellipses (i.e. the z -axis). The transformation formulae for these coordinates read

$$\begin{aligned} x &= \sqrt{1-\mu^2}\ \sqrt{1+\eta^2}\ \cos\vartheta \\ y &= \sqrt{1-\mu^2}\ \sqrt{1+\eta^2}\ \sin\vartheta \\ z &= \mu\eta \end{aligned} \tag{2,3,1}$$

where the range of variation of η is defined by $0 \leq \eta < \infty$, that of μ by $-1 \leq \mu \leq +1$ and that of ϑ by $0 \leq \vartheta < 2\pi$. The surface $\eta = 0$ is a disk of radius unity in the xy -plane, with centre at the origin. The surface $\mu = 1$ is the positive axis of z ; the surface $\mu = -1$ is the negative axis of z , and the surface $\mu = 0$ is the xy -plane, except for the region inside a circle of radius unity, centered at the origin (this region is just the surface $\eta = 0$). The surfaces $\eta =$ a positive constant are oblate spheroids with major axes given by $\frac{1}{2}\sqrt{1+\eta^2}$ and with minor axes given by $\frac{1}{2}\eta$; the surfaces $\mu =$ constant are hyperboloids of one sheet which are confocal with the spheroids $\eta =$ constant and orthogonal to them. The surface $\vartheta =$ constant is a plane passing through the axis of z at an angle ϑ with the xy -plane.

This transformation to spheroidal coordinates implies transformation formulae for the partial derivatives

$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$, viz.:

$$\begin{aligned} \frac{\partial}{\partial x} &= -\frac{\mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\cos\vartheta}{\mu^2+\eta^2}\frac{\partial}{\partial\mu} + \frac{\eta\sqrt{1-\mu^2}\sqrt{1+\eta^2}\cos\vartheta}{\mu^2+\eta^2}\frac{\partial}{\partial\eta} - \frac{\sin\vartheta}{\sqrt{1-\mu^2}\sqrt{1+\eta^2}}\frac{\partial}{\partial\vartheta} \\ \frac{\partial}{\partial y} &= -\frac{\mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\sin\vartheta}{\mu^2+\eta^2}\frac{\partial}{\partial\mu} + \frac{\eta\sqrt{1-\mu^2}\sqrt{1+\eta^2}\sin\vartheta}{\mu^2+\eta^2}\frac{\partial}{\partial\eta} + \frac{\cos\vartheta}{\sqrt{1-\mu^2}\sqrt{1+\eta^2}}\frac{\partial}{\partial\vartheta} \\ \frac{\partial}{\partial z} &= \frac{\eta(1-\mu^2)}{\mu^2+\eta^2}\frac{\partial}{\partial\mu} + \frac{\mu(1+\eta^2)}{\mu^2+\eta^2}\frac{\partial}{\partial\eta}. \end{aligned} \tag{2,3,2}$$

In the new coordinates the line element is expressed by

$$ds^2 = dx^2 + dy^2 + dx^2 = \frac{\mu^2+\eta^2}{1-\mu^2}\,d\mu^2 + \frac{\mu^2+\eta^2}{1+\eta^2}\,d\eta^2 + (1-\mu^2)(1+\eta^2)\,d\vartheta^2 . \tag{2,3,3}$$

Laplace's equation can now be written as

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\right\} + \frac{\partial}{\partial\eta}\left\{(1+\eta^2)\frac{\partial\Phi}{\partial\eta}\right\} + \frac{\partial^2\Phi}{\partial\vartheta^2}\left(\frac{1}{1-\mu^2} - \frac{1}{1+\eta^2}\right) = 0 . \tag{2,3,4}$$

In accordance with the method of separation of variables an expression of the form

$$\Phi(\eta,\mu,\vartheta) = F(\eta)\,G(\mu)\,H(\vartheta) \tag{2,3,5}$$

is assumed to be a solution of equation (2,3,4) wherein F denotes a function of η only, G a function of μ only and H a function of ϑ only. Solutions, which can be written in the form (2,3,5) are called normal solutions.
Substituting (2,3,5) into the equation (2,3,4) and dividing by FGH we obtain

$$\frac{1}{G}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dG}{d\mu}\right\} + \frac{1}{F}\frac{d}{d\eta}\left\{(1+\eta^2)\frac{dF}{d\eta}\right\} + \frac{1}{H}\frac{d^2H}{d\vartheta^2}\left\{\frac{1}{1-\mu^2} - \frac{1}{1+\eta^2}\right\} = 0 \tag{2,3,6}$$

Since the third term on the left-hand side is the only one which involves ϑ, it is impossible that the equation should be satisfied unless

$$\frac{1}{H}\frac{d^2H}{d\vartheta^2} = a \tag{2,3,7}$$

where a is a constant.
The equation (2,3,6) may now be written

$$\frac{1}{G}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dG}{d\mu}\right\} + \frac{1}{F}\frac{d}{d\eta}\left\{(1+\eta^2)\frac{dF}{d\eta}\right\} + a\left\{\frac{1}{1-\mu^2} - \frac{1}{1+\eta^2}\right\} = 0 \tag{2,3,8}$$

We see now that we must have

$$\frac{1}{G}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dG}{d\mu}\right\} + \frac{a}{1-\mu^2} = b \tag{2,3,9}$$

$$\frac{1}{F}\frac{d}{d\eta}\left\{(1+\eta^2)\frac{dF}{d\eta}\right\} - \frac{a}{1+\eta^2} = -b , \tag{2,3,10}$$

where b is again a constant.

The solution $\Phi(\eta,\mu,\vartheta)$ and, by consequence, also $H(\vartheta)$, must be a periodic function of ϑ with period 2π. Equation (2,3,7) yields such a solution provided the constant a assumes the value $-m^2$, where m is a positive integer or zero. The corresponding solutions are the functions $\cos m\vartheta$ and $\sin m\vartheta$

In the equation (2,3,9), μ varies from -1, which value it takes along the negative axis of z, to $+1$, which it takes along the positive axis of z. In order that the function $G(\mu)$ be finite over the range $(-1,+1)$, the constant b must be equal to $-n(n+1)$, where n is a positive integer (including zero) greater than, or equal to, m. The solution $G(\mu)$ is then proportional to the associated Legendre function of the first kind, viz. $P_n^m(\mu)$.

Replacing in the second equation (2,3,9) the independent variable μ by $i\eta$, the equation transforms into the equation (2,3,10). Hence the only possible solutions of (2,3,10) are $P_n^m(i\eta)$ and $Q_n^m(i\eta)$, representing the associated Legendre functions of the first and second kind respectively. The boundary condition at infinity prescribes that the potential Φ vanishes at infinity. This condition implies that the function $Q_n^m(i\eta)$ should be chosen for the solution of our problem.
Normal solutions of Laplace's equation showing the right behaviour at infinity are thus

$$P_n^m(\mu)\, Q_n^m(i\eta) \cos m\vartheta \quad \text{and} \quad P_n^m(\mu)\, Q_n^m(i\eta) \sin m\vartheta \,. \tag{2,3,11}$$

The functions $P_n^m(\mu)\cos m\vartheta$ and $P_n^m(\mu)\sin m\vartheta$, where n and m are positive integers, including zero, and $m \leq n$, are called surface harmonics. It is well known that these surface harmonics form a complete orthogonal system of functions defined on the region $0 \leq \vartheta < 2\pi$, $-1 \leq \mu \leq +1$ (ref.1).

The ortogonality relations read

$$\int_0^{2\pi}\int_{-1}^{+1} P_n^m(\mu) P_{n'}^{m'}(\mu) \cos m\vartheta \cos m'\vartheta \, d\mu \, d\vartheta = \begin{cases} 0 \text{ if } m \neq m' \text{ or } n \neq n' \\ \dfrac{2\pi}{2n+1}\dfrac{(n+m)!}{(n-m)!}\varepsilon_m \text{ if } m = m' \text{ and } n = n' \end{cases}$$

$$\int_0^{2\pi}\int_{-1}^{+1} P_n^m(\mu) P_{n'}^{m'}(\mu) \sin m\vartheta \sin m'\vartheta \, d\mu \, d\vartheta = \begin{cases} 0 \text{ if } m \neq m' \text{ or } n \neq n' \\ \dfrac{2\pi}{2n+1}\dfrac{(n+m)!}{(n-m)!} \text{ if } m = m' \text{ and } n = n' \end{cases}$$

$$\int_0^{2\pi}\int_{-1}^{+1} P_n^m(\mu) P_{n'}^{m'}(\mu) \cos m\vartheta \sin m'\vartheta \, d\mu \, d\vartheta = 0 \quad \text{for all values of } n, m, n' \text{ and } m' \tag{2,3,12}$$

ε_m represents the Neumann factor, viz. $\varepsilon_m = 1$ for $m > 0$, $\varepsilon_0 = 2$.

The following important theorem concerning these surface harmonics will now be used (ref.1).

If $g(\mu,\vartheta)$ is a function of the independent variables μ and ϑ, defined on the region $-1 \leq \mu \leq +1, 0 \leq \vartheta \leq 2\pi$ and if this function has continuous partial derivatives of the second order, then $g(\mu,\vartheta)$ can be expanded into a uniformly convergent series of surface harmonics.

Thus we may write

$$g(\mu,\vartheta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m P_n^m(\mu)\cos m\vartheta + \sum_{n=1}^{\infty}\sum_{m=1}^{n} B_n^m P_n^m(\mu)\sin m\vartheta \,. \tag{2,3,13}$$

In order to determine the coefficients A_n^m and B_n^m we multiply both sides of (2,3,13) with $P_l^k(\mu)\cos k\vartheta$ and $P_l^k(\mu)\sin k\vartheta$ respectively and thereupon integrate over μ from -1 to +1 and over ϑ from 0 to 2π. If the series on the right-hand side of (2,3,13) is assumed to be uniformly convergent, termswise integration is allowed. Performing this integration and applying the orthogonality relations (2,3,12), we find for the coefficients

$$A_l^k = \frac{1}{\varepsilon_k}\frac{2l+1}{2\pi}\frac{(l-k)!}{(l+k)!}\int_0^{2\pi}\int_{-1}^{+1} q(\mu_1,\vartheta_1)\, P_l^k(\mu_1)\cos k\vartheta_1\, d\mu_1\, d\vartheta_1$$

and

$$B_l^k = \frac{2l+1}{2\pi}\frac{(l-k)!}{(l+k)!}\int_0^{2\pi}\int_{-1}^{+1} q(\mu_1,\vartheta_1)\, P_l^k(\mu_1)\sin k\vartheta_1\, d\mu_1\, d\vartheta_1 \quad . \tag{2,3,14}$$

The condition of symmetry (2,1,9), expressing the odd character of the velocity potential with respect to the coordinate z or μ, restricts the values of the non-negative integers n and m to those values for which $n+m$ is an odd integer.

Let us now assume that a suitable solution of our boundary value problem can be written in the form

$$\Phi(\eta,\mu,\vartheta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' \alpha_n^m P_n^m(\mu) Q_n^m(i\eta)\cos m\vartheta + \sum_{n=1}^{\infty}\sum_{m=1}^{n}{}' \beta_n^m P_n^m(\mu) Q_n^m(i\eta)\sin m\vartheta \tag{2,3,15}$$

where the prime denotes that only those values of m are to be taken for which $n+m$ is an odd integer.

In order to determine the coefficients α_n^m and β_n^m we write down the boundary condition of prescribed downwash distribution over the wing surface in the spheroidal coordinates at $\eta=0$.

This condition is expressed by means of the formula

$$\left\{\frac{\partial\Phi}{\partial\eta}\right\}_{\eta=0} = \mu\, w(\mu,\vartheta) \quad . \tag{2,3,16}$$

The assumption that the function $w(\mu,\vartheta)$ is two times differentiable with continuous derivatives of the second order, implies that the function $\mu w(\mu,\vartheta)$ can be expanded into a uniformly convergent series of odd surface *harmonics*.

Thus we can write

$$\mu w(\mu,\vartheta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' A_n^m P_n^m(\mu)\cos m\vartheta + \sum_{n=1}^{\infty}\sum_{m=1}^{n}{}' B_n^m P_n^m(\mu)\sin m\vartheta \tag{2,3,17}$$

with

$$A_n^m = \frac{1}{\varepsilon_m}\frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_{-1}^{+1}\mu w(\mu,\vartheta)\, P_n^m(\mu)\cos m\vartheta\, d\mu\, d\vartheta$$

$$B_n^m = \frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_{-1}^{+1}\mu w(\mu,\vartheta)\, P_n^m(\mu)\sin m\vartheta\, d\mu\, d\vartheta \quad . \tag{2,3,18}$$

Differentiating the formal series solution (2,3,15) with respect to η and thereupon putting $\eta=0$, we obtain

$$\left\{\frac{\partial\Phi}{\partial\eta}\right\}_{\eta=0} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}{}' \alpha_n^m P_n^m(\mu) Q_n^{m\prime}(i0)\cos m\vartheta + \sum_{n=1}^{\infty}\sum_{m=1}^{n}{}' \beta_n^m P_n^m(\mu) Q_n^{m\prime}(i0)\sin m\vartheta \tag{2,3,19}$$

where $$Q_n^{m'}(i0) = \lim_{\eta \to 0} \frac{d}{d\eta} Q_n^m(i\eta)$$

Equating the corresponding coefficients in the expressions (2,3,17) and (2,3,19), we find

$$\alpha_n^m = \frac{A_n^m}{Q_n^{m'}(i0)} \qquad \text{and} \qquad \beta_n^m = \frac{B_n^m}{Q_n^{m'}(i0)} \qquad (2,3,20)$$

The solution $\Phi(\eta,\mu,\varphi)$ becomes thus

$$\Phi(\eta,\mu,\vartheta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' A_n^m P_n^m(\mu) \frac{Q_n^m(i\eta)}{Q_n^{m'}(i0)} \cos m\vartheta + \sum_{n=1}^{\infty} \sum_{m=1}^{n}{}' B_n^m P_n^m(\mu) \frac{Q_n^m(i\eta)}{Q_n^{m'}(i0)} \sin m\vartheta$$

or

$$\Phi(\eta,\mu,\vartheta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' \frac{1}{\varepsilon_m} \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} P_n^m(\mu) \frac{Q_n^m(i\eta)}{Q_n^{m'}(i0)} \cos m\vartheta \int_0^{2\pi}\int_{-1}^{+1} \mu_1 w(\mu_1,\vartheta_1) P_n^m(\mu_1) \cos m\vartheta_1 \, d\mu_1 \, d\vartheta_1$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{n}{}' \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} P_n^m(\mu) \frac{Q_n^m(i\eta)}{Q_n^{m'}(i0)} \sin m\vartheta \int_0^{2\pi}\int_{-1}^{+1} \mu_1 w(\mu_1,\vartheta_1) P_n^m(\mu_1) \sin m\vartheta_1 \, d\mu_1 \, d\vartheta_1 \qquad (2,3,21)$$

In the next paragraph it will be proved that the series occurring in the formula (2,3,21) are uniformly convergent for all admissible values of the variables $\mu, \mu_1, \vartheta, \vartheta_1$ and for η lying in the interval $\delta \leq \eta < \infty$, where δ is an arbitrary positive number. Under this condition of uniform convergence, it is allowable to interchange the order of summation and integration in (2,3,21). This leads to the expression

$$\Phi(\eta,\mu,\vartheta) =$$
$$\int_0^{2\pi}\int_{-1}^{+1} \mu_1 w(\mu_1,\vartheta_1) \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' \frac{1}{\varepsilon_m} \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} P_n^m(\mu) P_n^m(\mu_1) \frac{Q_n^m(i\eta)}{Q_n^{m'}(i0)} \cos m(\vartheta-\vartheta_1) \, d\mu_1 \, d\vartheta_1 \qquad (2,3,22)$$

Putting

$$G(\eta,\mu,\vartheta;\mu_1,\vartheta_1) = \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' \frac{1}{\varepsilon_m} \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} P_n^m(\mu) P_n^m(\mu_1) \frac{Q_n^m(i\eta)}{Q_n^{m'}(i0)} \cos m(\vartheta-\vartheta_1) \qquad (2,3,23)$$

we can write

$$\Phi(\eta,\mu,\vartheta) = \int_0^{2\pi}\int_{-1}^{+1} \mu_1 w(\mu_1,\vartheta_1) G(\eta,\mu,\vartheta;\mu_1,\vartheta_1) \, d\mu_1 \, d\vartheta_1 \qquad (2,3,24)$$

or in Cartesian coordinates

$$\Phi(x,y,z) = \iint_S \left(\frac{\partial \Phi}{\partial n}\right)_{z_1=0} G(x,y,z;x_1,y_1) \, dx_1 \, dy_1 \qquad (2,3,25)$$

where S denotes the surface formed by the two sides of the wing. Apart from a factor this formula (2,3,25) agrees with the formula (2,2,12) which expresses the potential in terms of its normal derivative at the boundary with the aid of Green's function of the second kind.
Because of the uniqueness of the solution we may conclude that the function $G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)$, defined by (2,3,23), represents a series expansion for Green's function of the second kind.
This series expansion shows that Green's function and consequently the potential Φ, vanishes in the xy-plane outside the wing.

4 The uniform convergence of the expansion of Green's function.

It will first be shown that the series in question

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' \frac{1}{\varepsilon_m} \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} P_n^m(\mu) P_n^m(\mu_1) \frac{Q_n^m(i\eta)}{Q_n^{m'}(i0)} \cos m(\vartheta-\vartheta_1) \tag{2,4,1}$$

is absolutely convergent in the region $\eta > \delta$, where δ is an arbitrary positive number.

We observe that the function $Q_n^m(z)$ for z not being on the real axis between +1 and $-\infty$ is defined by

$$Q_n^m(z) = (z^2-1)^{m/2} \frac{d^m}{dz^m} Q_n(z) \tag{2,4,2}$$

Hence it is clear that

$$Q_n^{m'}(i0) = \lim_{\eta \to 0} \frac{d}{d\eta} Q_n^m(i\eta) = Q_n^{m+1}(i0) \tag{2,4,3}$$

In order to give an estimation of the ratio $\dfrac{Q_n^m(i\eta)}{Q_n^{m'}(i0)}$ we apply the following integral representation of the function $Q_n^m(z)$ (ref.2)

$$Q_n^m(z) = e^{im\pi} \frac{n!}{(n-m)!} \int_0^{\infty} \frac{\cosh mu}{\left\{z+(z^2-1)^{1/2}\cosh u\right\}^{n+1}} du \tag{2,4,4}$$

wherein $m \leqq n$, and z is not a point on the real axis between +1 and $-\infty$.
We have

$$\frac{Q_n^m(i\eta)}{Q_n^{m'}(i0)} = \frac{Q_n^m(i\eta)}{Q_n^{m+1}(i0)} = e^{\pi i} \frac{(n-m-1)!}{(n-m)!} \frac{\displaystyle\int_0^{\infty} \frac{\cosh mu}{\left\{\eta + \sqrt{1+\eta^2}\cosh u\right\}^{n+1}} du}{\displaystyle\int_0^{\infty} \frac{\cosh (m+1)u}{\left\{\cosh u\right\}^{n+1}} du}$$

Because here $\eta > 0$ and $n-m \geqq 1$, it follows:

$$\left|\frac{Q_n^m(i\eta)}{Q_n^{m'}(i0)}\right| \leqq \frac{1}{\sqrt{1+\eta^2}^{\,n+1}} \tag{2,4,5}$$

Further use is made of the representation of the Legendre polynomials by their generating function, viz.

$$(1-2r\cos\theta + r^2)^{-1/2} = \sum_{n=0}^{\infty} r^n P_n(\cos\theta) \tag{2,4,6}$$

Another expansion for the left-hand side of (2,4,6) can be obtained by writing $(1-2r\cos\theta+r^2)^{-1/2}$ as

$$\left(1-re^{i\theta}\right)^{-1/2}\left(1-re^{-i\theta}\right)^{-1/2} \tag{2,4,7}$$

which can be expanded into the product

$$\left\{1+\frac{1}{2}re^{i\theta}+\frac{1.3}{2.4}r^2e^{2i\theta}+\cdots\right\}\left\{1+\frac{1}{2}re^{-i\theta}+\frac{1.3}{2.4}r^2e^{-2i\theta}+\cdots\right\} \tag{2,4,8}$$

These two binomial expansions are absolutely convergent for $|r|<1$ and thus their Cauchy product converges to the product of their sums. Isolating the coefficient of r^n we obtain the formula

$$P_n(\cos\theta)=2\,\frac{1.3.5.---(2n-1)}{2.4.6.---\;2n}\left\{\cos n\theta+\frac{1.n}{1\,(2n-1)}\cos(n-2)\theta+\right.$$

$$\left.+\frac{1.3.n(n-1)}{1.2\,(2n-1)(2n-3)}\cos(n-4)\theta+\cdots\right\} \tag{2,4,9}$$

From this expression for $P_n(\cos\theta)$ it is easily seen that, when θ is real, the maximum value of $P_n(\cos\theta)$ occurs for $\theta=0$, in which case $P_n=1$; thus $P_n(\cos\theta)$ never exceeds +1. It is also clear that $P_n(\cos\theta)$ is not less than -1; thus when x is between -1 and +1, $P_n(x)$ always lies between -1 and +1. The boundedness of $P_n(x)$ will now be applied to the integral relation

$$\int_0^{2\pi}\cos m\psi\,P_n(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\psi)\,d\psi=$$

$$2\pi\,\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)\,P_n^m(\cos\theta') \tag{2,4,10}$$

which follows immediately from the well-known addition theorem for the Legendre polynomials (ref. 7), viz.:

$$P_n(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\psi)=P_n(\cos\theta)P_n(\cos\theta')+$$

$$+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m\psi\,. \tag{2,4,11}$$

So we get the relation

$$\left|\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\right|\leqq 1 \tag{2,4,12}$$

With regard to the general term of the series (2,4,1) we can now conclude that for all values of the non-negative integer n the following relation holds:

$$\left|\frac{1}{\varepsilon_m}\,\frac{2n+1}{2\pi}\,\frac{(n-m)!}{(n+m)!}P_n^m(\mu)P_n^m(\mu_1)\,\frac{Q_n^m(i\eta)}{Q_n^{m'}(i0)}\cos m(\vartheta-\vartheta_1)\right|<C\,\frac{(2n+1)}{\sqrt{1+\eta^2}^{\;n+1}} \tag{2,4,13}$$

where C is a fixed constant.
Since each term of the series (2,4,1) is numerically less, in absolute value, than a fixed multiple of $(2n+1)(1+\eta^2)^{-\frac{n+1}{2}}$, it follows that the expansion of Green's function is absolutely convergent for $\eta > \delta$, where δ is an arbitrary positive number. Application of Weierstrass' test for uniform convergence shows that the series expansion for Green's function is uniformly convergent for all admissible values of μ, μ_1, ϑ and ϑ_1 and for all values of η lying in the range $\delta \leq \eta < \infty$, where δ is an arbitrary positive number.

5 Comparison of two acceleration potentials.

In chapter I it has been shown that in the linearized theory the acceleration potential satisfies Laplace's equation. There are now two alternatives for the determination of the acceleration potential.

At first we can derive a formula for the acceleration potential by aid of the steady form of the relation (1,3,7), which expresses the connection between velocity potential and acceleration potential. Denoting this acceleration potential by ψ^*, we get

$$\psi^* = U \frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x} U \iint_S w(x_1, y_1)\, G(x, y, z; x_1, y_1)\, dx_1\, dy_1 . \qquad (2,5,1)$$

On the other hand an expression for the acceleration potential can be found in a quite similar way as has been applied for the determination of the velocity potential Φ. Instead of the normal velocity w the normal acceleration a must now be prescribed at the wing surface. If the corresponding acceleration potential is designated by ψ, we have

$$\psi = \iint_S a(x_1, y_1)\, G(x, y, z; x_1, y_1)\, dx_1\, dy_1 \qquad (2,5,2)$$

In order to compare both expressions for the acceleration potential, we evaluate the potential at points of the wing surface. Inserting $\eta = 0$ into the formulas (2,5,1) and (2,5,2) we get

$$\psi(0, \mu, \vartheta) = \int_0^{2\pi} \int_{-1}^{+1} a(\mu_1, \vartheta_1)\, G(0, \mu, \vartheta; \mu_1, \vartheta_1)\, \mu_1\, d\mu_1\, d\vartheta_1 \qquad (2,5,3)$$

and

$$\psi^*(0, \mu, \vartheta) = -U \frac{\sqrt{1-\mu^2}}{\mu} \cos\vartheta\, \Phi_\mu(0, \mu, \vartheta) - U \frac{\sin\vartheta}{\sqrt{1-\mu^2}}\, \Phi_\vartheta(0, \mu, \vartheta) . \qquad (2,5,4)$$

As Green's function vanishes at the wing edge, i.e. for $\mu = \eta = 0$, it follows from equation (2,5,3) that the acceleration potential ψ also equals zero at the edge of the wing. This conclusion may only be made when the function $a(\mu, \vartheta)$ in (2,5,3) fulfils similar conditions as the normal velocity w, notably $a(\mu, \vartheta)$ must be continuously differentiable up to the second order. However, the formula (2,5,4) shows that the acceleration potential ψ^* becomes infinite as $1/\mu$ along the whole edge of the wing.

In physical terms the difference between the two potentials ψ and ψ^* can be explained in the following way.

The velocity potential Φ and consequently the acceleration potential ψ^* describes the flow over a circular wing without circulation, whereas the acceleration potential ψ represents the potential of a flow with a shock-free entrance (no pressure singularity at the leading edge). The only possible reason for the difference between the two potentials ψ and ψ^* must be ascribed to the fact that the normal acceleration a indeed does not fulfil the condition of continuous differentiability up to the second order. This aspect will be further elaborated in the next paragraph, in which an exact formula for the normal acceleration will be derived.

6 Complete expression for the normal acceleration at the wing surface.

In this paragraph a more detailed analysis for the evaluation of the normal acceleration at the wing surface will be given. Because of the three-dimensional characteristics of our problem it offers some advantages with respect to the surveyability to apply the technique of the tensor calculus. Moreover it will be supposed that some formulas that often occur in the tensor calculus are well known to the reader. For the derivation of those reference is made to text-books in common use (e.g. ref.20).

Let us consider a body that is placed in a homogeneous flow, x^i $(i=1,2,3)$ being the Cartesian space coordinates. The surface S of the body be given in parametric form, viz.:

$$x^i = x^i(u^\alpha)\ ,\quad \alpha = 1,2 \tag{2,6,1}$$

where u^α represent Gaussian surface coordinates.
The line-element on the surface can be expressed by

$$ds^2 = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta}du^\alpha du^\beta = a_{\alpha\beta}\, du^\alpha du^\beta \tag{2,6,2}$$

The system of numbers $a_{\alpha\beta}$ represents the so-called first fundamental tensor.

If $v_i = v^i$ denotes the velocity vector of the flow, one has

$$v^i = \frac{dx^i}{dt} = \frac{\partial x^i}{\partial u^\alpha}\frac{du^\alpha}{dt} = x^i_\alpha v^\alpha \quad , \tag{2,6,3}$$

where

$$v^\alpha = \frac{du^\alpha}{dt} \quad .$$

The acceleration vector a^i is found by differentiation of the velocity vector, hence,

$$a^i = \frac{dv^i}{dt} = \frac{dx^i_\alpha}{dt}v^\alpha + x^i_\alpha\frac{dv^\alpha}{dt}$$

or

$$a^i = x^i_{\alpha,\beta}\, v^\alpha v^\beta + x^i_\alpha \frac{dv^\alpha}{dt}\ . \tag{2,6,4}$$

In this expression the quantity $x^i_{\alpha,\beta}$ denotes:

$$x^i_{\alpha,\beta} = \frac{\partial x^i_\alpha}{\partial u^\beta} + \Gamma^i_{jk}\, x^j_\alpha x^k_\beta - \Gamma^\lambda_{\alpha\beta}\, x^i_\lambda \tag{2,6,5}$$

where Γ^i_{jk} is the Christoffel symbol formed from the metric coefficients g_{ij} associated with the coordinates x^i and $\Gamma^\lambda_{\alpha\beta}$ is the Christoffel symbol formed from the coefficients $a_{\alpha\beta}$ associated with the Gaussian surface coordinates u^α . The quantities Γ^i_{jk} vanish because x^i are Cartesian coordinates. The tensor $x^i_{\alpha,\beta}$ is called the tensor derivative of x^i_α with respect to u^β . It is well known that $x^i_{\alpha,\beta}$ is a space vector normal to the surface, hence it is directed along the unit normal n^i . Consequently, there exists a set of numbers $b_{\alpha\beta}$ such that

$$x^i_{\alpha,\beta} = b_{\alpha\beta} n^i \tag{2,6,6}$$

Equations (2,6,6) are known as the formulas of Gauss.
The quantities $b_{\alpha\beta}$ are the components of a symmetric surface tensor.
In the tensor calculus the quadratic form

$$b_{\alpha\beta}\, du^\alpha\, du^\beta$$

is called the second fundamental form.

Inserting the relation (2,6,6) into the expression (2,6,4) for the acceleration vector, we obtain

$$a^i = b_{\alpha\beta}\, v^\alpha v^\beta n^i + x^i_\alpha \frac{dv^\alpha}{dt} \quad . \tag{2,6,7}$$

The component of a^i in the direction of the unit normal n to the surface S thus becomes

$$a_n = a^i n_i = b_{\alpha\beta}\, v^\alpha v^\beta + x^i_\alpha \frac{dv^\alpha}{dt} n_i \quad . \tag{2,6,8}$$

Assuming that the velocity vector v^i satisfies the condition of tangential flow at the body surface we have

$$x^i_\alpha\, n_i = 0 \quad . \tag{2,6,9}$$

Accordingly equation (2,6,8) reduces to

$$a_n = b_{\alpha\beta} v^\alpha v^\beta \tag{2,6,10}$$

If s_1 and s_2 are the principal directions on the surface S , λ_1 and λ_2 the unit vectors in the directions s_1 and s_2 respectively and $\varkappa_1$ and $\varkappa_2$ the principal curvatures corresponding to the principal directions, the following relations, which determine the principal directions, hold

$$\begin{aligned} (b_{\alpha\beta} - \varkappa_1 a_{\alpha\beta})\lambda^\alpha_1 = 0 \\ (b_{\alpha\beta} - \varkappa_2 a_{\alpha\beta})\lambda^\alpha_2 = 0 \end{aligned} \tag{2,6,11}$$

From these equations (2,6,11) some other relations can be derived. Multiplying the first equation with λ^β_2 and the second one with λ^β_1 and subtracting thereupon, we obtain

$$a_{\alpha\beta}\, \lambda^\alpha_1 \lambda^\beta_2 = 0 \tag{2,6,12}$$

on the assumption that the principal curvatures are unequal.
This relation (2,6,12) expresses the fact that the unit vectors λ_1 and λ_2 are perpendicular.
Furthermore we see that

$$b_{\alpha\beta}\lambda_1^{\alpha}\lambda_2^{\beta}=0 \quad . \tag{2,6,13}$$

Multiplying the first equation (2,6,11) by λ_1^{β} and the second one by λ_2^{β}, addition of the results gives

$$\varkappa_1=b_{\alpha\beta}\lambda_1^{\alpha}\lambda_1^{\beta} \quad \text{and} \quad \varkappa_2=b_{\alpha\beta}\lambda_2^{\alpha}\lambda_2^{\beta} \quad . \tag{2,6,14}$$

The quantity v^{α} can now be transformed as given by

$$v^{\alpha}=\frac{du^{\alpha}}{dt}=\frac{\partial u^{\alpha}}{\partial s_1}\frac{ds_1}{dt}+\frac{\partial u^{\alpha}}{\partial s_2}\frac{ds_2}{dt}=\lambda_1^{\alpha}\frac{ds_1}{dt}+\lambda_2^{\alpha}\frac{ds_2}{dt} \quad . \tag{2,6,15}$$

Substitution of this result into the formula (2,6,10) yields:

$$a_n=b_{\alpha\beta}\left\{\lambda_1^{\alpha}\lambda_1^{\beta}\left(\frac{ds_1}{dt}\right)^2+\lambda_2^{\alpha}\lambda_2^{\beta}\left(\frac{ds_2}{dt}\right)^2+\lambda_1^{\alpha}\lambda_2^{\beta}\frac{ds_1}{dt}\frac{ds_2}{dt}+\lambda_2^{\alpha}\lambda_1^{\beta}\frac{ds_1}{dt}\frac{ds_2}{dt}\right\} \quad .$$

Using the relations (2,6,13) and (2,6,14) we find:

$$a_n=\varkappa_1\left(\frac{ds_1}{dt}\right)^2+\varkappa_2\left(\frac{ds_2}{dt}\right)^2 \quad . \tag{2,6,16}$$

In physical language the formula (2,6,16) expresses that the normal acceleration a_n at the surface S of the body is composed of the two centripetal accelerations of the fluid, each of them corresponding with a principal direction on the surface.

This general formula (2,6,16) will now be applied to the problem of the flow around an oblate spheroid. Such an oblate spheroid can be represented by the parametric equations

$$\begin{aligned} x^1 &= x=\sqrt{1-\mu^2}\sqrt{1+\eta_0^2}\ \cos\vartheta \\ x^2 &= y=\sqrt{1-\mu^2}\sqrt{1+\eta_0^2}\ \sin\vartheta \\ x^3 &= z=\mu\eta_0 \end{aligned} \tag{2,6,17}$$

where the quantity η_0 is a fixed positive number.

If η_0 tends to zero, the spheroid flattens down to the circular region in the xy -plane: $x^2+y^2\leq 1, z=0$. In other words, the oblate spheroid degenerates to our circular wing, if η_0 tends to zero. In fact η_0 gives a measure for the thickness of the wing. Until further notice we shall assume that η_0 is different from zero. In that case we can designate the Gaussian surface coordinates by

$$u^1=\mu \quad \text{and} \quad u^2=\vartheta$$

Elementary calculations yield the following expressions for the components of the first and second fundamental tensors

$$a_{11} = \frac{\mu^2+\eta_0^2}{1-\mu^2} \quad ; \quad a_{12} = a_{21} = 0 \quad ; \quad a_{22} = (1-\mu^2)\,(1+\eta_0^2)$$

$$b_{11} = -\frac{\eta_0\sqrt{1+\eta_0^2}}{(1-\mu^2)\sqrt{\mu^2+\eta_0^2}} \;;\quad b_{12} = b_{21} = 0 \;;\; b_{22} = -\frac{\eta_0(1-\mu^2)\sqrt{1+\eta_0^2}}{\sqrt{\mu^2+\eta_0^2}} \qquad (2,6,18)$$

The principal curvatures $\varkappa_n (n=1,2)$ are determined from the determinantal equation

$$\det.\left(b_{\alpha\beta} - \varkappa a_{\alpha\beta}\right) = 0 \quad . \qquad (2,6,19)$$

In our case equation (2,6,19) can be written in the form

$$\left(b_{11} - \varkappa a_{11}\right)\left(b_{22} - \varkappa a_{22}\right) = 0$$

and hence

$$\varkappa_1 = -\frac{\eta_0\sqrt{1+\eta_0^2}}{\sqrt{\mu^2+\eta_0^2}^{\,3}} \quad \text{and} \quad \varkappa_2 = -\frac{\eta_0}{\sqrt{1+\eta_0^2}\sqrt{\mu^2+\eta_0^2}} \quad . \qquad (2,6,20)$$

The relations (2,6,11) thus give

$$\lambda_1^2 = \frac{\partial\vartheta}{\partial s_1} = 0 \qquad \text{and} \qquad \lambda_2^1 = \frac{\partial\mu}{\partial s_2} = 0 \quad ,$$

or in other words: the coordinate net(μ,ϑ) coincides with the net of the principal directions (s_1, s_2).
The exact relations between μ and s_1, and ϑ and s_2 are found by aid of the formulas (2,6,14), viz.

$$\frac{\partial\mu}{\partial s_1} = \frac{1}{\sqrt{a_{11}}} = \frac{\sqrt{1-\mu^2}}{\sqrt{\mu^2+\eta_0^2}} \quad ; \quad \frac{\partial\vartheta}{\partial s_2} = \frac{1}{\sqrt{a_{22}}} = \frac{1}{\sqrt{(1-\mu^2)\,(1+\eta_0^2)}} \quad . \qquad (2,6,21)$$

If χ denotes the velocity potential of the field of flow around the spheroid, the normal acceleration a_n at the surface of the spheroid can be written, according to (2,6,16), in the form:

$$a_n = -\frac{\eta_0\sqrt{1+\eta_0^2}}{\sqrt{\mu^2+\eta_0^2}^{\,3}}\left(\frac{\partial\chi}{\partial s_1}\right)^2 - \frac{\eta_0}{\sqrt{(\mu^2+\eta_0^2)(1+\eta_0^2)}}\left(\frac{\partial\chi}{\partial s_2}\right)^2 \qquad (2,6.22)$$

Substitution of the relations (2,6,21) into (2,6,22) yields

$$a_n = -\frac{\eta_0\sqrt{1+\eta_0^2}\,(1-\mu^2)}{\sqrt{\mu^2+\eta_0^2}^{\,5}}\left(\frac{\partial\chi}{\partial\mu}\right)^2 - \frac{\eta_0}{(1-\mu^2)\sqrt{\mu^2+\eta_0^2}\sqrt{1+\eta_0^2}^{\,3}}\left(\frac{\partial\chi}{\partial\vartheta}\right)^2 \qquad (2,6,23)$$

We did assume that the spheroid is placed in a homogeneous flow with a velocity vector U directed along the positive axis of x . The velocity potential χ can be split in two parts, viz. the potential Ux corresponding

to the oncoming flow and a perturbation potential Φ .
Thus

$$X(\eta,\mu,\vartheta) = Ux + \Phi(\eta,\mu,\vartheta) = U\sqrt{1-\mu^2}\sqrt{1+\eta^2}\cos\vartheta + \Phi(\eta,\mu,\vartheta) \cdot \tag{2,6,24}$$

Application of the formula (2,6,23) to (2,6,24) yields for the normal acceleration a_n on the surface $\eta = \eta_o$

$$a_n = -\frac{\eta_o\sqrt{1+\eta_o^2}\,(1-\mu^2)}{\sqrt{\mu^2+\eta_o^2}^{\,5}}\left[-U\frac{\mu}{\sqrt{1-\mu^2}}\sqrt{1+\eta_o^2}\cos\vartheta + \Phi_\mu(\eta_o,\mu,\vartheta)\right]^2 +$$
$$-\frac{\eta_o}{(1-\mu^2)\sqrt{\mu^2+\eta_o^2}\sqrt{1+\eta_o^2}^{\,3}}\left[-U\sqrt{1-\mu^2}\sqrt{1+\eta_o^2}\sin\vartheta + \Phi_\vartheta(\eta_o,\mu,\vartheta)\right]^2 \tag{2,6,25}$$

Starting from this formula (2,6,25) we shall now investigate to what form the normal acceleration a_n degenerates when the spheroid flattens down to the circular wing.
Putting $\eta_o = 0$ in the expression (2,6,25) we see that a_n becomes zero, provided μ unequals zero. This means that the normal acceleration vanishes at the surface of the circular planform, except in the points of the edge of the wing.
In order to obtain an exact formula for the normal acceleration at the wing surface, including the points on the edge, we shall assume for a moment that η_o is a small positive number.
Moreover it can be stated that, within the scope of the linearized theory, terms which contain second powers of the derivatives of the velocity potential Φ , may be neglected in the expression (2,6,25).
The normal acceleration can thus be approximated by

$$a_n \approx -\frac{\eta_o(1-\mu^2)}{\sqrt{\mu^2+\eta_o^2}^{\,5}}\left[U^2\frac{\mu^2}{1-\mu^2}\cos^2\vartheta - 2U\frac{\mu}{\sqrt{1-\mu^2}}\cos\vartheta\,\Phi_\mu(0,\mu,\vartheta)\right] +$$
$$-\frac{\eta_o}{(1-\mu^2)\sqrt{\mu^2+\eta_o^2}}\left[U^2(1-\mu^2)\sin^2\vartheta - 2U\sqrt{1-\mu^2}\sin\vartheta\,\Phi_\vartheta(0,\mu,\vartheta)\right] \cdot \tag{2,6,26}$$

Before reducing further the formula (2,6,26), we consider the function

$$\Delta(\mu,\eta_o) = \frac{1}{\pi}\,\frac{\eta_o}{\mu^2+\eta_o^2} \quad . \tag{2,6,27}$$

We note that this function $\Delta(\mu,\eta_o)$ satisfies the relation

$$\int_{-\infty}^{+\infty}\Delta(\mu,\eta_o)\,d\mu = 1 \tag{2,6,28}$$

In order to investigate the properties of $\Delta(\mu,\eta_o)$ in case η_o and μ tend to zero we rewrite the function in the form

$$\Delta(\mu,\eta_0)=\frac{1}{\pi\eta_0}\,\frac{1}{1+\left(\frac{\mu}{\eta_0}\right)^2} \qquad (2,6,29)$$

This formula (2,6,29) shows, that if we put μ equal to zero, the function $\Delta(\mu,\eta_0)$ increases strongly if η_0 tends to zero. On the other hand, if μ differs from zero, and η_0 tends to zero, $\Delta(\mu,\eta_0)$ converges to zero. We thus see that $\lim\limits_{\eta_0\to 0}\Delta(\mu,\eta_0)$ is a "function" whose integral always equals unity, whose value at $\mu=0$ increases toward infinity and whose value for $\mu\neq 0$ converges to zero.
Therefore we may conclude that

$$\lim_{\eta_0\to 0}\Delta(\mu,\eta_0)=\delta(\mu) \quad . \qquad (2,6,30)$$

$\delta(\mu)$ represents the deltafunction of Dirac.
Furthermore we calculate now the derivative of the function $\Delta(\mu,\eta_0)$

$$\frac{\partial}{\partial\mu}\Delta(\mu,\eta_0)=-\frac{2}{\pi}\,\frac{\mu\eta_0}{(\mu^2+\eta_0^2)^2} \quad . \qquad (2,6,31)$$

If the function $f(\mu)$ is an arbitrary, differentiable function of the variable μ, we can write down the relation

$$\int_{-\infty}^{+\infty}\frac{\partial}{\partial\mu}\Delta(\mu,\eta_0)\,f(\mu)\,d\mu=-\int_{-\infty}^{+\infty}\Delta(\mu,\eta_0)\,f'(\mu)\,d\mu \quad . \qquad (2,6,32)$$

If η_0 tends to zero the right hand side of (2,6,32) tends to the value $-f'(0)$. This fact expresses that the "function" $\lim\limits_{\eta_0\to 0}\frac{\partial}{\partial\mu}\Delta(\mu,\eta_0)$ can be identified with the first derivative of the delta function.
Thus we have

$$\lim_{\eta_0\to 0}-\frac{2}{\pi}\,\frac{\mu\eta_0}{(\mu^2+\eta_0^2)^2}=\delta'(\mu) \quad . \qquad (2,6,33)$$

Taking now the limit $\eta_0\to 0$ in the formula (2,6,26) and applying the relation (2,6,33), we obtain the following expression for the normal acceleration at the circular wing surface

$$a_n\mu=\frac{\pi}{2}\delta'(\mu)\,(1-\mu^2)\left[\frac{\mu}{1-\mu^2}\,U^2\cos^2\vartheta-2\,U\cos\vartheta\,\frac{1}{\sqrt{1-\mu^2}}\,\Phi_\mu(0,\mu,\vartheta)\right]$$

or

$$a_n=\frac{\pi}{2}U^2\cos^2\vartheta\,\delta'(\mu)-\pi U\cos\vartheta\,\Phi_\mu(0,\mu,\vartheta)\sqrt{1-\mu^2}\;\frac{\delta'(\mu)}{\mu} \quad . \qquad (2,6,34)$$

This expression for the normal acceleration corresponds to the complete field of flow around the circular aerofoil. In fact we are interested in the normal acceleration, which corresponds to the disturbance part of the field of flow. This quantity can be found in the following way.
The acceleration vector corresponding with the complete field of flow can be resolved into two vectors, viz. the acceleration vector of the undisturbed flow and the acceleration vector which corresponds to the disturbance part of the flow. If v_n denotes the normal velocity at the wing surface ,

of the undisturbed flow, then the corresponding normal acceleration is given by $\frac{dv_n}{dt} = U\frac{\partial v_n}{\partial x}$. Thus the normal acceleration at the wing surface of the disturbance part of the field of flow becomes

$$a_n = \frac{\pi}{2}U^2\cos^2\vartheta\,\delta'(\mu) - \pi U\cos\vartheta\,\Phi_\mu(0,\mu,\vartheta)\sqrt{1-\mu^2}\,\frac{\delta(\mu)}{\mu} - U\frac{\partial v_n}{\partial x} \qquad (2,6,35)$$

or in terms of the perturbation normal velocity w

$$a_n = \frac{\pi}{2}U^2\cos^2\vartheta\,\delta'(\mu) - \pi U\cos\vartheta\,\Phi_\mu(0,\mu,\vartheta)\sqrt{1-\mu^2}\,\frac{\delta(\mu)}{\mu} + U\frac{\partial w}{\partial x} \qquad (2,6,36)$$

Equation (2,6,36) gives an exact formula for the normal disturbance acceleration at the wing surface in the linearized theory.
In the next section this expression (2,6,36) will be used to find the connection between the two acceleration potentials ψ and ψ^x.

7 The identity of the two acceleration potentials ψ and ψ^x.

The starting point of our considerations is the expression (2,6,35) for the normal acceleration a_n at the wing surface. We remark that the expression for the normal acceleration does not fulfil the condition that the second derivatives be continous, which is required for the application of the formula (2,3,25) to evaluate the corresponding acceleration potential. However, it is perhaps possible that the conditions mentioned may be weakened without affecting the ultimate expression for the potential in terms of its normal derivative at the boundary. In fact Laurent Schwarz's distribution theory has given the possibility to formulate several classical theorems of the potential theory under less stringent conditions. Halperin remarks in his abstract of the distribution theory (ref.6) that Green's theorems retain their validity by replacing the functions with the usual conditions of continuity and differentiability by the so-called distributions, e.g. the delta function and its derivatives.
Therefore it seems admissible to insert the expression (2,6,35) for the normal acceleration at the wing surface into the formula (2,3,24).
We then find

$$\begin{aligned}
\chi(\eta,\mu,\vartheta) &= \int_0^{2\pi}\int_{-1}^{+1}\Big[\mu_1 U\frac{\partial w}{\partial x_1} + \frac{\pi}{2}U^2\cos^2\vartheta_1\,\mu_1\,\delta'(\mu_1) + \\
&\qquad -\pi U\cos\vartheta_1\,\Phi_{\mu_1}(0,\mu_1,\vartheta_1)\sqrt{1-\mu_1^2}\,\delta'(\mu_1)\Big]\,G\,d\mu_1\,d\vartheta_1 \\
&= \int_0^{2\pi}\int_{-1}^{+1} U\frac{\partial w}{\partial x_1}\,G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,\mu_1\,d\mu_1\,d\vartheta_1 + \\
&\qquad + \int_0^{2\pi}\pi U\cos\vartheta_1\,\Phi_{\mu_1}(0,0,\vartheta_1)\,G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)\,d\mu_1 \\
&= \psi + \pi U\int_0^{2\pi}\cos\vartheta_1\,\Phi_{\mu_1}(0,0,\vartheta_1)\,G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)\,d\vartheta_1\,. \qquad (2,7,1)
\end{aligned}$$

The function ψ represents the regular acceleration potential which corresponds to the normal acceleration $a_n = U\frac{\partial w}{\partial x}$.

We shall now prove that the function χ is equal to the acceleration potential $\psi^x = U\frac{\partial \Phi}{\partial x}$.

For that purpose we consider the difference of the potentials ψ^x and χ :

$$\psi^x - \chi = \psi^x - \psi - \pi U \int_0^{2\pi} \cos\vartheta_1 \Phi_{\mu_1}(0,0,\vartheta_1) G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)\, d\vartheta_1 \; . \tag{2,7,2}$$

In order to calculate the integral in the right-hand side of (2,7,2) use must be made of some formulas which will be derived in the next paragraph. These formulas read as follows

$$\begin{aligned} \int_0^{2\pi} \cos m\vartheta_1 G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)\, d\vartheta_1 &= -\frac{1}{\pi}\frac{\mu}{\mu^2+\eta^2}\frac{\sqrt{1-\mu^2}^{\,m}}{\sqrt{1+\eta^2}^{\,m}}\cos m\vartheta \\ \int_0^{2\pi} \sin m\vartheta_1 G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)\, d\vartheta_1 &= -\frac{1}{\pi}\frac{\mu}{\mu^2+\eta^2}\frac{\sqrt{1-\mu^2}^{\,m}}{\sqrt{1-\eta^2}^{\,m}}\sin m\vartheta \; . \end{aligned} \tag{2,7,3}$$

Now let us assume that the expression $\cos\vartheta_1 \Phi_{\mu_1}(0,0,\vartheta_1)$ can be written in the form of a Fourier series

$$\sum_{n=0}^{\infty} \alpha_n \cos n\vartheta_1 + \sum_{n=1}^{\infty} \beta_n \sin n\vartheta_1 \; . \tag{2,7,4}$$

Then the integral in the right-hand side of (2,7,2) becomes

$$\pi U \int_0^{2\pi} \cos\vartheta_1 \Phi_{\mu_1}(0,0,\vartheta_1) G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)\, d\vartheta_1 =$$

$$= -U\frac{\mu}{\mu^2+\eta^2}\sum_{n=0}^{\infty}\alpha_n\frac{\sqrt{1-\mu^2}^{\,n}}{\sqrt{1+\eta^2}^{\,n}}\cos n\vartheta - U\frac{\mu}{\mu^2+\eta^2}\sum_{n=1}^{\infty}\beta_n\frac{\sqrt{1-\mu^2}^{\,n}}{\sqrt{1+\eta^2}^{\,n}}\sin n\vartheta \; . \tag{2,7,5}$$

It will be immediately clear that the difference potential $\psi^x - \chi$ vanishes for $\mu = 0$ and $\eta \neq 0$.

In the points of the edge of the wing, i.e. for $\mu = \eta = 0$, we have

$$\lim_{\mu\to 0} \mu\left[\psi^x(0,\mu,\vartheta) - \chi(0,\mu,\vartheta)\right] =$$

$$\lim_{\mu\to 0} \mu\left[\psi^x(0,\mu,\vartheta) - \psi(0,\mu,\vartheta) - \pi U\int_0^{2\pi}\cos\vartheta_1 \Phi_\mu(0,0,\vartheta_1) G_{\mu_1}(0,\mu_1,\vartheta;0,\vartheta_1)\, d\vartheta_1\right] =$$

$$\lim_{\mu\to 0} \mu\left[U\frac{\partial\Phi}{\partial x} + \frac{U}{\mu}\sum_{n=0}^{\infty}\alpha_n\sqrt{1-\mu^2}^{\,n}\cos n\vartheta + \frac{U}{\mu}\sum_{n=1}^{\infty}\beta_n\sqrt{1-\mu^2}^{\,n}\sin n\vartheta\right] =$$

$$= -U\cos\vartheta\,\Phi_\mu(0,0,\vartheta) + U\sum_{n=0}^{\infty}\alpha_n\cos n\vartheta + U\sum_{n=1}^{\infty}\beta_n\sin n\vartheta = 0 \; . \tag{2.7.6}$$

The relation (2,7,6) expresses the fact that the difference potential $\psi^{x}-\chi$ also vanishes at the edge of the wing. Thus the potential $\psi^{x}-\chi$ vanishes for $\mu=0$. Furthermore it may now be concluded that the potential $\psi^{x}-\chi$ is a regular solution of Laplace's equation in the whole space including the edge of the wing.
Because of the uniqueness of the regular solution of the Dirichlet problem we thus may identify the harmonic function $\psi^{x}-\chi$ with the zero solution in the entire space.
In this way we have proved the important identity

$$\psi^{x}(\eta,\mu,\vartheta)-\psi(\eta,\mu,\vartheta)=\pi U\int_0^{2\pi}\cos\vartheta_1\,\Phi_{\mu_1}(0,0,\vartheta_1)G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)\,d\vartheta_1 \qquad (2,7,7)$$

or

$$U\frac{\partial}{\partial x}\iint_S wG\,dx_1\,dy_1-U\iint_S\frac{\partial w}{\partial x_1}G\,dx_1\,dy_1=$$

$$=\pi U\int_0^{2\pi}\cos\vartheta_1\,\Phi_{\mu_1}(0,0,\vartheta_1)G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)\,d\vartheta_1\,. \qquad (2,7,8)$$

Partial integration of the second integral in (2,7,8) transforms this identity into

$$\iint_S w\frac{\partial G}{\partial x}\,dx_1\,dy_1+\iint_S w\frac{\partial G}{\partial x_1}\,dx_1\,dy_1=$$

$$=\pi\int_0^{2\pi}\cos\vartheta_1\,G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)\,d\vartheta_1\int_0^{2\pi}\int_{-1}^{+1}w(\mu_2,\vartheta_2)G_{\mu}(0,0,\vartheta_1;\mu_2,\vartheta_2)\,d\mu_2\,d\vartheta_2 \qquad (2,7,9)$$

This relation (2,7,9) can be rewritten as

$$\iint_S w\left[\frac{\partial G}{\partial x}+\frac{\partial G}{\partial x_1}\right]dx_1\,dy_1=$$

$$=\pi\iint_S w(x_2,y_2)\,dx_2\,dy_2\int_0^{2\pi}\cos\vartheta_1\,G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)G_{\mu}(0,0,\vartheta_1;\mu_2,\vartheta_2)\,d\vartheta_1 \qquad (2,7,10)$$

The last relation holds for every downwash distribution $w(x,y)$. Hence we may conclude that the identity (2,7,10) remains valid by omission of the integration over the wing surface.
Thus

$$\frac{\partial G}{\partial x}+\frac{\partial G}{\partial x_1}=\pi\int_0^{2\pi}\cos\vartheta_3\,G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_3)G_{\mu}(0,0,\vartheta_3;\mu_1,\vartheta_1)\,d\vartheta_3\,. \qquad (2,7,11)$$

8 Determination of the Fourier coefficients of the function $G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)$.

In order to be able to evaluate the Fourier coefficients of the function $\left[\frac{\partial G}{\partial\mu_1}\right]_{\mu_1=0}$ we start with the derivation of the following theorem:

If u and z are arbitrary real or complex numbers not situated on the real axis between $-\infty$ and $+1$ and if the inequality

$\left|u+(u^2-1)^{1/2}\right| < \left|z+(z^2-1)^{1/2}\right|$ holds, then the expansion

$$\frac{(-1)^m}{z-u}\,\frac{(u^2-1)^{m/2}}{(z^2-1)^{m/2}} = \sum_{n=m}^{\infty} (2n+1)\frac{(n-m)!}{(n+m)!}\,P_n^m(u)\,Q_n^m(z) \tag{2,8,1}$$

is valid.

In the case $m=0$, the expression (2,8,1) degenerates to the well-known formula of Heine, viz.

$$\frac{1}{z-u} = \sum_{n=0}^{\infty} (2n+1)\,P_n(u)\,Q_n(z) \tag{2,8,2}$$

for $\left|u+(u^2-1)^{1/2}\right| < \left|z+(z^2-1)^{1/2}\right|$.

It is known that the associated Legendre functions $P_n^m(z)$ and $Q_n^m(z)$ are single valued and regular in the z -plane, on the condition that the z -plane is cut along the real axis from +1 to $-\infty$. We assume that the following relations hold

$$|\arg(z\pm 1)| < \pi\,,\quad |\arg z| < \pi \tag{2,8,3}$$

and

$$(z^2-1)^\alpha = (z-1)^\alpha\,(z+1)^\alpha\ .$$

For the proof of the lemma we start from the known recurrence relations for Legendre's associated functions, viz.

$$(2r+1)\,uP_r^m(u)-(r-m+1)P_{r+1}^m(u)-(r+m)\,P_{r-1}^m(u)=0$$

$$(2r+1)\,zQ_r^m(z)-(r-m+1)\,Q_{r+1}^m(z)-(r+m)\,Q_{r-1}^m(z)=0\ . \tag{2,8,4}$$

Multiplying the first relation with $\frac{(r-m)!}{(r+m)!}Q_r^m(z)$ and the second one with $\frac{(r-m)!}{(r+m)!}P_r^m(u)$ and subtracting thereupon the two relations, we find

$$(2r+1)\,\frac{(r-m)!}{(r+m)!}(z-u)\,P_r^m(u)\,Q_r^m(z) = \frac{(r-m+1)!}{(r+m)!}\left\{Q_{r+1}^m(z)\,P_r^m(u)-P_{r+1}^m(u)\,Q_r^m(z)\right\} +$$
$$+\ \frac{(r-m)!}{(r+m-1)!}\left\{Q_{r-1}^m(z)\,P_r^m(u)-P_{r-1}^m(u)\,Q_r^m(z)\right\}\ .$$

Hence, by giving r the values n , $n-1$, $n-2$, ..., $m+1$, m and summing we obtain the expression

$$\sum_{r=m}^{n}(2r+1)\,\frac{(r-m)!}{(r+m)!}(z-u)\,P_r^m(u)\,Q_r^m(z) = \sum_{r=m}^{n}\frac{(r-m+1)!}{(r+m)!}\Big\{Q_{r+1}^m(z)\,P_r^m(u)\ +$$
$$-\,P_{r+1}^m(u)\,Q_r^m(z)\Big\}+\sum_{r=m}^{n}\frac{(r-m)!}{(r+m-1)!}\left\{Q_{r-1}^m(z)\,P_r^m(u)-P_{r-1}^m(u)\,Q_r^m(z)\right\} =$$
$$=\sum_{r=m}^{n}\frac{(r-m+1)!}{(r+m)!}\left\{Q_{r+1}^m(z)\,P_r^m(u)-P_{r+1}^m(u)\,Q_r^m(z)\right\} +$$

$$+ \sum_{r=m-1}^{n-1} \frac{(r-m+1)!}{(r+m)!} \left\{ Q_r^m(z) P_{r+1}^m(u) - P_r^m(u)\, Q_{r+1}^m(z) \right\} =$$

$$= \frac{(n-m+1)!}{(n+m)!} \left\{ Q_{n+1}^m(z) P_n^m(u) - P_{n+1}^m(u)\, Q_n^m(z) \right\} +$$

$$+ \frac{1}{(2m-1)!} \left\{ Q_{m-1}^m(z) P_m^m(u) - P_{m-1}^m(u)\, Q_m^m(z) \right\} . \tag{2,8,5}$$

From the definition of the Legendre function of the first kind, viz.:

$$P_r^m(u) = (u^2-1)^{m/2} \frac{d^m P_r(u)}{du^m} \tag{2,8,6}$$

it follows

$$P_{m-1}^m(u) = 0 \quad \text{and} \quad P_m^m(u) = (u^2-1)^{m/2} \frac{d^m P_m(u)}{du^m} \tag{2,8,7}$$

With Rodrigues' formula for the Legendre polynomials

$$P_m(u) = \frac{1}{2^m m!} \frac{d^m}{du^m} (u^2-1)^m \tag{2,8,8}$$

we find easily

$$P_m^m(u) = \frac{(2m)!}{2^m m!} (u^2-1)^{m/2} . \tag{2,8,9}$$

In order to determine an analytical expression for the function $Q_{m-1}^m(z)$ we use the following representation in terms of hypergeometric functions (ref.2)

$$e^{-i\mu\pi} Q_\nu^\mu(z) = 2^{1-\nu} \sqrt{\pi}\, \Gamma(1+\nu+\mu) (z^2-1)^{-1/2-1/2\nu} \frac{1}{\Gamma(\nu+3/2)}$$

$$F(1/2+1/2\nu-1/2\mu,\ 1/2+1/2\nu+1/2\mu\ ;\ \nu+3/2\ ;\ \frac{1}{1-z^2}) \tag{2,8,10}$$

In this formula it is assumed that

$$|\arg(z\pm1)| < \pi , \quad |\arg z| < \pi ,$$

$$(z^2-1)^\alpha = (z-1)^\alpha (z+1)^\alpha \qquad \text{and} \qquad |1-z^2| > 1 .$$

Putting $\nu = m-1$ and $\mu = m$, we get

$$e^{-im\pi} Q_{m-1}^m(z) = 2^{-m} \sqrt{\pi}\, \Gamma(2m) (z^2-1)^{-m/2} \frac{1}{\Gamma(m+1/2)} F(0, m; m+1/2; \frac{1}{1-z^2})$$

or

$$Q_{m-1}^m(z) = (-1)^m 2^{-m} \sqrt{\pi} \frac{\Gamma(2m)}{\Gamma(m+1/2)} (z^2-1)^{-m/2} . \tag{2,8,11}$$

As the left-hand side of (2,8,11) is an analytical function of z throughout the complex plane with cross-cut $(-\infty,1)$, we may conclude from a well-known theorem on analytical continuation that this relation holds without the restriction $|1-z^2| > 1$.

Inserting the relations (2,8,7), (2,8,9) and (2,8,11) into (2,8,5) we get

$$\sum_{r=m}^{n}(2r+1)\frac{(r-m)!}{(r+m)!}(z-u)P_r^m(u)Q_r^m(z)=\frac{(n-m+1)!}{(n+m)!}\left\{Q_{n+1}^m(z)P_n^m(u)-P_{n+1}^m(u)Q_n^m(z)\right\}+$$

$$+\frac{1}{(2m-1)!}(-1)^m 2^{-m}\sqrt{\pi}\frac{\Gamma(2m)}{\Gamma(m+1/2)}(z^2-1)^{-m/2}\quad\frac{(2m)!}{2^m m!}(u^2-1)^{m/2}=$$

$$=\frac{(n-m+1)!}{(n+m)!}\left\{Q_{n+1}^m(z)P_n^m(u)-P_{n+1}^m(u)Q_n^m(z)\right\}+(-1)^m\frac{(u^2-1)^{m/2}}{(z^2-1)^{m/2}}\quad. \tag{2,8,12}$$

We thus have found the relation

$$\sum_{r=m}^{n}(2r+1)\frac{(r-m)!}{(r+m)!}P_r^m(u)\,Q_r^m(z)=\frac{1}{z-u}\frac{(n-m+1)!}{(n+m)!}\left\{Q_{n+1}^m(z)P_n^m(u)-P_{n+1}^m(u)Q_n^m(z)\right\}+$$
$$+\frac{(-1)^m}{z-u}\frac{(u^2-1)^{m/2}}{(z^2-1)^{m/2}}\quad. \tag{2,8,13}$$

We shall show now that the first term on the right-hand side of (2,8,13) tends to zero when n becomes infinite. In order to find an estimation of the above-mentioned term of (2,8,13) for large values of n , use can be made of Laplace's definite integral expression for $P_n^m(u)$, notably

$$P_n^m(u)=\frac{\Gamma(n+m+1)}{\pi\Gamma(n+1)}\int_0^{\pi}\left[u+(u^2-1)^{1/2}\cos t\right]^n\cos mt\,dt\quad. \tag{2,8,14}$$

If n and m are positive integers, this integral representation is certainly valid for all values of u outside the cut $-\infty<u\leq+1$.
It can easily be seen that the maximum of the expression

$\left|u+(u^2-1)^{1/2}\cos t\right|$ is $\left|u+(u^2-1)^{1/2}\right|$ throughout the interval $0\leq t\leq\pi$. With the aid of this estimation it may be concluded that

$$\left|P_n^m(u)\right|\leq\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\left|u+(u^2-1)^{1/2}\right|^n\quad. \tag{2,8,15}$$

A similar formula can be found for the Legendre function of the second kind $Q_n^m(z)$. To find such inequality we start from the representation (ref.2)

$$Q_n^m(z)=e^{im\pi}\sqrt{\pi}\,2^m\frac{\Gamma(1+n+m)}{\Gamma(3/2+n)}(z^2-1)^{m/2}\mu^{-1-n-m}F(1/2+m,\,1+n+m;3/2+n;\frac{1}{\mu^2}), \tag{2,8,16}$$

where μ denotes $z+(z^2-1)^{1/2}$.
This expression holds good throughout the plane of z , with the exception of the interval on the real axis of z joining the points $-\infty,+1$.

We thus have

$$\left|Q_n^m(z)\right| \leq \sqrt{\pi}\, 2^m \frac{\Gamma(1+n+m)}{\Gamma(3/2+n)} \left|z^2-1\right|^{m/2} |\mu|^{-(1+n+m)} \left\{ 1+ \frac{(m+1/2)(1+n+m)}{1.(n+3/2)} \frac{1}{|\mu|^2} + \right.$$

$$\left. + \frac{(m+1/2)(m+3/2)(m+n+1)(m+n+2)}{1.2\,(n+3/2)(n+5/2)} \frac{1}{|\mu|^4} + \ldots\ldots \right\} \qquad (2,8,17)$$

As the integer m never exceeds n, we have

$$\left|Q_n^m(z)\right| < \sqrt{\pi}\, 2^m \frac{\Gamma(1+n+m)}{\Gamma(3/2+n)} (z^2-1)^{m/2} |\mu|^{-(1+n+m)} \left[1+(m+1/2)\frac{2}{|\mu|^2} + (m+1/2)(m+3/2)\frac{z^2}{|\mu|^4}+\ldots\right]$$

$$= \sqrt{\pi}\, 2^m \frac{\Gamma(1+n+m)}{\Gamma(3/2+n)} (z^2-1)^{m/2} |\mu|^{-(1+n+m)} F(m+1/2\,;\frac{2}{|\mu|^2})$$

or

$$\left|Q_n^m(z)\right| < \sqrt{\pi}\, 2^m \frac{\Gamma(1+n+m)}{\Gamma(3/2+n)} (z^2-1)^{m/2} |\mu|^{-(1+n+m)} \left(1-\frac{2}{|\mu|^2}\right)^{-m-1/2}, \qquad (2,8,18)$$

provided that $|\mu| > 1$, which is the case for all points z outside the segment joining the points +1 and -1 on the real axis of z .
From these two estimates for $P_n^m(u)$ and $Q_n^m(z)$ we conclude

$$\frac{(n-m+1)!}{(n+m)!}\left|Q_{n+1}^m(z)P_n^m(u)\right| < \frac{(n-m+1)!}{(n+m)!} \frac{\Gamma(n+m+1)}{\Gamma(n+1)} \left|u+(u^2-1)^{1/2}\right|^n \sqrt{\pi}\, 2^m \frac{\Gamma(2+n+m)}{\Gamma(5/2+n)}$$

$$\left|z^2-1\right|^{m/2} \left|z+(z^2-1)^{1/2}\right|^{-(2+n+m)} \left(1-\frac{2}{\left|z+(z^2-1)^{1/2}\right|^2}\right)^{-m-1/2} \qquad (2,8,19)$$

Using the well-known Stirling formula for the asymptotic behaviour of the gamma function it can be easily seen from this inequality (2,8,19), that

$$\frac{(n-m+1)!}{(n+m)!}\left|Q_{n+1}^m(z)P_n^m(u)\right|$$

converges to zero when n becomes infinite, if

$$\left|u+(u^2-1)^{1/2}\right| < \left|z+(z^2-1)^{1/2}\right| .$$

Similarly it can be proved that

$$\frac{(n-m+1)!}{(n+m)!}\left|P_{n+1}^m(u)\,Q_n^m(z)\right|$$

converges to zero when n tends to infinity, subject to the same condition. The conclusion can now be made that the right-hand side of (2,8,13) tends

to $\dfrac{(-1)^m}{z-u}\dfrac{(u^2-1)^{m/2}}{(z^2-1)^{m/2}}$, when n tends to infinity, provided that

$$\left|u+(u^2-1)^{1/2}\right| < \left|z+(z^2-1)^{1/2}\right| .$$

So we obtain the formula

$$\sum_{r=m}^{\infty} (2r+1) \frac{(r-m)!}{(r+m)!} \overset{m}{P}_r(u) \overset{m}{Q}_r(z) = \frac{(-1)^m}{z-u} \frac{(u^2-1)^{m/2}}{(z^2-1)^{m/2}} , \tag{2,8,20}$$

which had to be proved.

In section 4 of this chapter it has been shown that the expansion in series of Green's function G is uniformly convergent for all admissible values of $\mu, \mu_1, \vartheta, \vartheta_1$ and for η situated in the interval $\delta \leq \eta \leq R$, where δ is an arbitrary small positive number and R is an arbitrary large positive number. The equations $\eta = \delta$ and $\eta = R$ represent two confocal oblate spheroids. Thus it follows that the series expansion of the function G is uniformly convergent in the closed region which is bounded by the two spheroids mentioned. Now use will be made of a theorem of Harnack (ref.8), which reads as follows.

Let H be any closed region of space, and let $U_1, U_2, U_3, \ldots$ be an infinite sequence of harmonic functions in H. If the sequence converges uniformly on the boundary S of H, it converges uniformly throughout H and its limit U is a harmonic function in H. Furthermore, in any closed region H', entirely interior to H, the sequence of derivatives

$$\left[\frac{\partial^{i+j+k}}{\partial x^i \, \partial y^j \, \partial z^k} U_n\right], \quad n = 1, 2, 3 \ \ldots\ldots ,$$

i, j and k being fixed, converges uniformly to the corresponding derivative of U.

With respect to the function G

$$G = \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \frac{1}{\varepsilon_m} \overset{m}{P}_n(\mu) \overset{m}{P}_n(\mu_1) \frac{\overset{m}{Q}_n(i\eta)}{\overset{m'}{Q}_n(i0)} \cos m(\vartheta-\vartheta_1)$$

we put

$$U_j = \sum_{n=0}^{j} \sum_{m=0}^{n}{}' \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \frac{1}{\varepsilon_m} \overset{m}{P}_n(\mu) \overset{m}{P}_n(\mu_1) \frac{\overset{m}{Q}_n(i\eta)}{\overset{m'}{Q}_n(i0)} \cos m(\vartheta-\vartheta_1) . \tag{2,8,21}$$

In the closed region H bounded by the spheroids $\eta = \delta$ and $\eta = R$, the functions U_j, being finite sums of harmonic functions, are harmonic. Moreover, the sequence U_1, U_2, U_3, ...converges uniformly to the function G throughout the entire region H. On ground of Harnack's theorem, we can now decide that the sequence

$$\left[\frac{\partial U_j}{\partial \mu}\right]_{\mu=0}$$

converges uniformly to the function $\left[\frac{\partial G}{\partial \mu}\right]_{\mu=0}$ for all allowable values of μ_1, ϑ, ϑ_1 and for η lying in the interval $0<\delta \leq \eta < \infty$.
Therefore we have proved the formula

$$\left[\frac{\partial G}{\partial \mu}\right]_{\mu=0} = \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \frac{1}{\varepsilon_m} P_n^{m'}(0) P_n^{m}(\mu_1) \frac{Q_n^{m}(i\eta)}{Q_n^{m'}(i0)} \cos m(\vartheta-\vartheta_1). \qquad (2,8,22)$$

In order to simplify the right-hand side of (2,8,22) we consider the ratio $\dfrac{P_n^{m'}(0)}{Q_n^{m'}(i0)}$.

Using the relations (2,4,3) and (2,4,4) the expression $Q_n^{m'}(i0)$ can be written in the form

$$Q_n^{m'}(i0) = Q_n^{m+1}(i0) = (-1)^{\frac{2m-n+1}{2}} \frac{n!}{(n-m-1)!} \int_0^{\infty} \frac{\cosh(m+1)u}{(\cosh u)^{n+1}}\, du . \qquad (2,8,23)$$

Putting $e^{u}=x$, the integral in the right-hand side of (2,8,23) can be replaced by

$$\int_0^{\infty} \frac{\cosh(m+1)u}{(\cosh u)^{n+1}}\, du = 2^n \int_1^{\infty} \frac{x^{m+n+1}+x^{-m+n-1}}{(x^2+1)^{n+1}}\, dx .$$

The substitution $e^{-u}=x$ yields

$$\int_0^{\infty} \frac{\cosh(m+1)u}{(\cosh u)^{n+1}}\, du = 2^n \int_0^{1} \frac{x^{m+n+1}+x^{-m+n-1}}{(x^2+1)^{n+1}}\, dx .$$

Hence

$$\int_0^{\infty} \frac{\cosh(m+1)u}{(\cosh u)^{n+1}}\, du = 2^{n-1} \int_0^{\infty} \frac{x^{m+n+1}+x^{-m+n-1}}{(x^2+1)^{n+1}}\, dx . \qquad (2,8,24)$$

The integral in the right-hand side of (2,8,24) can be expressed in terms of the beta function by aid of the transformation

$$y = \frac{x^2}{x^2+1} .$$

This tranformation yields after some elementary calculations

$$\int_0^{\infty} \frac{\cosh(m+1)u}{(\cosh u)^{n+1}}\, du = 2^{n-2} \int_0^{1} y^{\frac{m+n}{2}} (1-y)^{\frac{-m+n-2}{2}}\, dy + 2^{n-2} \int_0^{1} y^{\frac{-m+n-2}{2}} (1-y)^{\frac{m+n}{2}}\, dy . \qquad (2,8,25)$$

Using the definition of the beta function, viz.

$$B(p,q) = \int_0^{1} x^{p-1}(1-x)^{q-1}\, dx$$

the expression (2,8,25) can be rewritten in the form

$$\int_0^{\infty} \frac{\cosh(m+1)u}{(\cosh u)^{n+1}}\, du = 2^{n-1} B\left(\frac{m+n+2}{2}\, ,\, \frac{-m+n}{2}\right) . \qquad (2,8,26)$$

Inserting this result into the relation (2,8,23), we get the formula

$$Q_n^{m'}(i0) = (-1)^{\frac{2m-n+1}{2}}\, 2^{n-1} \frac{n!}{(n-m-1)!} B\left(\frac{m+n+2}{2}\, ,\, \frac{-m+n}{2}\right) . \qquad (2,8,27)$$

On applying the well-known relation between the beta function and the gamma function, viz.

$$B(p,q) = B(q,p) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \qquad (2,8,28)$$

the expression (2.8.27) finally can be transformed into

$$Q_n^{m'}(io) = (-1)^{\frac{2m-n+1}{2}} 2^{n-1} \frac{\Gamma\left(\frac{m+n+2}{2}\right)\Gamma\left(\frac{-m+n}{2}\right)}{\Gamma(n-m)} . \tag{2,8,29}$$

In order to obtain a similar expression for the quantity $P_n^{m'}(0)$, we consider the definition of the function $P_n^m(z)$, where z is a point not lying on the real axis between $-\infty$ and $+1$. This definition is expressed by the formula

$$P_n^m(z) = (z^2-1)^{m/2} \frac{d^m P_n(z)}{dz^m} . \tag{2,8,30}$$

If x is a point on the interval $|x| < 1$, the function $P_n^m(x)$ is defined by

$$P_n^m(x) = \frac{1}{2}\left[e^{\frac{1}{2}im\pi} P_n^m(x+io) + e^{-\frac{1}{2}im\pi} P_n^m(x-io)\right] \tag{2,8,31}$$

where $P_n^m(x \pm io) = \lim\limits_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} P_n^m(x \pm i\varepsilon)$.

Replacing z^2-1 by $(1-x^2)e^{\pm\pi i}$, according as $z = x \pm io$, equation (2,8,30) yields

$$P_n^m(x) = \frac{1}{2}\left[e^{im\pi}(1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m} + e^{-im\pi}(1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}\right] =$$

$$= (-1)^m (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m} . \tag{2,8,32}$$

Differentiation of this formula (2,8,32) leads to

$$P_n^{m'}(x) = \frac{d}{dx} P_n^m(x) = (-1)^{m+1} mx (1-x^2)^{m/2-1} \frac{d^m P_n(x)}{dx^m} +$$

$$+ (-1)^m (1-x^2)^{m/2} \frac{d^{m+1} P_n(x)}{dx^m} . \tag{2,8,33}$$

Putting $x = 0$, the formula

$$P_n^{m'}(0) = - P_n^{m+1}(0) \tag{2,8,34}$$

is obtained.
With the aid of Rodrigues' formula (2,8,8) for the Legendre polynomials we easily derive

$$P_n^{m+1}(x) = \frac{(-1)^{m+1}}{2^n n!} (1-x^2)^{\frac{m+1}{2}} \frac{d^{n+m+1}}{dx^{n+m+1}} (x^2-1)^n . \tag{2,8,35}$$

If $n+m+1$ is an odd integer, it is directly clear that $P_n^{m+1}(0)$ equals zero. We are, however, more interested in the case that $n+m+1$ is an even integer.

For this case holds

$$P_n^{m+1}(0) = \frac{(-1)^{m+1}}{2^n n!} \lim_{x\to 0} \frac{d^{n+m+1}}{dx^{n+m+1}} (x^2-1)^n$$

$$= \frac{(-1)^{m+1}}{2^n n!} \lim_{x\to 0} \frac{d^{n+m+1}}{dx^{n+m+1}} \left\{ x^{2n} - \binom{n}{1} x^{2n-2} + \ldots\ldots \right.$$

$$\left. \ldots\ldots (-1)^{\frac{n-m-1}{2}} \binom{n}{\frac{n-m-1}{2}} x^{2n-(n-m-1)} + \ldots (-1)^n \right\}$$

$$= \frac{(-1)^{m+1}}{2^n n!} (-1)^{\frac{n-m-1}{2}} \binom{n}{\frac{n-m-1}{2}} (n+m+1)! \quad . \tag{2,8,36}$$

By elementary calculations it can now be derived

$$\frac{P_n^{m'}(0)}{Q_n^{m'}(i0)} = \frac{(-1)^m}{2^n n!} (-1)^{\frac{n-m-1}{2}} \binom{n}{\frac{n-m-1}{2}} (n+m+1)! (-1)^{\frac{-2m+n-1}{2}} 2^{1-n} \frac{\Gamma(n-m)}{\Gamma\left(\frac{m+n+2}{2}\right)\Gamma\left(\frac{-m+n}{2}\right)} = \frac{2}{\pi}(-1)^{\frac{-3m}{2}} \tag{2,8,37}$$

if $n+m$ is an odd integer.
Insertion of this result into the formula (2,8,22) gives

$$\left[\frac{\partial G}{\partial \mu}\right]_{\mu=0} = \sum_{n=0}^{\infty} \sum_{m=0}^{n}{}' \frac{2n+1}{\pi^2} e^{3/2\, m\pi i} \frac{(n-m)!}{(n+m)!} \frac{1}{\varepsilon_m} P_n^m(\mu_1) Q_n^m(i\eta) \cos m(\vartheta-\vartheta_1) . \tag{2,8,38}$$

We have already proved that the series in (2,8,38) is uniformly convergent for all values of μ_1 , ϑ , ϑ_1 and for η in the interval $0<\delta \leqq \eta<\infty$.
Hence the following termswise integration is allowed:

$$\int_0^{2\pi} \cos m\vartheta \left[\frac{\partial G}{\partial \mu}\right]_{\mu=0} d\vartheta = \sum_{n=m}^{\infty}{}' \frac{2n+1}{\pi} e^{3/2\, m\pi i} \frac{(n-m)!}{(n+m)!} P_n^m(\mu_1) Q_n^m(i\eta) \cos m\vartheta_1 . \tag{2,8,39}$$

The prime again denotes that $n+m$ must be an odd integer. The evaluation of the series in (2,8,39) can be performed by application of the theorem, mentioned in the beginning of this section. The formula (2,8,20) gives

$$\sum_{n=m}^{\infty} (2n+1) \frac{(n-m)!}{(n+m)!} P_n^m(\mu_1 + i0) Q_n^m(i\eta) = \frac{(-1)^m}{i\eta-\mu_1} \frac{(1-\mu_1^2)^{m/2} e^{m/2\,\pi i}}{(-\eta^2-1)^{m/2}}$$

and similarly

$$\sum_{n=m}^{\infty} (2n+1) \frac{(n-m)!}{(n+m)!} P_n^m(\mu_1 - i0) Q_n^m(i\eta) = \frac{(-1)^m}{i\eta-\mu_1} \frac{(1-\mu_1^2)^{m/2} e^{-m/2\,\pi i}}{(-\eta^2-1)^{m/2}} .$$

Thus according to the relation (2,8,31) we get

$$\sum_{n=m}^{\infty} (2n+1) \frac{(n-m)!}{(n+m)!} P_n^m(\mu_1) Q_n^m(i\eta) = \frac{e^{-m/2 \pi i}}{i\eta - \mu_1} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} . \qquad (2,8,40)$$

Replacing in (2,8,40) the variable μ_1 by $-\mu_1$, one obtains the formula

$$\sum_{n=m}^{\infty} (2n+1) \frac{(n-m)!}{(n+m)!} P_n^m(-\mu_1) Q_n^m(i\eta) = \sum_{n=m}^{\infty} (2n+1) \frac{(n-m)!}{(n+m)!} (-1)^{n+m} P_n^m(\mu_1) Q_n^m(i\eta) =$$

$$= \frac{e^{-m/2 \pi i}}{i\eta + \mu_1} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} . \qquad (2,8,41)$$

Subtraction of the relations (2,8,40) and (2,8,41) yields

$$\sum_{n=m}^{\infty}{}' (2n+1) \frac{(n-m)!}{(n+m)!} P_n^m(\mu_1) Q_n^m(i\eta) =$$

$$= \frac{1}{2} e^{-m/2 \pi i} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} \left[\frac{1}{i\eta - \mu_1} - \frac{1}{i\eta + \mu_1} \right] = e^{-m/2 \pi i} \frac{-\mu_1}{\mu_1^2 + \eta^2} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} . \qquad (2,8,42)$$

Hence the formula (2,8,39) can be written in the form

$$\int_0^{2\pi} \cos m\vartheta \left[\frac{\partial G}{\partial \mu}\right]_{\mu=0} d\vartheta = -\frac{1}{\pi} \frac{\mu_1}{\mu_1^2 + \eta^2} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} \cos m\vartheta_1 . \qquad (2,8,43)$$

In a quite similar way it can be proved

$$\int_0^{2\pi} \sin m\vartheta \left[\frac{\partial G}{\partial \mu}\right]_{\mu=0} d\vartheta = -\frac{1}{\pi} \frac{\mu_1}{\mu_1^2 + \eta^2} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} \sin m\vartheta_1 . \qquad (2,8,44)$$

From these two relations (2,8,43) and (2,8,44) it may easily be deduced that the function $\left[\frac{\partial G}{\partial \mu}\right]_{\mu=0}$ can be represented by the Fourier series

$$\left[\frac{\partial G}{\partial \mu}\right]_{\mu=0} = \sum_{m=0}^{\infty} - \frac{1}{\varepsilon_m \pi^2} \frac{\mu_1}{\mu_1^2 + \eta^2} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} \cos m\vartheta \cos m\vartheta_1 +$$

$$+ \sum_{m=1}^{\infty} - \frac{1}{\pi^2} \frac{\mu_1}{\mu_1^2 + \eta^2} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} \sin m\vartheta \sin m\vartheta_1 . \qquad (2,8,45)$$

This formula can now be simplified as follows

$$\left[\frac{\partial G}{\partial \mu}\right]_{\mu=0} = \sum_{m=0}^{\infty} - \frac{1}{\varepsilon_m \pi^2} \frac{\mu_1}{\mu_1^2+\eta^2} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} \cos m(\vartheta-\vartheta_1) =$$

$$= \sum_{m=0}^{\infty} - \frac{1}{\varepsilon_m \pi^2} \frac{\mu_1}{\mu_1^2+\eta^2} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} \frac{e^{im(\vartheta-\vartheta_1)} + e^{-im(\vartheta-\vartheta_1)}}{2}. \qquad (2,8,46)$$

The last series, occurring in (2,8,46), can be considered as the sum of two convergent geometric series, when η or μ differs from zero. In these cases the final result appears to be

$$\left[\frac{\partial G}{\partial \mu}\right]_{\mu=0} = -\frac{\mu_1}{2\pi^2} \frac{1}{(1-\mu_1^2)-2\sqrt{1-\mu_1^2}\sqrt{1+\eta_1^2}\cos(\vartheta-\vartheta_1)+(1+\eta_1^2)}. \qquad (2,8,47)$$

In the Appendix this formula (2,8,47) will be derived from the closed expression for Green's function.

9 Determination of the final acceleration potential.

In section 5 of this chapter we have shown that the velocity potential Φ, which fulfils the condition of a prescribed normal velocity w at the surface of the wing, yields a pressure distribution over the wing surface that is infinite along the whole edge of the wing. Furthermore it appeared that the acceleration potential ψ, the existence of which presupposed a normal acceleration $U\frac{\partial w}{\partial x}$ at the wing surface, gives a pressure distribution over the wing surface, which vanishes along the whole edge. Nevertheless none of these two pressure distributions agrees with the actual pressure distribution. In linearized aerofoil theory it is always required that the flow over the wing satisfies the Kutta condition, which implies that no velocity discontinuity occurs at the trailing edge of the wing. In terms of the pressure distribution the Kutta condition requires that the pressure difference between the upper and lower side of the wing vanishes at the trailing edge.

The ultimate physical problem thus can be formulated as follows: It is required to find a solution of Laplace's equation, which satisfies the condition of a given normal velocity at the wing surface, which furthermore yields a pressure distribution over the wing surface that vanishes at the trailing edge and which finally possesses a singularity at the leading edge. In order to obtain a solution of this physical problem we can add to the acceleration potential ψ a solution of Laplace's equation, which does not contribute to the normal acceleration at the wing surface with the exclusion of the points of the edge and which vanishes at the trailing edge of the wing, but which possesses a singularity at the leading edge.

Such solutions are given by the integrals

$$\sum_{n=m}^{\infty} (2n+1) \frac{(n-m)!}{(n+m)!} P_n^m(\mu_1)\, Q_n^m(i\eta) = \frac{e^{-\frac{m}{2}\pi i}}{i\eta-\mu_1} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} . \tag{2,8,40}$$

Replacing in (2,8,40) the variable μ_1 by $-\mu_1$, one obtains the formula

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Hence the formula (2,8,39) can be written in the form

$$\int_0^{2\pi} \cos m\vartheta \left[\frac{\partial G}{\partial \mu}\right]_{\mu=0} d\vartheta = -\frac{1}{\pi} \frac{\mu_1}{\mu_1^2+\eta^2} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} \cos m\vartheta_1 . \tag{2,8,43}$$

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This formula can now be simplified as follows

$$\left[\frac{\partial G}{\partial \mu}\right]_{\mu=0} = \sum_{m=0}^{\infty} - \frac{1}{\varepsilon_m \pi^2} \frac{\mu_1}{\mu_1^2+\eta^2} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} \cos m(\vartheta-\vartheta_1) =$$

$$= \sum_{m=0}^{\infty} - \frac{1}{\varepsilon_m \pi^2} \frac{\mu_1}{\mu_1^2+\eta^2} \frac{(1-\mu_1^2)^{m/2}}{(1+\eta^2)^{m/2}} \frac{e^{im(\vartheta-\vartheta_1)}+e^{-im(\vartheta-\vartheta_1)}}{2}. \tag{2,8,46}$$

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In order to obtain a solution of this physical problem we can add to the acceleration potential ψ a solution of Laplace's equation, which does not contribute to the normal acceleration at the wing surface with the exclusion of the points of the edge and which vanishes at the trailing edge of the wing, but which possesses a singularity at the leading edge.
Such solutions are given by the integrals

$$U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \quad . \qquad (2,9,1)$$

As the function of Green itself has a vanishing normal derivative at the wing surface, it is clear that expressions of the type (2,9,1) have a zero normal derivative at the wing surface.
Furthermore it follows from the analytical expression (2,8,47) for $\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0}$ that the function $\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0}$ is only singular for $\eta=0$, $\mu=0$ and $\vartheta=\vartheta_1$. Due to the fact that the integration in the expressions (2,9,1) extends only along the leading edge of the wing, the singular points of the expression (2,9,1) can only lie on the leading edge of the wing.
The integral relations (2,8,43) and (2,8,44) show that the singularity of the integrals (2,9,1) is of the type $\frac{1}{\mu}$. This singularity agrees completely with the square-root singularity, which occurs in the two-dimensional thin aerofoil theory.
At the trailing edge the integrals (2,9,1) vanish because the function G is equal to zero for $\mu=0$ and $\vartheta \neq \vartheta_1$.
We have thus shown that the expressions (2,9,1) indeed fulfil the required conditions.

The complete acceleration potential can now be represented by the expression

$$\psi\ (\eta,\mu,\vartheta)+U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \qquad (2,9,2)$$

This expression (2,9,2) satisfies all conditions stated in the above mentioned formulation of the boundary value problem, with the exception of the requirement of the prescribed normal velocity at the wing surface. However, this condition of the prescribed normal velocity enables us now to determine the unknown function $h(\vartheta)$.
Application of the identity (2,7,7) between the two potentials ψ and ψ^{x} yields for the complete acceleration potential the expression

$$U \frac{\partial \Phi}{\partial x} + \pi U \int_{0}^{2\pi} \cos\vartheta_1\ \Phi_\mu(0,0,\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 + U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \ . \qquad (2,9,3)$$

The first term in this formula denotes the acceleration potential, which corresponds with the regular velocity potential Φ , Consequently the normal velocity at the wing surface, which belongs to this acceleration potential exactly equals the prescribed normal velocity $w(x,y)$. Hence the following conclusion can be made:

The normal velocity at the wing surface corresponding to the acceleration potential

$$\Omega\,(\eta,\mu,\vartheta) = \pi U \int_0^{2\pi} \cos\vartheta_1\, \Phi_\mu\,(0,0,\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 + U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h\,(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \,. \tag{2,9,4}$$

must be equal to zero.

In chapter I we have proved that an acceleration potential ψ and the corresponding velocity potential φ are related by the formula

$$\varphi\,(x,y,z,t) = \frac{1}{U} \int_{-\infty}^{x} \psi\,\left(x',y,z,t - \frac{x-x'}{U}\right) dx' \,. \tag{2,9,5}$$

In steady flow this formula degenerates to

$$\varphi\,(x,y,z) = \frac{1}{U} \int_{-\infty}^{x} \psi\;(x',y,z)\,dx' \tag{2,9,6}$$

If the normal velocity at the wing surface which corresponds to the velocity potential φ is denoted by $w\,(x,y)$, we have

$$w\,(x,y) = \left[\frac{\partial\varphi}{\partial z}\right]_{z=0} = \lim_{z\to 0} \frac{\partial}{\partial z}\,\frac{1}{U} \int_{-\infty}^{x} \psi\;(x',y,z)\,dx' \tag{2,9,7}$$

It is clear that the equation for the unknown weight function $h\,(\vartheta)$ takes the form

$$\lim_{z\to 0} \frac{\partial}{\partial z}\,\frac{1}{U} \int_{-\infty}^{x} \Omega\;(x',y,z)\,dx' = 0 \tag{2,9,8}$$

or explicitly

$$\lim_{z\to 0} \frac{\partial}{\partial z} \int_{-\infty}^{x} dx' \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h\,(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 +$$

$$+ \lim_{z\to 0} \frac{\partial}{\partial z} \int_{-\infty}^{x} dx' \int_0^{2\pi} g\,(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 = 0 \,.$$

where $g\,(\vartheta) = \pi \cos\vartheta\,\Phi_\mu\,(0,0,\vartheta)$. (2,9,9)

If this equation (2,9,8) or (2,9,9) for the function $h(\vartheta)$ has been solved, it will be possible to evaluate the actual acceleration potential with the aid of the formula (2,9,2).

10 Transformation of the equation (2,9,9) into an infinite system of linear equations.

In this section it is our main objective to transform the equation (2,9,9) for the unknown weight function $h(\vartheta)$ into a system of linear equations with an infinite number of unknowns by means of Fourier series expansion.

The acceleration potential $\Omega\,(\eta,\mu,\vartheta)$, which has been defined by the formula (2,9,4), will be written here in the form

$$\Omega\,(\eta,\mu,\vartheta) = U \int_0^{2\pi} g^{x}\,(\vartheta_1) \cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 + U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h^{x}\,(\vartheta_1) \cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \tag{2,10,1}$$

wherein

$$g^{\times}(\vartheta) = \pi\, \Phi_{\mu}(0,0,\vartheta) \quad \text{and} \quad h(\vartheta) = h^{\times}(\vartheta) \cos\vartheta \quad . \qquad (2,10,2)$$

For shortening the mathematical expressions, we distinguish between problems, which are symmetric with respect to the axis of ψ and problems, which are anti-symmetric with respect to the axis of ψ. The property of symmetry designates that the given downwash distribution over the wing surface is represented by an even function of the variable ψ, whereas anti-symmetry assumes that the downwash distribution is expressed by an odd function of ψ. Deduced quantities such as velocity potential, acceleration potential or pressure distribution retain this property of symmetry or anti-symmetry, according as the corresponding downwash at the wing surface is symmetric or anti-symmetric with respect to ψ. Our problem is now to calculate the function $h^{\times}(\vartheta)$ in the range $\frac{\pi}{2} \leqslant \vartheta \leqslant \frac{3\pi}{2}$.

It is well known that the functions

$$1,\ \sin\vartheta,\ \cos 2\vartheta,\ \sin 3\vartheta,\ \cos 4\vartheta,\ \sin 5\vartheta,\ \cos 6\vartheta,\ \ldots\ldots \qquad (2,10,3)$$

form a complete orthogonal set of functions in the range $\frac{\pi}{2} \leqslant \vartheta \leqslant \frac{3\pi}{2}$.

Hence we assume that the function $h^{\times}(\vartheta)$ can be represented by the Fourier series

$$h^{\times}(\vartheta) = \sum_{n=0}^{\infty} a_n \cos 2n\vartheta \qquad (2,10,4)$$

for symmetric problems, and by the series

$$h^{\times}(\vartheta) = \sum_{n=0}^{\infty} b_n \sin(2n+1)\vartheta \qquad (2,10,5)$$

for anti-symmetric problems.

First of all we treat the symmetric case. The given weight-function $g^{\times}(\vartheta)$ then can be written into the form

$$g^{\times}(\vartheta) = \sum_{n=0}^{\infty} c_n \cos n\vartheta \quad . \qquad (2,10,6)$$

Inserting the series (2,10,4) and (2,10,6) into the formula (2,10,1), we obtain

$$\Omega(\eta,\mu,\vartheta) = U \int_0^{2\pi} \sum_{n=0}^{\infty} c_n \cos n\vartheta_1 \cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 +$$
$$+ U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sum_{n=0}^{\infty} a_n \cos 2n\vartheta_1 \cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \quad . \qquad (2,10,7)$$

On the assumption that the series for $g^{\times}(\vartheta)$ and $h^{\times}(\vartheta)$ are uniformly convergent it is legitimate to invert the order of summation and integration in the expression (2,10,7). In most practical cases the Fourier series for the function $g^{\times}(\vartheta)$ breaks off.

The potential Ω can so be written in the form

$$\Omega(\eta,\mu,\vartheta) = U \sum_{n=0}^{\infty} c_n \int_0^{2\pi} \cos n\vartheta_1 \cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 +$$

$$+ U \sum_{n=0}^{\infty} a_n \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos 2n\vartheta_1 \cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 . \qquad (2,10,8)$$

Using the Fourier series (2,8,45) for the function $\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0}$ it is easy to find a similar expansion for the product $\cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0}$. For we have

$$\cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} = \sum_{m=0}^{\infty} - \frac{1}{\varepsilon_m \pi^2} \frac{\mu}{\mu^2+\eta^2} \frac{(1-\mu^2)^{m/2}}{(1+\eta^2)^{m/2}} \cos m\vartheta \cos m\vartheta_1 \cos\vartheta_1 +$$

$$+ \sum_{m=1}^{\infty} - \frac{1}{\pi^2} \frac{\mu}{\mu^2+\eta^2} \frac{(1-\mu^2)^{m/2}}{(1+\eta^2)^{m/2}} \sin m\vartheta \sin m\vartheta_1 \cos\vartheta_1$$

$$= \sum_{m=0}^{\infty} - \frac{1}{2\varepsilon_m \pi^2} \frac{\mu}{\mu^2+\eta^2} \frac{(1-\mu^2)^{m/2}}{(1+\eta^2)^{m/2}} \cos m\vartheta \left\{\cos(m+1)\vartheta_1 + \cos(m-1)\vartheta_1\right\} +$$

$$+ \sum_{m=1}^{\infty} - \frac{1}{2\pi^2} \frac{\mu}{\mu^2+\eta^2} \frac{(1-\mu^2)^{m/2}}{(1+\eta^2)^{m/2}} \sin m\vartheta \left\{\sin(m+1)\vartheta_1 + \sin(m-1)\vartheta_1\right\} =$$

$$= -\frac{1}{2\pi^2} \frac{\mu}{\mu^2+\eta^2} \frac{(1-\mu^2)^{1/2}}{(1+\eta^2)^{1/2}} \cos\vartheta - \frac{1}{2\pi^2} \frac{\mu}{\mu^2+\eta^2} \sum_{m=1}^{\infty} \left\{ \frac{(1-\mu^2)^{\frac{m-1}{2}}}{(1+\eta^2)^{\frac{m-1}{2}}} \cos(m-1)\vartheta + \right.$$

$$\left. + \frac{(1-\mu^2)^{\frac{m+1}{2}}}{(1+\eta^2)^{\frac{m+1}{2}}} \cos(m+1)\vartheta \right\} \cos m\vartheta_1 +$$

$$- \frac{1}{2\pi^2} \frac{\mu}{\mu^2+\eta^2} \sum_{m=1}^{\infty} \left\{ \frac{(1-\mu^2)^{\frac{m-1}{2}}}{(1+\eta^2)^{\frac{m-1}{2}}} \sin(m-1)\vartheta + \frac{(1-\mu^2)^{\frac{m+1}{2}}}{(1+\eta^2)^{\frac{m+1}{2}}} \sin(m+1)\vartheta \right\} \sin m\vartheta_1$$

$$(2,10,9)$$

Introducing the expressions

$$-\frac{U}{2\pi^2}\,\frac{\mu}{\mu^2+\eta^2}\,\frac{(1-\mu^2)^{1/2}}{(1+\eta^2)^{1/2}}\cos\vartheta = \overset{s}{\Psi}_0$$

$$-\frac{U}{2\pi^2}\,\frac{\mu}{\mu^2+\eta^2}\left\{\frac{(1-\mu^2)^{\frac{m-1}{2}}}{(1+\eta^2)^{\frac{m-1}{2}}}\cos(m-1)\vartheta + \frac{(1-\mu^2)^{\frac{m+1}{2}}}{(1+\eta^2)^{\frac{m+1}{2}}}\cos(m+1)\vartheta\right\} = \overset{s}{\Psi}_m \quad \text{for } m\geq 1 \qquad (2,10,10)$$

$$-\frac{U}{2\pi^2}\,\frac{\mu}{\mu^2+\eta^2}\left\{\frac{(1-\mu^2)^{\frac{m-1}{2}}}{(1+\eta^2)^{\frac{m-1}{2}}}\sin(m-1)\vartheta + \frac{(1-\mu^2)^{\frac{m+1}{2}}}{(1+\eta^2)^{\frac{m+1}{2}}}\sin(m+1)\vartheta\right\} = \overset{a}{\Psi}_m \quad \text{for } m\geq 1$$

it is evident that the formula (2,10,9) can be rewritten into the form

$$U\cos\vartheta_1\left[\frac{\partial G}{\partial\mu_1}\right]_{\mu_1=0} = \sum_{m=0}^{\infty}\overset{s}{\Psi}_m\cos m\vartheta_1 + \sum_{m=1}^{\infty}\overset{a}{\Psi}_m\sin m\vartheta_1 \,. \qquad (2,10,11)$$

Furthermore it can easily be shown that the series on the right-hand side of the last formula are uniformly convergent for all admissible values of the variables $\vartheta, \vartheta_1, \mu$ and η, except for $\mu=\eta=0$.
Inserting this series (2,10,11) into the relation (2,10,8) and reversing the order of summation and integration, we obtain

$$\begin{aligned}\Omega(\eta,\mu,\vartheta) = &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n\overset{s}{\Psi}_m\int_0^{2\pi}\cos n\vartheta_1\cos m\vartheta_1\,d\vartheta_1 + \\ &+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_n\overset{a}{\Psi}_m\int_0^{2\pi}\cos n\vartheta_1\sin m\vartheta_1\,d\vartheta_1 + \\ &+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\overset{s}{\Psi}_m\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos 2n\vartheta_1\cos m\vartheta_1\,d\vartheta_1 + \\ &+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_n\overset{a}{\Psi}_m\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos 2n\vartheta_1\sin m\vartheta_1\,d\vartheta_1\end{aligned}$$

or

$$\Omega(\eta,\mu,\vartheta) = \pi\sum_{n=0}^{\infty}\varepsilon_n c_n\overset{s}{\Psi}_n + \sum_{n=0}^{\infty}a_n\left\{p_{2n}^{2n}\,\overset{s}{\Psi}_{2n} + \sum_{m=0}^{\infty}p_{2n}^{2m+1}\,\overset{s}{\Psi}_{2m+1}\right\} \qquad (2,10,12)$$

where

$$p_{2n}^{2n} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2 2n\vartheta_1 \, d\vartheta_1 = \varepsilon_n \frac{\pi}{2}$$

and

$$p_{2n}^{2m+1} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos 2n\vartheta_1 \cos (2m+1)\vartheta_1 \, d\vartheta_1 = \frac{(-1)^{m+n+1}(4m+2)}{(2m+1)^2 - 4n^2} \; . \qquad (2,10,13)$$

In the formula (2,10,12) the functions $\overset{a}{\psi}_m$ do not enter. This fact agrees completely with the symmetry property of the problems considered. The equation for the coefficients a_n is found by requiring that the acceleration potential Ω has a vanishing normal velocity at the wing surface.
If the normal velocity at the wing surface that corresponds with the acceleration potential $\overset{s}{\psi}_n$ is denoted by $\overset{s}{w}_n$, the equation for the coefficients a_n takes the form

$$\pi \sum_{n=0}^{\infty} \varepsilon_n c_n \overset{s}{w}_n + \sum_{n=0}^{\infty} a_n \left\{ p_{2n}^{2n} \overset{s}{w}_{2n} + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \overset{s}{w}_{2m+1} \right\} = 0 \; . \qquad (2,10,14)$$

The evaluation of the normal velocity $\overset{s}{w}_n$ will be performed in the next paragraph. We know that the acceleration potential $\overset{s}{\psi}_n$ has a zero normal acceleration at the surface of the wing. This implies that the normal velocity $\overset{s}{w}_n$ is a function of the variable y only because of the connection between normal acceleration and normal velocity. The equation (2,10,14) must be interpreted as an identity in the variable y .
It is obvious to transform the identity (2,10,14) into an infinite system of linear equations by means of some orthogonal system of functions of the variable y . The variable y lying in the interval $-1 \leq y \leq +1$, we choose the orthogonal system of the Legendre polynomials $P_n(y)$. Because of the symmetry of the problems, we can restrict the set of the Legendre polynomials to the half of the system, viz. the Legendre polynomials of even order, $P_{2n}(y)$.
Multiplying the identity (2,10,14) with $P_{2n}(y)$ $(n=0,1,2,\ldots)$ and integrating thereupon over the variable y from -1 to $+1$, we obtain the system of equations

$$\pi \sum_{n=0}^{\infty} \varepsilon_n c_n \int_{-1}^{+1} \overset{s}{w}_n P_{2\ell}(y)\,dy + \sum_{n=0}^{\infty} a_n \left\{ p_{2n}^{2n} \int_{-1}^{+1} \overset{s}{w}_{2n} P_{2\ell}(y)\,dy + \right.$$

$$\left. + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \int_{-1}^{+1} \overset{s}{w}_{2m+1} P_{2\ell}(y)\,dy \right\} = 0$$

$$\text{for } \ell = 0,1,2,\ldots\ldots \qquad (2,10,15)$$

Putting

$$-\pi\ \varepsilon_n \int_{-1}^{+1} \overset{s}{w}_n P_{2\ell}(y)\,dy = \overset{s}{\delta}{}_\ell^n$$

and (2,10,16)

$$p_{2n}^{2n} \int_{-1}^{+1} \overset{s}{w}_{2n} P_{2\ell}(y)\,dy + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \int_{-1}^{+1} \overset{s}{w}_{2m+1} P_{2\ell}(y)\,dy = \overset{s}{\tau}_\ell{}^n$$

the system of equations transforms into

$$\sum_{n=0}^{\infty} \overset{s}{\tau}{}_\ell^n a_n = \sum_{n=0}^{\infty} \overset{s}{\delta}{}_\ell^n c_n \qquad \ell = 0,1,2,\ldots\ldots \tag{2,10,17}$$

The coefficients $\overset{s}{\tau}{}_\ell^n$ *and* $\overset{s}{\delta}{}_\ell^n$ will be evaluated in section 12.
The system (2,10,17) represents an infinite set of linear algebraic equations for the unknown coefficients a_n. In order to arrive at numerical results it is necessary to truncate the infinite series in (2,10,17) to get a finite system of linear equations, which can be solved.

The mathematical treatment of the anti-symmetric problem follows the same lines as in the symmetric case. The weight-functions $g(\vartheta)$ and $k(\vartheta)$ are written in the form

$$g(\vartheta) = \cos\vartheta\ g^*(\vartheta) = \cos\vartheta \sum_{n=1}^{\infty} d_n \sin n\vartheta \quad \text{and} \quad k(\vartheta) = \cos\vartheta\ k^*(\vartheta) = \cos\vartheta \sum_{n=0}^{\infty} b_n \sin(2n+1)\vartheta$$

The acceleration potential Ω reads in this case

$$\Omega(\eta,\mu,\vartheta) = U \sum_{n=1}^{\infty} d_n \int_0^{2\pi} \sin n\vartheta_1 \cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 +$$

$$+ U \sum_{n=0}^{\infty} b_n \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(2n+1)\vartheta_1 \cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \tag{2,10,18}$$

or on substituting the Fourier expansion (2,10,11) of $U \cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0}$

$$\Omega(\eta,\mu,\vartheta) = \pi \sum_{n=1}^{\infty} d_n \overset{a}{\psi}_n + \sum_{n=0}^{\infty} b_n \left\{ q_{2n+1}^{2n+1} \overset{a}{\psi}_{2n+1} + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \overset{a}{\psi}_{2m} \right\} \tag{2,10,19}$$

wherein

$$q_{2n+1}^{2n+1} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin^2(2n+1)\vartheta_1\,d\vartheta_1 = \frac{\pi}{2}$$

and (2,10,20)

$$q_{2n+1}^{2m} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin 2m\vartheta_1 \sin(2n+1)\vartheta_1\,d\vartheta_1 = \frac{(-1)^{m+n}\,4m}{4m^2-(2n+1)^2}.$$

The condition, that the normal velocity at the wing surface which corresponds to the acceleration potential Ω vanishes, yields the identity in the variable y :

$$\pi \sum_{n=1}^{\infty} d_n \overset{a}{w}_n + \sum_{n=0}^{\infty} b_n \left\{ q_{2n+1}^{2n+1} \overset{a}{w}_{2n+1} + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \overset{a}{w}_{2m} \right\} = 0 \,. \qquad (2,10,21)$$

The quantity $\overset{a}{w}_n$ denotes the normal velocity at the wing surface that corresponds to the acceleration potential $\overset{a}{\psi}_n$.

Multiplication of (2,10,21) with the Legendre polynomials $P_{2k+1}(y)$ $(k=0,1,2,\ldots)$ and taking thereupon the integral over y from -1 to $+1$, yields the system

$$\pi \sum_{n=1}^{\infty} d_n \int_{-1}^{+1} \overset{a}{w}_n P_{2k+1}(y)\,dy + \sum_{n=0}^{\infty} b_n \left\{ q_{2n+1}^{2n+1} \int_{-1}^{+1} \overset{a}{w}_{2n+1} P_{2k+1}(y)\,dy + \right.$$

$$\left. + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \int_{-1}^{+1} \overset{a}{w}_{2m} P_{2k+1}(y)\,dy \right\} = 0 \qquad (2,10,22)$$

or written in a simpler way

$$\sum_{n=0}^{\infty} \overset{a}{\tau}{}_k^n b_n = \sum_{n=1}^{\infty} \overset{a}{\sigma}{}_k^n d_n \qquad k=0,1,2,\ldots \qquad (2,10,23)$$

where

$$\overset{a}{\sigma}{}_k^n = -\pi \int_{-1}^{+1} \overset{a}{w}_n P_{2k+1}(y)\,dy$$

and (2,10,24)

$$\overset{a}{\tau}{}_k^n = q_{2n+1}^{2n+1} \int_{-1}^{+1} \overset{a}{w}_{2n+1} P_{2k+1}(y)\,dy + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \int_{-1}^{+1} \overset{a}{w}_{2m} P_{2k+1}(y)\,dy \,.$$

Quite similar as in the symmetric case, approximate values of the unknown coefficients b_n are found by truncating the infinite system (2,10,23) and solving the resulting finite system of linear algebraic equations.

11 Determination of the downwashes $\overset{s}{w}_n$ and $\overset{a}{w}_n$.

In this section the normal velocities at the wing surface, which correspond with the singular acceleration potentials $\overset{s}{\psi}_n$ and $\overset{a}{\psi}_n$ will be calculated. The usual method of determining such velocities is based on the relation between acceleration potential and velocity potential as indicated by the formula (1,3,10). This method will be applied in the second part of this paper, which deals with the unsteady problem. However, in the steady case it is possible to evaluate the normal velocities mentioned by application of some formulas from the ordinary lifting-surface theory, in which downwash and vorticity components are connected by an integral equation. In this approach to the lifting surface theory a three-dimensional planform is represented by a vortex sheet, which covers the projection of the wing onto the xy -plane. The pattern of velocities induced by this vortex sheet must fulfil the linearized boundary condition

$$w = U \frac{\partial z}{\partial x} \text{ for } z=0,\ (x,y) \text{ in } S \,, \qquad (2,11,1)$$

where S denotes the projection of the wing surface onto the xy -plane and $z=z(x,y)$ is the analytical representation of the surface of the wing.

The vorticity vector $\vec{\gamma}$ is defined as the vector product of the unit vector normal to the vortex sheet and the difference vector of the velocities just above and below the vortex sheet. The components of this vector $\vec{\gamma}$ can be expressed in the perturbation velocities, viz.

$$\gamma_x = -(v_+ - v_-)\ ,\quad \gamma_y = u_+ - u_-\ ,\quad \gamma_z = 0 \qquad (2,11,2)$$

where u is the velocity component in the direction of the x -axis and v the component in the direction of the y -axis. The + sign denotes the upper side of the xy -plane and the - sign the lower side. The vortex lines of the vector field $\vec{\gamma}$ are defined by the requirement that in every point the tangent to the vortex line coincides with the vorticity vector $\vec{\gamma}$ in that point.

The equations (2,11,2) also furnish a simple means of expressing quantitatively the continuity of vortex lines.
For, if φ_+ and φ_- represent the values of the disturbance potential just above and below the vortex sheet, we have

$$\frac{\partial \gamma_y}{\partial y} = \frac{\partial}{\partial y}(u_+ - u_-) = \frac{\partial^2}{\partial y\,\partial x}(\varphi_+ - \varphi_-) = \frac{\partial}{\partial x}(v_+ - v_-) = -\frac{\partial \gamma_x}{\partial x}$$

or

$$\frac{\partial \gamma_x}{\partial x} + \frac{\partial \gamma_y}{\partial y} = 0 \ . \qquad (2,11,3)$$

A close examination of the vortex pattern shows that the vortex sheet cannot be limited to the region S alone. If it were, the vortex pattern would have to consist of a series of closed rings according to Helmholtz' law. Integration of (2,11,3) over the wing chord at any spanwise station of the wing then yields

$$\int_{x_\ell}^{x_t} \frac{\partial \gamma_y}{\partial y}\,dx = -\int_{x_\ell}^{x_t} \frac{\partial \gamma_x}{\partial x} = \gamma_x(x_\ell, y) - \gamma_x(x_t, y) \qquad (2,11,4)$$

wherein $x = x_\ell(y)$ represents the leading edge of the wing and $x = x_t(y)$ the trailing edge. The right-hand side (2,11,4) can be put equal to zero. The condition of zero $\gamma_x(x_\ell, y)$ and $\gamma_x(x_t, y)$ is justified when we assert that all integrations over S extend a short distance beyond the actual wing edge into the region of zero vorticity. This artifice is completely consistent with the analysis given here.
The left-hand side of (2,11,4) can be reduced as follows:

$$\int_{x_\ell}^{x_t} \frac{\partial \gamma_y}{\partial y}\,dx = \frac{d}{dy}\int_{x_\ell}^{x_t} \gamma_y\,dx - \gamma_y(x_t, y)\frac{dx_t}{dy} + \gamma_y(x_\ell, y)\frac{dx_\ell}{dy} =$$

$$= \frac{d}{dy}\int_{x_\ell}^{x_t} \gamma_y\,dx = \frac{d\Gamma}{dy}\ , \qquad (2,11,5)$$

where $\Gamma(y)$ is the circulation at the spanwise station y .
For the second equality in (2,11,5) the same reasoning holds as above.
Thus we have the relation

$$\frac{d\Gamma}{dy} = 0 \qquad (2,11,6)$$

or in other words, the circulation $\Gamma(y)$ is a constant over the span of the wing.

Because at the tips of the wing the circulation falls down to zero, this constant must be equal to zero. We can thus say, that if the vortex sheet were restricted to the projection S of the wing onto the xy -plane, the wing could not develop a lift force.
The need for a solution which gives a non-zero lift, requires that the vortex lines somehow extend away from S to infinity. Since the vortex lines have their source at the wing, the only direction in which they can move while still obeying the rule that they must always be attached to the same particles in the open flow, is downstream along the wake.
To be consistent with the idea of small disturbances, we therefore assume a wake region S' which lies in the xy -plane between the downstream projections of the wing tips, and we fill it with a vortex sheet having vorticity components γ_x and γ_y similarly as on the wing.

The condition of zero pressure discontinuity across S' implies that the vorticity component γ_y must vanish in the wake.

It then follows from equation (2,11,3) that $\frac{\partial \gamma_x}{\partial x} = 0$ everywhere in S' or that

γ_x is a function of y only throughout the wake.

We consider now the vortices contained within a small rectangular element $dxdy$ of the sheet, centered at (x,y) and in particular the vertical velocity dw induced by them at an arbitrary spatial point $P(x_o, y_o, z_o)$. Therefore we apply Biot-Swart's law, which states that an elementary length dx of a vortex line with vorticity-strength γ induces a velocity

$$dq = \frac{\gamma \vec{ds} \times \vec{r}}{4\pi r^3} , \qquad (2,11,7)$$

at a point P located a vector distance r from ds . The direction of ds must be taken such that the circulation γ is positive around it, in accordance with the right-hand rule. The scalar form of this law reads

$$dq = \frac{\gamma \sin \beta \, ds}{4\pi r^2} \qquad (2,11,8)$$

where β denotes the angle between r and the element ds .
Elementary calculations show that the total velocity due to vortices within the rectangular element $dxdy$ is given by

$$dw = + \frac{\gamma_y (x-x_o)\, dx\, dy}{4\pi \sqrt{(x-x_o)^2 + (y-y_o)^2 + z_o^2}^{\,3}} - \frac{\gamma_x (y-y_o)\, dx\, dy}{4\pi \sqrt{(x-x_o)^2 + (y-y_o)^2 + z_o^2}^{\,3}} . \qquad (2,11,9)$$

To calculate the effect of the entire sheet, we simply sum the elementary contributions by integrating over wing and wake regions.
We find

$$w(x_o, y_o, z_o) = + \frac{1}{4\pi} \iint_S \frac{\gamma_y (x-x_o) - \gamma_x (y-y_o)}{\sqrt{(x-x_o)^2 + (y-y_o)^2 + z_o^2}^{\,3}} dx\, dy +$$

$$- \frac{1}{4\pi} \iint_{S'} \frac{\gamma_x (y-y_o)}{\sqrt{(x-x_o)^2 + (y-y_o)^2 + z_o^2}^{\,3}} dx\, dy . \qquad (2,11,10)$$

We can now construct an integral equation by letting z_0 approach zero, while the point $(x_0, y_0, 0)$ belongs to the region S .

In the limit the singular integral over the wing S assumes a principal value, analogous as in the two-dimensional case (ref.17)

$$w(x_0, y_0) = +\frac{1}{4\pi}\iint_S \frac{\gamma_y(x-x_0)-\gamma_x(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}^{\,3}}\,dx\,dy +$$

$$-\frac{1}{4\pi}\iint_{S'} \frac{\gamma_x(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}^{\,3}}\,dx\,dy \quad . \qquad (2,11,11)$$

Let us now return to the circular wing. In order to express the vorticity component γ_x in terms of γ_y , we consider a strip dy of the wing as indicated in figure 2.

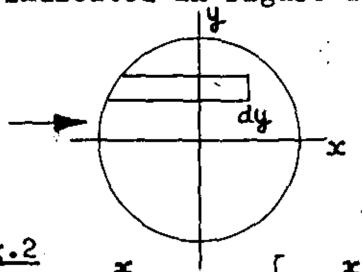

Fig.2

Due to the law of conservation no vorticity can vanish within this strip.
This fact expressed mathematically reads

$$\int_{-\sqrt{1-y^2}}^{x} \gamma_y\,dx - \left[\int_{-\sqrt{1-y^2}}^{x} \gamma_y\,dx + \frac{\partial}{\partial y}\int_{-\sqrt{1-y^2}}^{x} \gamma_y\,dx\,dy\right] - \gamma_x\,dy = 0$$

or

$$\gamma_x = -\frac{\partial}{\partial y}\int_{-\sqrt{1-y^2}}^{x} \gamma_y\,dx \quad . \qquad (2,11,12)$$

Hence for a point x in the wake one has

$$\gamma_x = -\frac{\partial}{\partial y}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \gamma_y\,dx \quad . \qquad (2,11,13)$$

Putting

$$\Gamma(x,y) = \int_{-\sqrt{1-y^2}}^{x} \gamma_y\,dx \qquad (2,11,14)$$

we can write

$$\gamma_x = -\frac{\partial \Gamma(x,y)}{\partial y} \text{ and for a point } x \text{ in the wake } \gamma_x = -\frac{d\Gamma(y)}{dy} . \qquad (2,11,15)$$

Moreover it follows from the formula (2,11,14)

$$\gamma_y = \frac{\partial \Gamma(x,y)}{\partial x} \quad . \qquad (2,11,16)$$

The expression (2,11,11) for the downwash thus can be written in the form

$$w(x_0,y_0)=\frac{1}{4\pi}\iint_S \frac{x-x_0}{r^3}\frac{\partial\Gamma}{\partial x}dxdy+\frac{1}{4\pi}\iint_S \frac{y-y_0}{r^3}\frac{\partial\Gamma}{\partial y}dxdy+$$

$$+\frac{1}{4\pi}\iint_{S'} \frac{y-y_0}{r^3}\frac{d\Gamma}{dy}dxdy\,. \qquad (2,11,17)$$

Our problem being the evaluation of the downwashes $\overset{s}{w}_n$ and $\overset{a}{w}_n$, which correspond to the singular acceleration potentials $\overset{s}{\psi}_n$ and $\overset{a}{\psi}_n$ resp., we remember that these singular potentials have vanishing normal accelerations at the wing surface. Consequently the normal velocities $\overset{s}{w}_n$ and $\overset{a}{w}_n$ at the wing surface must be independent of the variable x. This means that we may put x_0 equal to zero in the formula (2,11,17) without changing the value of the downwash.
Thus

$$w(y_0)=\frac{1}{4\pi}\iint_S \frac{x}{r^3}\frac{\partial\Gamma}{\partial x}dxdy+\frac{1}{4\pi}\iint_S \frac{y-y_0}{r^3}\frac{\partial\Gamma}{\partial y}dxdy+$$

$$+\frac{1}{4\pi}\iint_{S'} \frac{y-y_0}{r^3}\frac{d\Gamma}{dy}dxdy\,. \qquad (2,11,18)$$

In particular we shall suppose that the acceleration potential ψ and thus the corresponding vorticity component γ_y is an even function of the variable x.
The formula (2,11,18) then reduces to

$$w(y_0)=\frac{1}{4\pi}\iint_S \frac{y-y_0}{r^3}\frac{\partial\Gamma}{\partial y}dxdy+\frac{1}{4\pi}\iint_{S'} \frac{y-y_0}{r^3}\frac{d\Gamma}{dy}dxdy\,. \qquad (2,11,19)$$

Due to the connection between the quantities γ_y and $\Gamma(x,y)$ we put

$$\Gamma(x,y)=f(y)+g(x,y) \qquad (2,11,20)$$

where $g(x,y)$ is an odd function of the variable x.
In the wake the function $\Gamma(x,y)$ reduces to the function $\Gamma(y)$ and thus the following relation holds

$$\Gamma(y)=f(y)+g\left(\sqrt{1-y^2},\,y\right)\,. \qquad (2,11,21)$$

For $x=-\sqrt{1-y^2}$ the function $\Gamma(x,y)$ vanishes, thus

$$0=f(y)+g\left(-\sqrt{1-y^2},y\right)=f(y)-g\left(\sqrt{1-y^2},y\right)\,. \qquad (2,11,22)$$

These two relations (2,11,21) and (2,11,22) lead to the conclusion

$$f(y)=\frac{1}{2}\Gamma(y) \qquad (2,11,23)$$

Hence equation (2,11,20) can be written as

$$\Gamma(x,y)=\frac{1}{2}\Gamma(y)+g(x,y) \qquad (2,11,24)$$

where $g(x,y)$ is an odd function of x.

Insertion of this result into (2,11,19) yields

$$w(y_0) = \frac{1}{8\pi} \iint_S \frac{y-y_0}{r^3} \frac{d\Gamma}{dy} \, dx\,dy + \frac{1}{4\pi} \iint_{S'} \frac{y-y_0}{r^3} \frac{d\Gamma}{dy} \, dx\,dy$$

or

$$w(y_0) = \frac{1}{4\pi} \int_{-1}^{+1} \frac{d\Gamma}{dy} \, dy \int_0^{\infty} \frac{y-y_0}{r^3} \, dx = \frac{1}{4\pi} \int_{-1}^{+1} \frac{d\Gamma}{dy} \frac{1}{y-y_0} \, dy \, . \tag{2,11,25}$$

Assuming that γ_y is an even function of the variable x, it is now possible to calculate the corresponding downwash by aid of the formula (2,11,25).
Here this method will be applied to two cases, viz.

$$\gamma_y = -\frac{1}{\pi^2} \frac{\sqrt{1-\mu^2}^{\,2m}}{\mu} \cos 2m\vartheta \quad \text{and} \quad \gamma_y = -\frac{1}{\pi^2} \frac{\sqrt{1-\mu^2}^{\,2m+1}}{\mu} \sin (2m+1)\vartheta \tag{2,11,26}$$

Hence we must calculate the two integrals

$$L_{2m} = -\frac{1}{\pi^2} \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{\sqrt{1-\mu^2}^{\,2m}}{\mu} \cos 2m\vartheta \, dx \quad \text{and}$$

$$L_{2m+1} = -\frac{1}{\pi^2} \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{\sqrt{1-\mu^2}^{\,2m+1}}{\mu} \sin (2m+1)\vartheta \, dx \, . \tag{2,11,27}$$

Let us consider the real part of the infinite series

$$\sum_{m=0}^{\infty} (i\rho)^m e^{im\vartheta}$$

with $0 < \rho < 1$.
It can then be derived that

$$\mathrm{Re}\left\{\sum_{m=0}^{\infty} (i\rho)^m e^{im\vartheta}\right\} = \sum_{m=0}^{\infty} (-1)^m \rho^{2m} \cos 2m\vartheta + \sum_{m=0}^{\infty} (-1)^{m+1} \rho^{2m+1} \sin (2m+1)\vartheta =$$

$$= \mathrm{Re}\left\{\frac{1}{1-i\rho e^{i\vartheta}}\right\} = \mathrm{Re}\left\{\frac{1}{1+\rho \sin\vartheta - i\rho\cos\vartheta}\right\} = \frac{1+\rho\sin\vartheta}{1+\rho^2+2\rho\sin\vartheta} \, . \tag{2,11,28}$$

Putting $\rho = r\sqrt{1-\mu^2}$ we get

$$-\frac{1}{\pi^2}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}}\sum_{m=0}^{\infty}(-1)^m r^{2m}\frac{\sqrt{1-\mu^2}^{\,2m}}{\mu}\cos 2m\vartheta\, dx+$$

$$-\frac{1}{\pi^2}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}}\sum_{m=0}^{\infty}(-1)^{m+1} r^{2m+1}\frac{\sqrt{1-\mu^2}^{\,2m+1}}{\mu}\sin(2m+1)\vartheta\, dx=$$

$$=\sum_{m=0}^{\infty}(-1)^m r^{2m}L_{2m}+\sum_{m=0}^{\infty}(-1)^{m+1}r^{2m+1}L_{2m+1}=-\frac{1}{\pi^2}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}}\frac{1+r\sqrt{1-\mu^2}\sin\vartheta}{1+r^2(1-\mu^2)+2r\sqrt{1-\mu^2}\sin\vartheta}\frac{dx}{\mu}. \tag{2,11,29}$$

Using the transformation formulas

$$x=\sqrt{1-\mu^2}\cos\vartheta\quad,\qquad y=\sqrt{1-\mu^2}\sin\vartheta$$

we find

$$\sum_{m=0}^{\infty}(-1)^m r^{2m}L_{2m}+\sum_{m=0}^{\infty}(-1)^{m+1}r^{2m+1}L_{2m+1}=$$

$$=-\frac{1+yr}{\pi^2}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}}\frac{dx}{\left\{1+r^2(x^2+y^2)+2ry\right\}\sqrt{1-x^2-y^2}}\ . \tag{2,11,30}$$

The substitution $x=\sqrt{1-y^2}\sin\theta$ transforms the right-hand side of (2,11,30) into

$$-\frac{1+yr}{\pi^2}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}}\frac{dx}{\left\{1+r^2(x^2+y^2)+2ry\right\}\sqrt{1-x^2-y^2}}=-\frac{2(1+yr)}{\pi^2}\int_0^{\pi/2}\frac{d\theta}{(1+yr)^2+r^2(1-y^2)\sin^2\theta}\ .$$

Inserting in this integral $\lambda=\tan\theta$, we get

$$-\frac{1+yr}{\pi^2}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}}\frac{dx}{\left\{1+r^2(x^2+y^2)+2ry\right\}\sqrt{1-x^2-y^2}}=-\frac{2(1+yr)}{\pi^2}\int_0^{\infty}\frac{d\lambda}{(1+\lambda^2)\left\{(1+yr)^2+r^2(1-y^2)\dfrac{\lambda^2}{1-\lambda^2}\right\}}=$$

$$=-\frac{2}{\pi^2(1+yr)}\int_0^{\infty}\frac{d\lambda}{1+\dfrac{1+2yr+r^2}{(1+yr)^2}\lambda^2}=-\frac{1}{\pi\sqrt{1+2yr+r^2}}\ . \tag{2,11,31}$$

The right-hand side of (2,11,31) agrees with the generating function of the Legendre polynomials, viz.

$$\frac{1}{\sqrt{1+2yr+r^2}} = \sum_{m=0}^{\infty} (-r)^m P_m(y) \quad , \tag{2,11,32}$$

where $P_m(y)$ denotes the Legendre polynomial of order m.
We have now proved the identity

$$\sum_{m=0}^{\infty} (-1)^m r^{2m} L_{2m} + \sum_{m=0}^{\infty} (-1)^{m+1} r^{2m+1} L_{2m+1} = -\frac{1}{\pi} \sum_{m=0}^{\infty} (-r)^m P_m(y) \, . \tag{2,11,33}$$

From this relation it follows

$$L_{2m} = \frac{(-1)^{m+1}}{\pi} P_{2m}(y) \quad \text{and} \quad L_{2m+1} = \frac{(-1)^{m+1}}{\pi} P_{2m+1}(y) \, . \tag{2,11,34}$$

The downwash corresponding with the vorticity component γ_y, first mentioned in (2,11,26) can now be found by application of formula (2,11,25)

$$w(y_0) = \frac{(-1)^{m+1}}{4\pi^2} \int_{-1}^{+1} \frac{dP_{2m}}{dy} \, \frac{1}{y-y_0} \, dy \, . \tag{2,11,35}$$

In order to determine the downwash $\overset{s}{w}_{2m+1}$ we must take for the vorticity component γ_y

$$\gamma_y = -\frac{1}{\pi^2} \left\{ \frac{\sqrt{1-\mu^2}^{\,2m}}{\mu} \cos 2m\vartheta + \frac{\sqrt{1-\mu^2}^{\,2m+2}}{\mu} \cos (2m+2)\vartheta \right\} . \tag{2,11,36}$$

Application of the formula (2,11,35) yields

$$\overset{s}{w}_{2m+1} = \frac{(-1)^{m+1}}{4\pi^2} \int_{-1}^{+1} \left\{ \frac{dP_{2m}}{dy} - \frac{dP_{2m+2}}{dy} \right\} \frac{1}{y-y_0} \, dy \, . \tag{2,11,37}$$

Using the relation

$$(2n+1) P_n(y) = \frac{dP_{n+1}}{dy} - \frac{dP_{n-1}}{dy} \tag{2,11,38}$$

equation (2,11,37) can be written in the form

$$\overset{s}{w}_{2m+1} = \frac{(-1)^m}{4\pi^2} (4m+3) \int_{-1}^{+1} \frac{P_{2m+1}(y)}{y-y_0} \, dy \, . \tag{2,11,39}$$

The integral in this expression (2,11,39) can be calculated with the well-known formula from the theory of the Legendre functions, viz.

$$Q_n(x) = \frac{1}{2} \int_{-1}^{+1} \frac{P_n(y)}{x-y} \, dy \tag{2,11,40}$$

where $Q_n(x)$ denotes the Legendre function of the second kind.
Equation (2,11,39) becomes thus

$$\overset{s}{w}_{2m+1} = \frac{(-1)^{m+1}}{2\pi^2} (4m+3) \; Q_{2m+1}(y) \, . \tag{2,11,41}$$

In the same way it can be derived that the downwash $\overset{a}{w}_{2m}$, which corresponds to the acceleration potential $\overset{a}{\psi}_{2m}$ is expressed by the formula

$$\overset{a}{w}_{2m} = \frac{(-1)^m}{2\pi^2} (4m+1) \, Q_{2m}(y) \, . \tag{2,11,42}$$

In connection with the evaluation of the downwashes $\overset{s}{w}_{2m}$ and $\overset{a}{w}_{2m+1}$ we remark that the corresponding vorticity component γ_y is represented by an odd function of the variable x . This fact implies that the circulation $\Gamma(y)$ is equal to zero for these cases. From the definition of the function $\Gamma(y)$ it follows that the circulation $\Gamma(y)$ is proportional to the velocity potential in the plane $z=0$ behind the wing $\left(x>\sqrt{1-y^2}\right)$. Thus the velocity potentials corresponding to the singular potentials $\overset{s}{\psi}_{2m}$ and $\overset{a}{\psi}_{2m+1}$ vanish in the plane $z=0$ outside the wing surface. Such velocity potentials can be written in the form

$$\Phi(\eta,\mu,\vartheta)=\int_0^{2\pi}\int_{-1}^{+1} w(\mu_1,\vartheta_1)\, G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,\mu_1\, d\mu_1\, d\vartheta_1 \;, \qquad (2,11,43)$$

where $w(\mu,\vartheta)$ is a still unknown downwash.
In this way we get the following two integral equations for the unknown downwashes $\overset{s}{w}_{2m}$ and $\overset{a}{w}_{2m+1}$

$$\overset{s}{\psi}_{2m}=\frac{\partial}{\partial x}U\int_0^{2\pi}\int_{-1}^{+1}\overset{s}{w}_{2m}\, G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,\mu_1\, d\mu_1\, d\vartheta_1 \qquad (2,11,44)$$

and

$$\overset{a}{\psi}_{2m+1}=\frac{\partial}{\partial x}U\int_0^{2\pi}\int_{-1}^{+1}\overset{a}{w}_{2m+1}\, G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,\mu_1\, d\mu_1\, d\vartheta_1 \,. \qquad (2,11,45)$$

The first integral equation (2,11,44) completely written down, reads for $m\geq 1$

$$-\frac{U}{2\pi^2}\,\frac{\mu}{\mu^2+\eta^2}\left\{\frac{\sqrt{1-\mu^2}^{\,2m-1}}{\sqrt{1+\eta^2}^{\,2m-1}}\cos(2m-1)\vartheta+\frac{\sqrt{1-\mu^2}^{\,2m+1}}{\sqrt{1+\eta^2}^{\,2m+1}}\cos(2m+1)\vartheta\right\}=$$

$$=U\frac{\partial}{\partial x}\int_0^{2\pi}\int_{-1}^{+1}\overset{s}{w}_{2m}\, G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,\mu_1\, d\mu_1\, d\vartheta_1 \,. \qquad (2,11,46)$$

At the wing surface the equation (2,11,46) reduces to

$$-\frac{1}{2\pi^2}\,\frac{1}{\mu}\left\{\sqrt{1-\mu^2}^{\,2m-1}\cos(2m-1)\vartheta+\sqrt{1-\mu^2}^{\,2m+1}\cos(2m+1)\vartheta\right\}=$$

$$=-\int_0^{2\pi}\int_{-1}^{+1}\overset{s}{w}_{2m}\left\{\frac{\sqrt{1-\mu^2}}{\mu}\cos\vartheta\,\frac{\partial G}{\partial\mu}+\frac{\sin\vartheta}{\sqrt{1-\mu^2}}\,\frac{\partial G}{\partial\vartheta}\right\}\mu_1\, d\mu_1\, d\vartheta_1 \,. \qquad (2,11,47)$$

Multiplying both sides of (2,11,47) with μ and taking thereupon the limit for μ tending to zero, wet get

$$\frac{1}{2\pi^2}\left\{\cos(2m-1)\vartheta+\cos(2m+1)\vartheta\right\}=$$

$$=\int_0^{2\pi}\int_{-1}^{+1}\overset{s}{w}_{2m}\cos\vartheta\, G_\mu\left(0,0,\vartheta;\mu_1,\vartheta_1\right)\mu_1\, d\mu_1\, d\vartheta_1$$

or

$$\frac{1}{\pi^2}\cos 2m\vartheta=2\int_{-1}^{+1}\overset{s}{w}_{2m}(y_1)\,dy_1\int_{-\sqrt{1-y_1^2}}^{+\sqrt{1-y_1^2}} G_\mu(0,0,\vartheta;\mu_1,\vartheta_1)\,dx_1 \qquad (2,11,48)$$

where use has been made of the fact that $\overset{s}{w}_{2m}$ is a function of the variable y only.
In order to simplify the relation (2,11,48) we substitute the Fourier series for $\left[\frac{\partial G}{\partial \mu}\right]_{\substack{\mu=0\\ \eta=0}}$, viz.

$$\left[\frac{\partial G}{\partial \mu}\right]_{\substack{\mu=0\\ \eta=0}} = \sum_{m=0}^{\infty} -\frac{1}{\varepsilon_m \pi^2} \frac{\sqrt{1-\mu_1^2}^{\,m}}{\mu_1} \cos m\vartheta \cos m\vartheta_1 +$$

$$+\sum_{m=1}^{\infty} -\frac{1}{\pi^2} \frac{\sqrt{1-\mu_1^2}^{\,m}}{\mu_1} \sin m\vartheta \sin m\vartheta_1 \qquad (2,11,49)$$

into the right-hand side of (2,11,48).
Moreover we apply the following formulas, which already have been given partly in the expressions (2,11,27) and (2,11,34)

$$\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{\sqrt{1-\mu^2}^{\,2m}}{\mu} \cos 2m\vartheta \, dx = (-1)^m \pi P_{2m}(y)$$

$$\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{\sqrt{1-\mu^2}^{\,2m+1}}{\mu} \cos (2m+1)\vartheta \, dx = 0$$

$$\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{\sqrt{1-\mu^2}^{\,2m}}{\mu} \sin 2m\vartheta \, dx = 0 \qquad (2,11,50)$$

$$\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{\sqrt{1-\mu^2}^{\,2m+1}}{\mu} \sin (2m+1)\vartheta \, dx = (-1)^m \pi P_{2m+1}(y),$$

The equation (2,11,48) reduces now to

$$\cos 2m\vartheta = -2 \int_{-1}^{+1} \overset{s}{w}_{2m}(y_1) \left\{ \sum_{k=0}^{\infty} \frac{1}{\varepsilon_k} (-1)^k \pi P_{2k}(y_1) \cos 2k\vartheta + \right.$$

$$\left. + \sum_{k=0}^{\infty} (-1)^k \pi P_{2k+1}(y_1) \sin (2k+1)\vartheta \right\} dy_1 \,. \qquad (2,11,51)$$

Let us now assume that $w_{2m}(y)$ can be represented by a series of Legendre polynomials, viz.

$$\overset{s}{w}_{2m}(\psi) = \sum_{n=0}^{\infty} a_n P_n(\psi) \quad . \tag{2,11,52}$$

Inserting this series (2,11,52) into the equation (2,11,51) and performing the integration, we get for $m \geq 1$

$$\cos 2m\vartheta = -2\sum_{n=0}^{\infty} a_{2n}\,\frac{1}{\varepsilon_n}\,(-1)^n\,\pi\cos 2n\vartheta\,\frac{2}{4n+1} +$$

$$-2\sum_{n=0}^{\infty} a_{2n+1}\,(-1)^n\,\pi\,\sin(2n+1)\vartheta\,\frac{2}{4n+3} \quad . \tag{2,11,53}$$

We can now conclude

$$a_{2n} = 0 \text{ for } n \neq m \;;\; a_{2n+1} = 0 \text{ and } a_{2m} = (-1)^{m+1}\,\frac{4m+1}{4\pi} \;.$$

Thus

$$\overset{s}{w}_{2m}(\psi) = (-1)^{m+1}\,\frac{4m+1}{4\pi} P_{2m}(\psi) \quad \text{for } m \geq 1 \quad . \tag{2,11,54}$$

For $m=0$ it can easily be derived along the same lines that

$$\overset{s}{w}_0(\psi) = -\frac{1}{4\pi} P_0(\psi) \quad . \tag{2,11,55}$$

The downwashes $\overset{a}{w}_{2m+1}$, which correspond to the potentials $\overset{a}{\psi}_{2m+1}$ can be determined with the same method and the result is

$$\overset{a}{w}_{2m+1}(\psi) = (-1)^{m+1}\,\frac{4m+3}{4\pi} P_{2m+1}(\psi) \;. \tag{2,11,56}$$

12 Numerical evaluation of the coefficients in the linear systems.

In this section some integral properties of Legendre's functions will be applied frequently. These formulas read

$$\int_{-1}^{+1} P_n(x) P_m(x)\,dx = 0 \quad \text{for } n \neq m \;;\; \int_{-1}^{+1} \left\{P_n(x)\right\}^2 dx = \frac{2}{2n+1} \tag{2,12,1}$$

$$\int_{-1}^{+1} P_n(x) Q_m(x)\,dx = \frac{1-(-1)^{n+m}}{(n-m)(n+m+1)} \quad \text{for } n \neq m \;;\; \int_{-1}^{+1} P_n(x) Q_n(x)\,dx = 0 \;. \tag{2,12,2}$$

The properties expressed by (2,12,1) are well-known and will therefore not be proved here. However, the proof of the formulas (2,12,2) will be given. Because $P_n(x)$ and $Q_m(x)$ satisfy Legendre's differential equation, the following relations hold

$$(1-x^2)\,\frac{d^2P_n}{dx^2} - 2x\,\frac{dP_n}{dx} + n(n+1)\,P_n = 0$$

$$(1-x^2)\,\frac{d^2Q_m}{dx^2} - 2x\,\frac{dQ_m}{dx} + m(m+1)\,Q_m = 0 \quad . \tag{2,12,3}$$

Multiplying the first expression by $Q_m(x)$, and the second one by $P_n(x)$ and subtracting the results, we find

$$Q_m \frac{d}{dx}\left\{(1-x^2)\ \frac{dP_n}{dx}\right\} - P_n \frac{d}{dx}\left\{(1-x^2)\ \frac{dQ_m}{dx}\right\} = -(n-m)(n+m+1) P_n Q_m \, . \quad (2,12,4)$$

Integrating both sides of this equation between the limits $+1$ and -1, we obtain

$$(n-m)\ (n+m+1)\int_{-1}^{+1} P_n(x)\, Q_m(x)\, dx = \int_{-1}^{+1}\left[P_n \frac{d}{dx}\left\{(1-x^2)\ \frac{dQ_m}{dx}\right\} + \right.$$

$$\left. - Q_m \frac{d}{dx}\left\{(1-x^2)\ \frac{dP_n}{dx}\right\}\right] dx = \left[(1-x^2)\left\{P_n \frac{dQ_m}{dx} - Q_m \frac{dP_n}{dx}\right\}\right]_{x=-1}^{x=+1} . \quad (2,12,5)$$

Hence the value of $\int_{-1}^{+1} P_n(x)\, Q_m(x)\, dx$ for $n \neq m$ is given by

$$\int_{-1}^{+1} P_n(x)\, Q_m(x)\, dx = \frac{1}{(n-m)(n+m+1)}\left[(1-x^2)\left\{P_n \frac{dQ_m}{dx} - Q_m \frac{dP_n}{dx}\right\}\right]_{x=-1}^{x=+1} . \quad (2,12,6)$$

In order to evaluate the right-hand side of (2,12,6) use can be made of the formula (ref.2):

$$Q_m(x) = \tfrac{1}{2} P_m(x) \log \frac{1+x}{1-x} - W_{n-1}(x)\, , \quad -1 < x < +1$$

with

$$W_{n-1}(x) = \frac{2n-1}{1.n} P_{n-1}(x) + \frac{2n-5}{3(n-1)} P_{n-3}(x) + \frac{2n-9}{5(n-2)} P_{n-5}(x) + \cdots\cdots \quad (2,12,7)$$

We see easily that

$$\int_{-1}^{+1} P_n(x)\, Q_m(x)\, dx = \frac{1}{(n-m)(n+m+1)}\left[P_n(x)\, P_m(x)\right]_{-1}^{+1} = \frac{1-(-1)^{n+m}}{(n-m)(n+m+1)} . \quad (2,12,8)$$

In the case $n=m$, we have

$$\int_{-1}^{+1} P_n(x)\, Q_n(x)\, dx = \tfrac{1}{2}\int_{-1}^{+1} P_n(x)\, P_n(x) \log \frac{1+x}{1-x}\, dx =$$

$$\tfrac{1}{2}\int_{0}^{1} P_n(x)\, P_n(x) \log \frac{1+x}{1-x}\, dx + \tfrac{1}{2}\int_{-1}^{0} P_n(x)\, P_n(x) \log \frac{1+x}{1-x}\, dx =$$

$$= \tfrac{1}{2}\int_{0}^{1} P_n(x)\, P_n(x) \log \frac{1+x}{1-x}\, dx - \tfrac{1}{2}\int_{0}^{1} P_n(x)\, P_n(x) \log \frac{1+x}{1-x}\, dx = 0 . \quad (2,12,9)$$

It is now easy to calculate the coefficients $\overset{s}{\sigma}{}_{\ell}^{n}$ and $\overset{a}{\sigma}{}_{k}^{n}$ which occur in the right-hand sides of the systems (2,10,17) and (2,10,23) respectively. In fact we have ,

$$\overset{s}{\sigma}{}_{\ell}^{2n} = -\pi\varepsilon_n \int_{-1}^{+1} \overset{s}{w}_{2n} P_{2\ell}(y)\,dy = \varepsilon_n \frac{(-1)^n (4n+1)}{4} \int_{-1}^{+1} P_{2n}(y) P_{2\ell}(y)\,dy = \begin{cases} 0 \text{ for } n \neq \ell \\ \tfrac{1}{2}\varepsilon_\ell (-1)^\ell \text{ for } n = \ell \end{cases}$$

$$\overset{s}{\sigma}{}_{\ell}^{2n+1} = -\pi \int_{-1}^{+1} \overset{s}{w}_{2n+1} P_{2\ell}(y)\,dy = \frac{(-1)^n (4n+3)}{2\pi} \int_{-1}^{+1} Q_{2n+1}(y) P_{2\ell}(y)\,dy = \frac{(-1)^n (4n+3)}{\pi (2\ell-2n-1)(2\ell+2n+2)}$$

$$\overset{a}{\sigma}{}_{k}^{2n} = -\pi \int_{-1}^{+1} \overset{a}{w}_{2n} P_{2k+1}(y)\,dy = \frac{(-1)^{n+1} (4n+1)}{2\pi} \int_{-1}^{+1} Q_{2n}(y) P_{2k+1}(y)\,dy = \frac{(-1)^{n+1}(4n+1)}{\pi (2k-2n+1)(2k+2n+2)}$$

$$\overset{a}{\sigma}{}_{k}^{2n+1} = -\pi \int_{-1}^{+1} \overset{a}{w}_{2n+1} P_{2k+1}(y)\,dy = \frac{(-1)^n (4n+3)}{4} \int_{-1}^{+1} P_{2n+1}(y) P_{2k+1}(y)\,dy = \begin{cases} 0 \text{ for } n \neq k \\ \tfrac{1}{2}(-1)^k \text{ for } n = k \end{cases} \quad (2,12,10)$$

The determination of the coefficients on the left-hand side of the linear systems requires more elaboration than the quantities mentioned above. The definition of the coefficient $\overset{s}{\tau}{}_{\ell}^{n}$ is given by the formula (2,10,6), viz.

$$\overset{s}{\tau}{}_{\ell}^{n} = p_{2n}^{2n} \int_{-1}^{+1} \overset{s}{w}_{2n} P_{2\ell}(y)\,dy + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \int_{-1}^{+1} \overset{s}{w}_{2m+1} P_{2\ell}(y)\,dy \; .$$

After substitution of the expressions for the coefficients p_n^m and the downwashes $\overset{s}{w}_n$ and by application of the integral relations (2,12,1) and (2,12,2) we can write for this coefficient $\overset{s}{\tau}{}_{\ell}^{n}$

$$\overset{s}{\tau}{}_{\ell}^{n} = \sum_{m=0}^{\infty} \frac{(-1)^n (4m+2)(4m+3)}{\pi^2 \left\{(2m+1)^2 - 4n^2\right\} (2\ell-2m-1)(2\ell+2m+2)} \quad \text{for } n \neq \ell$$

and (2,12,11)

$$\overset{s}{\tau}{}_{\ell}^{\ell} = \varepsilon_\ell \frac{(-1)^{\ell+1}}{4} + \sum_{m=0}^{\infty} \frac{(-1)^\ell (4m+2)(4m+3)}{\pi^2 \left\{(2m+1)^2 - 4\ell^2\right\} (2\ell-2m-1)(2\ell+2m+2)}$$

The further elaboration of the coefficients is simplified by splitting up the expressions under the summation signs into four terms. We write

$$\overset{s}{\tau}{}_{\ell}^{n} = \frac{(-1)^{n+1}}{\pi^2} \sum_{m=0}^{\infty} \left[\frac{1}{(2m-2n+1)(2m-2\ell+1)} + \frac{1}{(2m+2n+1)(2m-2\ell+1)} + \right.$$

$$\left. + \frac{1}{(2m-2n+1)(2m+2\ell+2)} + \frac{1}{(2m+2n+1)(2m+2\ell+2)} \right] \text{for } n \neq \ell$$

and

$$\overset{s}{\tau}{}_{\ell}^{\ell} = \varepsilon_{\ell} \frac{(-1)^{\ell+1}}{4} + \frac{(-1)^{\ell+1}}{\pi^2} \sum_{m=0}^{\infty} \left[\frac{1}{(2m-2\ell+1)^2} + \frac{1}{(2m+2\ell+1)(2m-2\ell+1)} + \right.$$

$$\left. + \frac{1}{(2m-2\ell+1)(2m+2\ell+2)} + \frac{1}{(2m+2\ell+1)(2m+2\ell+2)} \right] . \qquad (2,12,12)$$

The infinite sums in (2,12,12) can be written in a convenient form by aid of the so-called ψ -function, which is usually derived from the gamma function. Here we introduce the gamma function $\Gamma(z)$ by Weierstrasz'definition

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} \right] \qquad (2,12,13)$$

where γ denotes Euler's constant.
The function $\psi(z)$ is the logarithmic derivative of the gamma function

$$\psi(z) = \frac{d \log \Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} . \qquad (2,12,14)$$

Insertion of the expression (2,12,13) into the definition of the ψ -function yields the formula

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)} . \qquad (2,12,15)$$

We see at once that the ψ -function is meromorphic with simple poles at $z = 0, -1, -2, \ldots$. If a and b differ from $0, -1, -2, \ldots$, we have according to (2,12,15)

$$\psi(b) - \psi(a) = \frac{1}{a} - \frac{1}{b} + \sum_{n=1}^{\infty} \left\{ \frac{b}{n(b+n)} - \frac{a}{n(a+n)} \right\} =$$

$$= \frac{b-a}{ab} + \sum_{n=1}^{\infty} \frac{b-a}{(n+a)(n+b)} = (b-a) \sum_{n=0}^{\infty} \frac{1}{(n+a)(n+b)} .$$

Thus we have proved the relation

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{\psi(b)-\psi(a)}{b-a} \text{ for } a, b \neq 0, -1, -2, \ldots \text{ and } a \neq b \qquad (2,12,16)$$

Differentiation of the relation (2,12,15) yields

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2} \; . \tag{2,12,17}$$

The formulas (2,12,16) and (2,12,17) enable us to express the coefficients $\overset{s}{\tau}{}_{\ell}^{n}$ in terms of the ψ -function. We find

$$\overset{s}{\tau}{}_{\ell}^{n} = \frac{(-1)^{n+1}}{4\pi^2}\left[\frac{\psi(-n+1/2)-\psi(-\ell+1/2)}{\ell-n} + \frac{\psi(n+1/2)-\psi(-\ell+1/2)}{\ell+n} + \right.$$

$$\left. - \frac{\psi(-n+1/2)-\psi(\ell+1)}{\ell+n+1/2} + \frac{\psi(n+1/2)-\psi(\ell+1)}{n-\ell-1/2}\right] \qquad \text{for } n \neq \ell$$

and

$$\overset{s}{\tau}{}_{\ell}^{\ell} = \frac{(-1)^{\ell+1}}{4} + \frac{(-1)^{\ell+1}}{4\pi^2}\left[\psi'(-\ell+1/2) + \frac{\psi(\ell+1/2)-\psi(-\ell+1/2)}{2\ell} + \right.$$

$$\left. - \frac{\psi(-\ell+1/2)-\psi(\ell+1)}{2\ell+1/2} - \frac{\psi(\ell+1/2)-\psi(\ell+1)}{1/2}\right] \qquad \text{for } \ell \neq 0 \tag{2,12,18}$$

The coefficient $\overset{s}{\tau}{}_{0}^{0}$ will be treated separately.

The right-hand sides of (2,12,18) can be further reduced by application of some relations known from the theory of the gamma function. For that purpose we start from the formula (ref.2)

$$\Gamma(x)\;\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad x \neq 0, \pm 1, \pm 2, \ldots\ldots \tag{2,12,19}$$

Differentiation with respect to x yields

$$\Gamma'(x)\,\Gamma(1-x) - \Gamma(x)\,\Gamma'(1-x) = -\frac{\pi^2 \cos \pi x}{\sin^2 \pi x}$$

or

$$\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(1-x)}{\Gamma(1-x)} = -\frac{\pi^2 \cos \pi x}{\sin^2 \pi x}\;\frac{\sin \pi x}{\pi} = \pi \operatorname{cotg} \pi x .$$

Thus

$$\psi(x) - \psi(1-x) = -\pi \operatorname{cotg} \pi x \quad x \neq 0, \pm 1, \pm 2, \ldots\ldots . \tag{2,12,20}$$

Putting $x = 1/2 + n$ where n is an integer, we get

$$\psi(1/2+n) = \psi(1/2-n) \; . \tag{2,12,21}$$

Application of this formula transforms the expressions (2,12,18) into

$$\overset{s}{\tau}{}_{\ell}^{n} = \frac{(-1)^{n+1}}{4\pi^2}\left[\frac{2\ell}{\ell^2-n^2}\left\{\psi(n+1/2)-\psi(\ell+1/2)\right\} + \right.$$

$$\left. + \frac{8\ell+4}{(2n+2\ell+1)(2n-2\ell-1)}\left\{\psi(n+1/2)-\psi(\ell+1)\right\}\right] \qquad \text{for } n \neq \ell$$

and

$$\overset{s}{\tau}{}_{\ell}^{\ell} = \frac{(-1)^{\ell+1}}{4} + \frac{(-1)^{\ell+1}}{4\pi^2}\left[\psi'(-\ell+1/2) + \frac{8\ell+4}{4\ell+1}\left\{\psi(\ell+1)-\psi(\ell+1/2)\right\}\right] \qquad \text{for } \ell \neq 0 \; . \tag{2,12,22}$$

We consider now the special case $l=0$. The coefficient $\overset{s}{\tau}{}^{o}_{0}$ becomes according to (2,12,12).

$$\overset{s}{\tau}{}^{o}_{0} = -1/2 - \frac{1}{\pi^2} \sum_{m=0}^{\infty} \left[\frac{2}{(2m+1)^2} + \frac{2}{(2m+1)(2m+2)} \right] . \qquad (2,12,23)$$

This can be written as

$$\overset{s}{\tau}{}^{o}_{0} = -1/2 - \frac{1}{2\pi^2} \left[\psi'(1/2) + 2\left\{ \psi(1) - \psi(1/2) \right\} \right] . \qquad (2,12,24)$$

Substituting $x=1$ *and* $x=1/2$ *into* (2,12,15) *we get*

$$\therefore \psi(1) = -\gamma - 1 + \sum_{n=1}^{\infty} \frac{1}{n(1+n)} = -\gamma - 1 + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{n+1} \right\} = -\gamma$$

and

$$\psi(1/2) = -\gamma - 2 + 1/2 \sum_{n=1}^{\infty} \frac{1}{n(1/2+n)} = \psi(1) - 2 + \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} .$$

Thus we have

$$\psi(1) - \psi(1/2) = 2 - \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} = 2 - \left\{ \frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots\ldots \right\} =$$

$$= 2\left\{ 1 - \frac{1}{2.3} - \frac{1}{4.5} - \frac{1}{6.7} - \ldots\ldots \right\} = 2\left\{ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \ldots\ldots \right\} = 2 \log 2$$

Furthermore

$$\psi'(1/2) = \sum_{m=0}^{\infty} \frac{1}{(m+1/2)^2} = 4 \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = 4\,\frac{\pi^2}{8} = \frac{\pi^2}{2} .$$

In this way the coefficient $\overset{s}{\tau}{}^{o}_{0}$ becomes

$$\overset{s}{\tau}{}^{o}_{0} = -\frac{1}{2} - \frac{1}{2\pi^2}\left\{ \frac{\pi^2}{2} + 4 \log 2 \right\} = -\frac{3}{4} - \frac{2 \log 2}{\pi^2} . \qquad (2,12,25)$$

In order to determine numerical values for the other coefficients $\overset{s}{\tau}{}^{n}_{l}$ we derive the following recurrence relations:

$$\psi(x+1) = \frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{\frac{d}{dx}\left\{ x\Gamma(x) \right\}}{x\Gamma(x)} = \frac{\Gamma(x) + x\Gamma'(x)}{x\Gamma(x)} = \frac{1}{x} + \frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{x} + \psi(x) \qquad (2,12,26)$$

and by differentiation

$$\psi'(x+1) = \psi'(x) - \frac{1}{x^2} . \qquad (2,12,27)$$

These two formulas (2,12,26) and (2,12,27) enable us to evaluate all coefficients $\overset{s}{\tau}{}^{n}_{l}$. Numerical values are given in table I.

In a quite similar way expressions can be derived for the coefficients $\overset{a}{\tau}{}^{n}_{k}$ which correspond to the anti-symmetric problems. This coefficient $\overset{a}{\tau}{}^{n}_{k}$ has been defined by the formula (2,10,24):

$$\overset{a}{\tau}{}^{n}_{k} = q^{2n+1}_{2n+1} \int_{-1}^{+1} \overset{a}{w}_{2n+1} P_{2k+1}(y)\,dy + \sum_{m=1}^{\infty} q^{2m}_{2n+1} \int_{-1}^{+1} \overset{a}{w}_{2m} P_{2k+1}(y)\,dy .$$

If $k \neq n$, the expression in the right-hand side of this formula can be reduced as follows

$$\overset{a_n}{\tau_k} = \sum_{m=1}^{\infty} q_{2n+1}^{2m} \int_{-1}^{+1} \overset{a}{w}_{2m} P_{2k+1}(y)\,dy =$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{n+1}\; 4m\,(4m+1)}{\pi^2 \left[4m^2-(2n+1)^2\right](2m-2k-1)\,(2m+2k+2)} =$$

$$= \frac{(-1)^{n+1}}{\pi^2} \sum_{m=0}^{\infty} \left[\frac{1}{(2m-2n-1)(2m-2k-1)} + \frac{1}{(2m-2n-1)(2m+2k+2)} + \right.$$

$$\left. + \frac{1}{(2m+2n+1)\,(2m-2k-1)} + \frac{1}{(2m+2n+1)\,(2m+2k+2)}\right] =$$

$$= \frac{(-1)^{n+1}}{4\pi^2} \left[\frac{\psi(-n-1/2)-\psi(-k-1/2)}{k-n} - \frac{\psi(-n-1/2)-\psi(k+1)}{n+k+\frac{3}{2}} + \right.$$

$$\left. + \frac{\psi(n+1/2)-\psi(-k-1/2)}{n+k+1} + \frac{\psi(n+1/2)-\psi(k+1)}{n-k-1/2}\right] =$$

$$= \frac{(-1)^{n+1}}{4\pi^2} \left[\frac{\psi\left(n+\frac{3}{2}\right)-\psi\left(k+\frac{3}{2}\right)}{k-n} - \frac{\psi\left(n+\frac{3}{2}\right)-\psi(k+1)}{n+k+\frac{3}{2}} + \right.$$

$$\left. + \frac{\psi(n+1/2)-\psi\left(k+\frac{3}{2}\right)}{n+k+1} + \frac{\psi(n+1/2)-\psi(k+1)}{n-k-1/2}\right] =$$

$$= \frac{(-1)^{n+1}}{4\pi^2} \left[\frac{4n+3}{(k-n)(2n+2k+3)}\,\psi\left(n+\frac{3}{2}\right) + \frac{4n+1}{(n+k+1)\,(2n-2k-1)}\,\psi(n+1/2) + \right.$$

$$\left. - \frac{2k+1}{(k-n)\,(n+k+1)}\,\psi\left(k+\frac{3}{2}\right) - \frac{8(k+1)}{(2n+2k+3)(2n-2k-1)}\,\psi(k+1)\right] \qquad (2,12,28)$$

The case $k=n$ results into

$$\overset{a_k}{\tau_k} = q_{2k+1}^{2k+1} \int_{-1}^{+1} \overset{a}{w}_{2k+1} P_{2k+1}(y)\,dy + \sum_{m=1}^{\infty} q_{2k+1}^{2m} \int_{-1}^{+1} \overset{a}{w}_{2m} P_{2k+1}(y)\,dy =$$

$$= \frac{(-1)^{k+1}}{4} + \frac{(-1)^{k+1}}{\pi^2} \sum_{m=1}^{\infty} \frac{4m\,(4m+1)}{\left[4m^2-(2k+1)^2\right]\,(2m-2k-1)(2m+2k+2)} =$$

$$= \frac{(-1)^{k+1}}{4} + \frac{(-1)^{k+1}}{\pi^2} \sum_{m=0}^{\infty} \left[\frac{1}{(2m-2k-1)^2} + \frac{1}{(2m-2k-1)(2m+2k+2)} + \right.$$

$$\left. + \frac{1}{(2m+2k+1)(2m-2k-1)} + \frac{1}{(2m+2k+1)(2m+2k+2)} \right] =$$

$$= \frac{(-1)^{k+1}}{4} + \frac{(-1)^{k+1}}{4\pi^2} \left[\psi'(-k-\tfrac{1}{2}) - \frac{\psi(-k-\frac{1}{2}) - \psi(k+1)}{2k+\frac{3}{2}} + \frac{\psi(k+\frac{1}{2}) - \psi(-k-\frac{1}{2})}{2k+1} - \frac{\psi(k+\frac{1}{2}) - \psi(k+1)}{\frac{1}{2}} \right] =$$

$$= \frac{(-1)^{k+1}}{4} + \frac{(-1)^{k+1}}{4\pi^2} \left[\psi'(-k-\tfrac{1}{2}) + \frac{2}{4k+3} \left\{ \psi(k+1) - \psi(k+\tfrac{3}{2}) \right\} + \right.$$

$$\left. - \frac{1}{2k+1} \left\{ \psi(k+\tfrac{3}{2}) - \psi(k+\tfrac{1}{2}) \right\} + 2 \left\{ \psi(k+1) - \psi(k+\tfrac{1}{2}) \right\} \right] =$$

$$= \frac{(-1)^{k+1}}{4} + \frac{(-1)^{k+1}}{4\pi^2} \left[\psi'(-k-\tfrac{1}{2}) + \frac{8(k+1)}{4k+3} \left\{ \psi(k+1) - \psi(k+\tfrac{1}{2}) \right\} - \frac{16k+10}{(4k+3)(2k+1)^2} \right] . \tag{2,12,29}$$

Putting $k=0$ into (2,12,29), we get

$$\overset{a}{\tau}{}_0^0 = -\frac{1}{4} - \frac{1}{4\pi^2} \left[\psi'(-\tfrac{1}{2}) + \frac{8}{3} \quad \psi(1) - \psi(\tfrac{1}{2}) \quad - \frac{10}{3} \right]$$

$$= -\frac{1}{4} - \frac{1}{4\pi^2} \left[\frac{\pi^2}{2} + 4 + \frac{16}{3} \log 2 - \frac{10}{3} \right] = -\frac{3}{8} - \frac{1+8\log 2}{6\pi^2} . \tag{2,12,30}$$

Applying the recurrence relations (2,12,26) and (2,12,27) it is possible to determine all coefficients $\overset{a}{\tau}{}_k^n$. Numerical values for these coefficients are given in table II.

13 Derivation of formulas for pressure distribution, forces, moments and spanwise lift distribution.

If the complete acceleration potential, corresponding to the physical boundary value problem is denoted by $\tilde{\psi}$, the pressure p can be found by aid of the formula (1,3,5), which reads

$$\tilde{\psi} = \frac{p_\infty - p}{\rho}$$

where p_∞ represents the pressure in the undisturbed flow and ρ the density of the fluid. Because of the odd character with respect to the variable z of the pressure p , the pressure jump Π over the wing surface can be expressed by the formula

$$\Pi = p_- - p_+ = \rho \left[\tilde{\psi}_+ - \tilde{\psi}_-\right] = 2\rho\tilde{\psi} \tag{2,13,1}$$

The complete acceleration potential has been given in a general form by the formula (2,9,2), which reads

$$\tilde{\psi} = \psi\ (\eta,\mu,\vartheta) + U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k\,(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1$$

The pressure distribution Π over the wing surface becomes thus

$$\Pi\ (0,\mu,\vartheta) = 2\rho\psi\,(0,\mu,\vartheta) + 2\rho U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k\,(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}} d\vartheta_1 \,. \tag{2,13,2}$$

The substitution of the Fourier series (2,10,4) and (2,10,5), and the substitution of the analytical expression (2,8,47) for the quantity $\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0}$ into the formula (2,13,2), yield for the pressure distribution over the wing surface, the expression

$$\Pi\,(0,\mu,\vartheta) = 2\rho\psi\,(0,\mu,\vartheta) - \frac{U\rho\mu}{\pi^2} \sum_{n=0}^{j} a_n \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos\vartheta_1\ \cos 2n\vartheta_1}{2-\mu^2-2\sqrt{1-\mu^2}\ \cos\,(\vartheta-\vartheta_1)}\, d\vartheta_1 \tag{2,13,3}$$

for symmetric problems and

$$\Pi\,(0,\mu,\vartheta) = 2\rho\psi\,(0,\mu,\vartheta) - \frac{U\rho\mu}{\pi^2} \sum_{n=0}^{j} b_n \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos\vartheta_1\ \sin\,(2n+1)\,\vartheta_1}{2-\mu^2-2\sqrt{1-\mu^2}\ \cos\,(\vartheta-\vartheta_1)}\, d\vartheta_1 \tag{2,13,4}$$

for anti-symmetric problems.

In the two relations given above, the Fourier series have been truncated after the $(j+1)$th term. Consequently the expressions (2,13,3) and (2,13,4) give approximations for the pressure distribution. For every arbitrary downwash distribution prescribed at the wing surface, the pressure distribution can now be evaluated by aid of these formulas (2,13,3) or (2,13,4). The regular acceleration potential $\psi(\eta,\mu,\vartheta)$ is found by application of formula (2,5,2):

$$\psi\left(\eta,\mu,\vartheta\right)=\int_{0}^{2\pi}\int_{-1}^{+1} a\left(\mu_1,\vartheta_1\right) G\left(\eta,\mu,\vartheta;\mu_1,\vartheta_1\right)\mu_1\, d\mu_1\, d\vartheta_1$$

and the coefficients a_n and b_n must be solved from the systems of linear equations which have been derived in section 10 of this chapter.

The lift-force and the moments acting on the aerofoil are obtained by suitable integrations. The lift L is given by the formula

$$L=\iint_{S}\Pi\,(x,y)\,dx\,dy=\int_{0}^{2\pi}\int_{0}^{1}\Pi\left(0,\mu,\vartheta\right)\mu d\mu d\vartheta\ . \qquad (2,13,5)$$

S denotes the area of the projected wing surface onto the xy -plane. Inserting the expression (2,13,2) for the pressure distribution into (2,13,5), we get

$$L=2\rho\int_{0}^{2\pi}\int_{0}^{1}\psi\left(0,\mu,\vartheta\right)\mu d\mu d\vartheta+2\rho U\int_{0}^{2\pi}\int_{0}^{1}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k\left(\vartheta_1\right)\left[\frac{\partial G}{\partial\mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}}\mu d\vartheta_1\, d\mu d\vartheta\ . \qquad (2,13,6)$$

Interchanging the order of integration in the second term on the right-hand side and applying the integral relations (2,8,43), the lift L can be written in the form

$$L=2\rho\int_{0}^{2\pi}\int_{0}^{1}\psi\left(0,\mu,\vartheta\right)\mu d\mu d\vartheta+2\rho U\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k\left(\vartheta_1\right)d\vartheta_1\int_{0}^{1}\mu d\mu\int_{0}^{2\pi}\left[\frac{\partial G}{\partial\mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}} d\vartheta=$$

$$=2\rho\int_{0}^{2\pi}\int_{0}^{1}\psi\left(0,\mu,\vartheta\right)\mu d\mu d\vartheta+2\rho U\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k\left(\vartheta_1\right)d\vartheta_1\int_{0}^{1}-\frac{1}{\pi}\,d\mu$$

$$=2\rho\int_{0}^{2\pi}\int_{0}^{1}\psi\left(0,\mu,\vartheta\right)\mu d\mu d\vartheta-\frac{2\rho U}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k\left(\vartheta_1\right)d\vartheta_1\ . \qquad (2,13,7)$$

Taking into account $(j+1)$ terms of the Fourier series of the weight function $k^{x}(\vartheta)$, the lift L is approximated in the symmetric case by the formula

$$L\simeq 2\rho\int_{0}^{2\pi}\int_{0}^{1}\psi\left(0,\mu,\vartheta\right)\mu d\mu d\vartheta-\frac{2\rho U}{\pi}\sum_{n=0}^{j}a_n\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos\ 2n\vartheta_1\cos\vartheta_1\,d\vartheta_1$$

$$=2\rho\int_{0}^{2\pi}\int_{0}^{1}\psi\left(0,\mu,\vartheta\right)\mu d\mu d\vartheta-\frac{2\rho U}{\pi}\sum_{n=0}^{j}a_n\ P_{2n}^{1}\ . \qquad (2,13,8)$$

The moment about the y -axis, denoted by M_y , is obtained in a similar way. In fact we have

$$M_y = \iint_S \Pi(x,y)\, x\, dx\, dy = \int_0^{2\pi}\int_0^1 \Pi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\cos\vartheta\, d\mu\, d\vartheta =$$

$$= 2\rho \int_0^{2\pi}\int_0^1 \psi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\cos\vartheta\, d\mu\, d\vartheta +$$

$$+ 2\rho U \int_0^{2\pi}\int_0^1 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k(\vartheta_1)\left[\frac{\partial G}{\partial\mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}} \mu\sqrt{1-\mu^2}\,\cos\vartheta\, d\vartheta_1\, d\mu\, d\vartheta$$

$$= 2\rho \int_0^{2\pi}\int_0^1 \psi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\cos\vartheta\, d\mu\, d\vartheta +$$

$$+ 2\rho U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k(\vartheta_1)\, d\vartheta_1 \int_0^1 \mu\sqrt{1-\mu^2}\, d\mu \int_0^{2\pi} \left[\frac{\partial G}{\partial\mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}} \cos\vartheta\, d\vartheta$$

$$= 2\rho \int_0^{2\pi}\int_0^1 \psi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\cos\vartheta\, d\mu\, d\vartheta - \frac{4\rho U}{3\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k(\vartheta_1)\cos\vartheta_1\, d\vartheta_1 \,. \tag{2,13,9}$$

This moment can thus be approximated by the expression

$$M_y \approx 2\rho \int_0^{2\pi}\int_0^1 \psi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\cos\vartheta\, d\mu\, d\vartheta +$$

$$- \frac{4\rho U}{3\pi}\sum_{n=0}^{j} a_n \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos 2n\vartheta_1 \cos^2\vartheta_1\, d\vartheta_1$$

$$= 2\rho \int_0^{2\pi}\int_0^1 \psi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\cos\vartheta\, d\mu\, d\vartheta - \frac{2\rho U}{3\pi}\sum_{n=0}^{j} a_n \left[p_{2n}^0 + p_{2n}^2\right]. \tag{2,13,10}$$

In the anti-symmetric case the lift L and the moment M_y vanish; the main quantity in this case is the moment about the x -axis. This moment M_x, mostly called the roll-moment, is defined by the formula

$$M_x = \iint_S \Pi(x,y)\, y\, dx\, dy = \int_0^{2\pi}\int_0^1 \Pi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\sin\vartheta\, d\mu\, d\vartheta. \tag{2,13,11}$$

Inserting the formula (2,13,2) for the pressure distribution into (2,13,11), we obtain

$$M_x = 2\rho \int_0^{2\pi}\int_0^1 \psi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\sin\vartheta\, d\mu\, d\vartheta +$$

$$+ 2\rho U \int_0^{2\pi}\int_0^1 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k(\vartheta_1)\left[\frac{\partial G}{\partial \mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}} \mu\sqrt{1-\mu^2}\,\sin\vartheta\, d\vartheta_1\, d\mu\, d\vartheta$$

$$= 2\rho \int_0^{2\pi}\int_0^1 \psi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\sin\vartheta\, d\mu\, d\vartheta +$$

$$+ 2\rho U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k(\vartheta_1)\, d\vartheta_1 \int_0^1 \mu\sqrt{1-\mu^2}\, d\mu \int_0^{2\pi} \left[\frac{\partial G}{\partial \mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}} \sin\vartheta\, d\vartheta$$

$$= 2\rho \int_0^{2\pi}\int_0^1 \psi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\sin\vartheta\, d\mu\, d\vartheta - \frac{4\rho U}{3\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k(\vartheta_1)\sin\vartheta_1\, d\vartheta_1 \,. \tag{2,13,12}$$

Truncating the series for the weight function $k^x(\vartheta_1)$ after the $(j+1)$ th term, the moment M_x is approximated by

$$M_x = 2\rho \int_0^{2\pi}\int_0^1 \psi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\sin\vartheta\, d\mu\, d\vartheta +$$

$$- \frac{2\rho U}{3\pi}\sum_{n=0}^{j} b_n \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(2n+1)\vartheta_1\, \sin 2\vartheta_1\, d\vartheta_1 = \tag{2,13,13}$$

$$= 2\rho \int_0^{2\pi}\int_0^1 \psi(0,\mu,\vartheta)\,\mu\sqrt{1-\mu^2}\,\sin\vartheta\, d\mu\, d\vartheta - \frac{2\rho U}{3\pi}\sum_{n=0}^{j} b_n\, q^2_{2n+1} \,.$$

In the symmetrical case the moment M_x is equal to zero. As is customary in aerodynamics the forces and moments are indicated by their dimensionless lift- and moment coefficients. Therefore we introduce the following coefficients

$$c_a = \frac{L}{\pi\rho U^2}$$

$$c_{m_y} = \frac{M_y}{\pi\rho U^2} \tag{2,13,14}$$

$$c_{m_x} = \frac{M_x}{\pi\rho U^2}$$

Besides the forces and moments, the so-called spanwise lift distribution plays an important role in the theory of lifting surfaces. This spanwise lift distribution $C(y)$ is defined by the formula

$$C(y) = \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \Pi\ (x,y)\,dx \quad . \tag{2,13,15}$$

The quantity $C(y)$ is connected with the circulation $\Gamma(y)$ by means of the formula

$$C(y) = \rho U \Gamma(y) \quad .$$

Substitution of the expression for the pressure Π into this formula (2,13,15) yields

$$C(y) = 2\rho \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \psi\ (0,\mu,\vartheta)\,dx + 2\rho U \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} dx \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k\,(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}} d\vartheta_1 \ . \tag{2,13,16}$$

The second integral on the right of (2,13,16) can be further reduced by interchanging the order of integration. We find

$$C(y) = 2\rho \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \psi\ (0,\mu,\vartheta)\,dx + 2\rho U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k\,(\vartheta_1)\,d\vartheta_1 \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \left[\frac{\partial G}{\partial \mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}} dx \ . \tag{2,13,17}$$

Replacing the function $\left[\frac{\partial G}{\partial \mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}}$ by its Fourier series (2,8,45) and applying the integral relations (2,8,43) and (2,8,44), we get after termswise integration

$$C(y) = 2\rho \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \psi\ (0,\mu,\vartheta)\,dx + 2\rho U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k\,(\vartheta_1) \left[\sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{\pi \varepsilon_m} P_{2m}(y) \cos\ 2m\vartheta_1 + \right.$$

$$\left. + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{\pi} P_{2m+1}(y) \sin\ (2m+1)\,\vartheta_1 \right] d\vartheta_1 \ . \tag{2,13,18}$$

The introduction of the new variable $\alpha = \vartheta_1 - \frac{\pi}{2}$, transforms this formula (2,13,18) into

$$C(y) = 2\rho \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \psi\ (0,\mu,\vartheta)\,dx - \frac{2\rho U}{\pi} \int_0^{\pi} k\ \left(\alpha + \frac{\pi}{2}\right) \sum_{m=0}^{\infty} \frac{1}{\varepsilon_m} P_m(y) \cos m\alpha .\, d\alpha \, . \tag{2,13,19}$$

It is now possible to evaluate the sum of the series in the right-hand side of this formula by aid of the relation between the Legendre polynomials and their generating function, viz.

$$\frac{1}{\sqrt{1 - 2k\cos\theta + k^2}} = \sum_{n=0}^{\infty} k^n P_n\ (\cos\theta) \ . \tag{2,13,20}$$

Putting $k = e^{i\alpha}$, we get

$$\sum_{n=0}^{\infty} e^{in\alpha} P_n(\cos\theta) = \begin{cases} \dfrac{1}{e^{\frac{1}{2}i\alpha}\sqrt{2(\cos\alpha-\cos\theta)}} & \text{if } \alpha<\theta \\[2ex] \dfrac{1}{e^{\frac{1}{2}i(\alpha+\pi)}\sqrt{2(\cos\theta-\cos\alpha)}} & \text{if } \alpha>\theta \end{cases} \tag{2,13,21}$$

On equating the real parts on both sides of the equation (2,13,21), we obtain the relation

$$\sum_{n=0}^{\infty} P_n(\cos\theta)\cos n\alpha \begin{cases} \dfrac{\cos\frac{1}{2}\alpha}{\sqrt{2(\cos\alpha-\cos\theta)}} & \text{if } \alpha<\theta \\[2ex] \dfrac{\sin\frac{1}{2}\alpha}{\sqrt{2(\cos\theta-\cos\alpha)}} & \text{if } \alpha>\theta \end{cases} \tag{2,13,22}$$

Using this formula (2,13,22), we can transform the expression for $C(y)$ into

$$C(y) = 2\rho\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \psi(0,\mu,\vartheta)\,dx - \frac{2\rho U}{\pi}\int_0^\theta k\left(\alpha+\frac{\pi}{2}\right)\frac{\cos\frac{\alpha}{2}}{\sqrt{2(\cos\alpha-\cos\theta)}}\,d\alpha +$$

$$-\frac{2\rho U}{\pi}\int_\theta^\pi k\left(\alpha+\frac{\pi}{2}\right)\frac{\sin\frac{\alpha}{2}}{\sqrt{2(\cos\theta-\cos\alpha)}}\,d\alpha + \frac{\rho U}{\pi}\int_0^\pi k\left(\alpha+\frac{\pi}{2}\right)d\alpha\ , \tag{2,13,23}$$

where the parameter θ is defined by $y=\cos\theta$. At the tips of the wing the variable y equals $+1$ or -1, and thus $\theta=0$ or $\theta=\pi$. From the formula (2,13,23) it follows at once, that the spanwise lift distribution $C(y)$ vanishes at the tips of the wing. Inserting into (2,13,23) the truncated series for the weight-function $k(\vartheta)$, we obtain for the spanwise lift distribution in the symmetric case the approximation

$$C(y) \approx 2\rho\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \psi(0,\mu,\vartheta)\,dx + \frac{2\rho U}{\pi}\sum_{n=0}^{j}(-1)^n a_n\int_0^\theta \frac{\sin\alpha\cos 2n\alpha\cos\frac{\alpha}{2}}{\sqrt{2(\cos\alpha-\cos\theta)}}\,d\alpha +$$

$$+\frac{2\rho U}{\pi}\sum_{n=0}^{j}(-1)^n a_n\int_\theta^\pi \frac{\sin\alpha\cos 2n\alpha\sin\frac{\alpha}{2}}{\sqrt{2(\cos\theta-\cos\alpha)}}\,d\alpha + \frac{\rho U}{\pi}\sum_{n=0}^{j} a_n P_{2n}^1 \tag{2,13,24}$$

and in the anti-symmetric case

$$C(y) \approx 2\rho\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \psi(0,\mu,\vartheta)\,dx + \frac{2\rho U}{\pi}\sum_{n=0}^{j}(-1)^n b_n\int_0^\theta \frac{\sin\alpha\cos(2n+1)\alpha\cos\frac{\alpha}{2}}{\sqrt{2(\cos\alpha-\cos\theta)}}\,d\alpha +$$

$$+\frac{2\rho U}{\pi}\sum_{n=0}^{j}(-1)^n b_n\int_\theta^\pi \frac{\sin\alpha\cos(2n+1)\alpha\sin\frac{\alpha}{2}}{\sqrt{2(\cos\theta-\cos\alpha)}}\,d\alpha\ . \tag{2,13,25}$$

These two relations enable us to determine numerical values of the spanwise lift distribution.

Next we consider the x -component D of the aerodynamic force which acts on the wing according to potential theory. In order to obtain the total drag on a wing in a real fluid we have to add D to the profile drag which is due to the effect of viscosity. By contra-distinction, D is called the induced drag, since it is associated with, or induced by the region of disturbed motion of the fluid behind the wing. The expression for the induced resistance is obtained by means of the momentum theorem. A surface enclosing the wing is denoted by S ; the momentum law applied to the wing then leads to the expression

$$D = \iint_S p \cos(n,x)\, d\sigma + \iint_S \rho \left\{ U \cos(n,x) + \frac{\partial\varphi}{\partial n} \right\} \left\{ U + \frac{\partial\varphi}{\partial x} \right\} d\sigma \qquad (2,13,26)$$

(n,x) represents the angle between the outer normal to the surface S and the axis of x , whereas φ denotes the disturbance velocity potential. Hence $U + \frac{\partial\varphi}{\partial x}$ is the x -component of the fluid velocity and $U\cos(n,x) + \frac{\partial\varphi}{\partial n}$ the component of the fluid velocity normal to the surface S . The expression (2,13,26) can be written in another form by application of Bernoulli's equation:

$$p + \frac{1}{2}\rho \left[\left(U + \frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 \right] = p_\infty + \frac{1}{2}\rho U^2 \quad . \qquad (2,13,27)$$

Equation (2,13,26) becomes now

$$D = \iint_S \left\{ p_\infty - \rho U \frac{\partial\varphi}{\partial x} - \frac{1}{2}\rho \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 \right] \right\} \cos(n,x)\, d\sigma +$$

$$+ \rho \iint_S \left\{ U \cos(n,x) + \frac{\partial\varphi}{\partial n} \right\} \left\{ U + \frac{\partial\varphi}{\partial x} \right\} d\sigma =$$

$$= p_\infty \iint_S \cos(n,x)\, d\sigma - \frac{1}{2}\rho \iint_S \left\{ \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 \right\} \cos(n,x)\, d\sigma +$$

$$+ \rho U^2 \iint_S \cos(n,x)\, d\sigma + \rho U \iint_S \frac{\partial\varphi}{\partial n}\, d\sigma + \rho \iint_S \frac{\partial\varphi}{\partial n} \frac{\partial\varphi}{\partial x}\, d\sigma \, . \qquad (2,13,28)$$

In order to simplify the relation (2,13,28) use can be made of the well-known theorems of Gauss and Green, which read respectively

$$\iiint_D \operatorname{div} \vec{v}\, d\tau = \iint_S v_n\, d\sigma \qquad (2,13,29)$$

$$\iiint_D \left\{ \psi_1 \Delta \psi_2 - \psi_2 \Delta \psi_1 \right\} d\tau = \iint_S \left\{ \psi_1 \frac{\partial\psi_2}{\partial n} - \psi_2 \frac{\partial\psi_1}{\partial n} \right\} d\sigma \, . \qquad (2,13,30)$$

D denotes a region in space which is closed by a surface S . In particular we consider the case $\vec{v} = (1,0,0)$. Then equation (2,13,29) gives

$$\iint_S \cos(n,x)\, d\sigma = 0 \, . \qquad (2,13,31)$$

With respect to the identity (2,13,30) we assume that ψ_1 is a harmonic function and ψ_2 is equal to one. Then we obtain the relation

$$\iint_S \frac{\partial\psi_1}{\partial n}\, d\sigma = 0 \, . \qquad (2,13,32)$$

Application of the formulas (2,13,31) and (2,13,32) transforms the expression (2,13,28) into

$$D=-\frac{1}{2}\rho\iint\limits_S\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right\}\cos(n,x)\,d\sigma+\rho\iint\limits_S\frac{\partial\varphi}{\partial n}\frac{\partial\varphi}{\partial x}\,d\sigma. \qquad (2,13,33)$$

The surface S is now chosen in a particular way. In fact we take a hemisphere of large radius R with center at the point $x=a$ of the x -axis enclosing the wing and the circle cut out by this hemisphere on the plane $x=a$. If the radius R increases to infinity the corresponding parts of the integrals in (2,13,33) tend to zero. On the surface $x=a$ holds

$$\cos(n,x)=-1 \quad \text{and} \quad \frac{\partial\varphi}{\partial n}=-\frac{\partial\varphi}{\partial x} \quad . \qquad (2,13,34)$$

Thus

$$D=-\frac{1}{2}\rho\iint\limits_{x=a}\left\{\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2-\left(\frac{\partial\varphi}{\partial x}\right)^2\right\}dy\,dz\ . \qquad (2,13,35)$$

For $a\rightarrow\infty$ the following equation is obtained:

$$D=-\frac{1}{2}\rho\iint\limits_{x=\infty}\left\{\left(\frac{\partial\varphi_\infty}{\partial y}\right)^2+\left(\frac{\partial\varphi_\infty}{\partial z}\right)^2\right\}dy\,dz \qquad (2,13,36)$$

where $\varphi_\infty=\lim\limits_{x\to\infty}\varphi(x,y,z)$.

In fact φ_∞ denotes the velocity potential in the so-called Trefftz-plane. Inserting in the theorem of Gauss for two dimensions $\vec{v}=\psi\,\mathrm{grad}\,\psi$, we obtain the relation

$$\iint\limits_S\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+\psi\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right]\right\}d\tau=\int\limits_L\psi\frac{\partial\psi}{\partial n}\,d\sigma\ .$$

If ψ is harmonic this formula degenerates to

$$\iint\limits_S\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right\}d\tau=\int\limits_L\psi\frac{\partial\psi}{\partial n}\,d\sigma \quad . \qquad (2,13,37)$$

Application of this relation (2,13,37) to the expression (2,13,36) for the induced drag yields

$$D=-\rho\int_{-1}^{+1}\varphi_\infty\frac{\partial\varphi_\infty}{\partial z}\,dy=-\rho\int_{-1}^{+1}\varphi_\infty w_\infty\,dy\ . \qquad (2,13,38)$$

It is now possible to express the induced resistance in terms of the circulation $C(y)$. According to formula (2,13,15) we have

$$\rho U\Gamma(y)=C(y)=\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}}\Pi(x,y)\,dx=2\rho\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}}\bar{\psi}(x,y)\,dx=2\rho U\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}}\frac{\partial\varphi}{\partial x}\,dx=2\rho U\varphi_\infty$$

or

$$\Gamma(y)=2\varphi_\infty(x,y) \quad . \qquad (2,13,39)$$

In order to find an expression for the downwash w_∞, use can be made of the integral equation (2,11,11), which expresses the downwash w in terms of the vorticity components. This equation reads

$$w = -\frac{1}{4\pi}\int_{-1}^{+1}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{\gamma_x (y-y_0) - \gamma_y (x-x_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}^3}\, dx\, dy +$$

$$-\frac{1}{4\pi}\int_{-1}^{+1}\int_{\sqrt{1-y^2}}^{\infty} \frac{\gamma_x (y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}^3}\, dx\, dy\,.$$

If x_0 increases indefinitely, the downwash w becomes

$$w_\infty = \lim_{x_0\to\infty} -\frac{1}{4\pi}\int_{-1}^{+1}\int_{\sqrt{1-y^2}}^{\infty} \frac{\gamma_x (y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}^3}\, dx\, dy =$$

$$= \lim_{x_0\to\infty} + \frac{1}{4\pi}\int_{-1}^{+1}\frac{d\Gamma}{dy}\,dy\int_{\sqrt{1-y^2}}^{\infty}\frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}^3}\,dx$$

$$= \lim_{x_0\to\infty} \frac{1}{4\pi}\int_{-1}^{+1}\frac{d\Gamma}{dy}\left[\frac{1}{y-y_0}\,\frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right]_{x=\sqrt{1-y^2}}^{x=\infty}\cdot dy$$

$$= \lim_{x_0\to\infty} \frac{1}{4\pi}\int_{-1}^{+1}\frac{d\Gamma}{dy}\left[\frac{1}{y-y_0} - \frac{1}{y-y_0}\,\frac{\sqrt{1-y^2}-x_0}{\sqrt{\left(\sqrt{1-y^2}-x_0\right)^2+(y-y_0)^2}}\right]dy$$

$$= \frac{1}{2\pi}\int_{-1}^{+1}\frac{d\Gamma}{dy}\,\frac{1}{y-y_0}\,dy\,. \qquad (2,13,40)$$

Substituting the expressions (2,13,39)and (2,13,40) into (2,13,38), we obtain the formula for the induced drag:

$$D = -\rho\int_{-1}^{+1}\frac{1}{2}\Gamma(y)\,dy\int_{-1}^{+1}\frac{1}{2\pi}\,\frac{d\Gamma}{dy}\,\frac{1}{\eta-y}\,d\eta =$$

$$= +\frac{\rho}{4\pi}\int_{-1}^{+1}\Gamma(y)\,dy\int_{-1}^{+1}\frac{d\Gamma}{d\eta}\,\frac{1}{y-\eta}\,dy\,. \qquad (2,13,41)$$

It is very remarkable that this formula for the induced drag agrees completely with the formula for the induced drag in lifting-line theory. In fact this corresponds with the meaning of Munk's so-called stagger-theorem. For the evaluation of the induced drag it is convenient to replace the circulation $\Gamma(y)$ by its Fourier series, viz.

$$\Gamma(y) = \sum_{n=1}^{\infty} A_n \sin n\theta \quad , \qquad (2,13,42)$$

with $y = \cos\theta$. Substitution into (2,13,41) yields

$$D = \frac{\rho}{4\pi}\int_0^{\pi}\sum_{n=1}^{\infty} A_n \sin n\theta \sin\theta\, d\theta\int_0^{\pi}\sum_{m=1}^{\infty} m A_m \cos m\psi\,\frac{1}{\cos\theta-\cos\psi}\,d\psi$$

$$= -\frac{\rho}{4\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m A_m A_n\int_0^{\pi}\sin n\theta\,\sin\theta\, d\theta\int_0^{\pi}\frac{\cos m\psi}{\cos\theta-\cos\psi}\,d\psi$$

$$= \frac{\rho}{4}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m A_m A_n\int_0^{\pi}\sin n\theta\,\sin m\theta\, d\theta = \frac{\pi\rho}{8}\sum_{n=1}^{\infty} n A_n^2\,. \qquad (2,13,43)$$

14 Examples.

14.1 The circular plate at a finite angle of attack.

If α is the angle of attack and U is the undisturbed fluid velocity, then the normal velocity w at the wing surface is given by the expression

$$w = -\alpha U \quad . \tag{2,14,1}$$

The regular velocity potential Φ is found by aid of the formula (2,3,24)

$$\Phi(\eta,\mu,\vartheta) = -\int_0^{2\pi}\int_{-1}^{+1} \alpha U G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,\mu_1\,d\mu_1\,d\vartheta_1 = -\alpha U P_1(\mu)\,\frac{Q_1(i\eta)}{Q_1'(io)} =$$

$$= -\alpha U\mu \frac{1-\eta\arctan\frac{1}{\eta}}{\left[\frac{d}{d\eta}\left(1-\eta\arctan\frac{1}{\eta}\right)\right]_{\eta=0}} = \frac{2}{\pi}\alpha U\mu\left(1-\eta\arctan\frac{1}{\eta}\right) . \tag{2,14,2}$$

The regular acceleration potential ψ is equal to zero because the normal acceleration $U\frac{\partial w}{\partial x}$ vanishes at the wing surface. According to formula (2,9,2) the complete acceleration potential $\widetilde{\psi}$ can thus be put in the form

$$\widetilde{\psi} = U\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1)\left[\frac{\partial G}{\partial\mu_1}\right]_{\mu_1=0} d\vartheta_1 \tag{2,14,3}$$

wherein the function $h(\vartheta_1)$ is still unknown. The other weight function $g(\vartheta)$ which plays an important role in our theory can easily be derived by means of the formula (2,9,9)

$$g(\vartheta) = -\pi\cos\vartheta\,\Phi_\mu(0,0,\vartheta) = -2\alpha U\cos\vartheta \quad . \tag{2,14,4}$$

The functions $g(\vartheta)$ and $h(\vartheta)$ must now be written in the form

$$g(\vartheta) = \cos\vartheta\, g^x(\vartheta) = \cos\vartheta\sum_{n=0}^{\infty} c_n\cos n\vartheta$$

$$h(\vartheta) = \cos\vartheta\, h^x(\vartheta) = \cos\vartheta\sum_{n=0}^{\infty} a_n\cos 2n\vartheta \quad ,$$

The coefficients c_n follow at once from (2,14,4):

$$c_0 = -2\alpha U\,,\quad c_1 = c_2 = c_3 = \ldots\ldots = 0 \tag{2,14,5}$$

In accordance with section 10 of this chapter, the equations for the unknown coefficients a_n read

$$\begin{cases} \sum_{n=0}^{j} \tau_0^{2n} a_n = -2\alpha U\sigma_0^0 \\ \sum_{n=0}^{j} \tau_\ell^{2n} a_n = 0 \qquad \ell = 1,2,3,\ldots \end{cases} \tag{2,14,6}$$

The integer j indicates that the Fourier series of the functions $g^x(\vartheta)$ and $h^x(\vartheta)$ are truncated after the $(j+1)$ th term. The system (2,14,6) has been solved for different values of j , viz. $j = 2,3,4,5$ and 9 . In this particular case the coefficients a_n are given in table III for the different values of j and apart from a factor αU .

Table III

	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 9$
a_0	+2,2435	+2,2433	+2,2433	+2,2433	+2,2432
a_1	-0,092964	-0,092916	-0,092890	-0,092875	-0,092852
a_2	+0,040863	+0,040851	+0,040842	+0,040836	+0,040824
a_3	-	-0,023476	-0,023473	-0,023471	-0,023464
a_4	-	-	+0,015436	+0,015435	+0,015431
a_5	-	-	-	-0,011011	-0,011009
a_6	-	-	-	-	+0,0082956
a_7	-	-	-	-	-0,0065009
a_8	-	-	-	-	+0,0052479
a_9	-	-	-	-	-0,0043354

These coefficients enable us to evaluate the weight function $k(\vartheta)$. The result is given in figure 3. It appears that the successive approximation for $k(\vartheta)$ coincide nearly completely. Further we can determine the lift L and the moment about the axis of y, M_y. We find

$$L = 2{,}812\ \rho\alpha U^2 \quad \text{and} \quad M_y = -1{,}465\ \rho\alpha U^2 .$$

These quantities can also be expressed in terms of their corresponding coefficients, viz.

$$\frac{\partial c_a}{\partial\alpha} = 0{,}8951 \qquad \text{and} \qquad \frac{\partial c_{my}}{\partial\alpha} = -0{,}4663 .$$

The center of pressure of the whole wing is in the middle section $y = 0$ on a distance $1 + \frac{M_y}{L} = 1 - \frac{1{,}465}{2{,}812} = 0{,}479$ aft of the leading edge. The pressure distribution over the wing surface can be calculated by aid of the formula (2,13,3). Here we restrict ourselves to the calculation of the pressure distribution on three sections of the wing, notably the middle section $y = 0$ and the two sections $y = 1/2$ and $y = 1/2\sqrt{3}$. Graphs of the pressure distributions are given in figures 9a, 9b and 9c. Moreover the sectional center of pressure is indicated in these figures. Using the formula (2,13,24) it is possible to evaluate the spanwise lift distribution $C(y)$. The result is given in figure 15. With the aid of the formula (2,13,43) we calculate the value of the induced resistance and find

$$D = 1{,}260\ \rho U^2 .$$

14.2 The spherical cap.

We shall now treat the problem of the spherical cap, which is placed in a homogeneous field of flow with velocity vector U, directed along the positive x-axis. The equation of the wing surface can be written in the form

$$(z-a)^2 + x^2 + y^2 = 1 + a^2, \quad z > 0 . \tag{2,14,7}$$

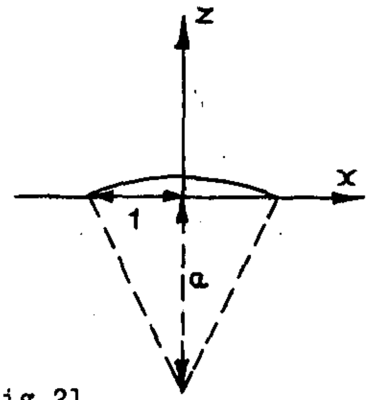

Fig.21

Assuming that $|a|$ is large compared with unity, linearization of the boundary conditions of the boundary value problem is allowed. The equation of the wing surface can be approximated by

$$z-a=-a\sqrt{1+\frac{1-x^2-y^2}{a^2}}\approx -a-\frac{1-x^2-y^2}{2a} .$$

Thus we can represent the wing surface by the equation

$$z=-\frac{1-x^2-y^2}{2a} \quad , \; z>0 \quad , \; x^2+y^2\leqq 1 . \tag{2,14,8}$$

The downwash distribution, which is prescribed on the projected wing surface, becomes so

$$w=U\frac{\partial z}{\partial x}=U\frac{x}{a} \quad \text{for } z=0 \quad \text{and} \quad x^2+y^2\leqq 1 . \tag{2,14,9}$$

The regular velocity potential Φ is found by substitution of the expression (2,14,9) into (2,3,24):

$$\begin{aligned}\Phi(\eta,\mu,\vartheta)&=\frac{U}{a}\int_0^{2\pi}\int_{-1}^{+1}\mu_1\sqrt{1-\mu_1^2}\cos\vartheta_1\, G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,d\mu_1\,d\vartheta_1=\\ &=-\frac{U}{3a}\int_0^{2\pi}\int_{-1}^{+1}P_2^1(\mu_1)\cos\vartheta_1\, G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,d\mu_1\,d\vartheta_1=\\ &=-\frac{U}{3a}P_2^1(\mu)\frac{Q_2^1(i\eta)}{Q_2^{1\prime}(i0)}\cos\vartheta=\\ &=-\frac{2U}{3a\pi}\mu\sqrt{1-\mu^2}\cos\vartheta\sqrt{1+\eta^2}\left\{3-\frac{1}{1+\eta^2}-3\eta\arctan\frac{1}{\eta}\right\} .\end{aligned} \tag{2,14,10}$$

The regular acceleration potential is easily found by aid of the formula (2,5,2)

$$\begin{aligned}\Psi(\eta,\mu,\vartheta)&=U\iint_S\frac{\partial w}{\partial x_1}G\,dx_1\,dy_1=\frac{U^2}{a}\int_0^{2\pi}\int_{-1}^{+1}G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\mu_1\,d\mu_1\,d\vartheta_1=\\ &=-\frac{2}{\pi a}U^2\mu\left\{1-\eta\arctan\frac{1}{\eta}\right\} .\end{aligned} \tag{2,14,11}$$

The complete acceleration potential $\widetilde{\Psi}$ can thus be written in the form

$$\widetilde{\Psi}(\eta,\mu,\vartheta)=-\frac{2U^2}{\pi a}\mu\left\{1-\eta\arctan\frac{1}{\eta}\right\}+U\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}h(\vartheta_1)\left[\frac{\partial G}{\partial\mu_1}\right]_{\mu_1=0}d\vartheta_1 . \tag{2,14,12}$$

The weight function $g(\vartheta)$ can be determined by means of (2,14,10):

$$g(\vartheta) = -\pi \cos\vartheta \, \Phi_\mu (0,0,\vartheta) = \frac{4U}{3a} \cos^2\vartheta \; . \qquad (2,14,13)$$

The Fourier coefficients c_n of the function $g^x(\vartheta)$ become thus

$$c_0 = 0 \, , \quad c_1 = \frac{4U}{3a} \, , \quad c_2 = c_3 = c_4 = \ldots\ldots = 0 \; . \qquad (2,14,14)$$

According to formula (2,10,7) the system of linear equations for the Fourier coefficients a_n of the unknown function $h^x(\vartheta)$ can be put into the form

$$\sum_{n=0}^{j} \overset{s_n}{\tau_\ell} \, a_n = \frac{4U}{3a} \, \overset{s_1}{\sigma_\ell} \qquad \ell = 0, 1, 2, \ldots\ldots \qquad (2,14,15)$$

The coefficients $\overset{s_1}{\sigma_\ell}$ can be easily found by aid of the formula (2,12,10). Solving the system of equations for $j = 4$, we find the numerical values to be apart from a factor U/a

$$a_0 = 0{,}73073 \quad a_1 = 0{,}60617 \quad a_2 = -0{,}12958 \quad a_3 = 0{,}057417 \quad a_4 = -0{,}032556$$

The function $h(\vartheta)$ is drawn in figure 4 for $j = 2$ and $j = 9$. The points which have been evaluated, again coincide nearly completely. The lift L and the moment M_y have the values

$$L = -1{,}465 \, \rho \frac{U^2}{a} \qquad\qquad M_y = -0{,}6892 \, \rho \frac{U^2}{a}$$

and their respective coefficients

$$c_a = -0{,}4663 \, \frac{1}{a} \qquad\qquad c_{m_y} = -0{,}2194 \, \frac{1}{a}$$

The position of the center of pressure of the whole wing is in the middle section $y = 0$ on a distance $1 + \frac{0{,}6892}{1{,}465} = 1{,}471$ aft of the leading edge. The pressure distributions on the sections $y = 0$, $y = \frac{1}{2}$ and $y = \frac{1}{2}\sqrt{s}$ are given in the figures 10a, 10b and 10c. The sectional center of pressure is also indicated in these figures. Finally the spanwise lift distribution is given in figure 16. The induced drag becomes in this case

$$D = 0{,}3725 \, \frac{\rho U^2}{a} \; .$$

14.3 The wing with a downwash distribution proportional to y .

We shall now apply our theory to an anti-symmetrical problem. In particular we consider the surface

$$z = xy \, , \quad x^2 + y^2 \leq 1 \qquad (2,14,16)$$

which is placed in a homogeneous field of flow with velocity U directed along the positive x -axis. The downwash distribution on the projected wing surface follows immediately from (2,14,16):

$$w = U \frac{\partial z}{\partial x} = Uy \quad \text{for } z = 0, \; x^2 + y^2 \leq 1 \; . \qquad (2,14,17)$$

The regular velocity potential becomes here

$$\Phi(\eta,\mu,\vartheta) = U\int_0^{2\pi}\int_{-1}^{-1}\mu_1\sqrt{1-\mu_1^2}\,\sin\vartheta_1\,G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,d\mu_1\,d\vartheta_1$$

$$= -\frac{U}{3}\int_0^{2\pi}\int_{-1}^{+1}P_2^1(\mu_1)\sin\vartheta_1\,G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,d\mu_1\,d\vartheta_1$$

$$= -\frac{U}{3}P_2^1(\mu)\,\frac{Q_2^1(i\eta)}{Q_2^{1'}(io)}\sin\vartheta =$$

$$= -\frac{2U}{3\pi}\,\mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\left\{3-\frac{1}{1+\eta^2}-3\eta\arctan\frac{1}{\eta}\right\}\sin\vartheta\,. \qquad (2,14,18)$$

As $\frac{\partial w}{\partial x}=0$, the regular acceleration potential $\psi(\eta,\mu,\vartheta)$ vanishes. Thus the complete acceleration potential $\tilde{\Psi}$ assumes the form

$$\tilde{\Psi}(\eta,\mu,\vartheta) = U\int_{\pi/2}^{3\pi}k(\vartheta_1)\left[\frac{\partial G}{\partial\mu_1}\right]_{\mu_1=0}d\vartheta_1\,. \qquad (2,14,19)$$

The weight function $g(\vartheta)$ reads in this case

$$g(\vartheta) = -\pi\cos\vartheta\,\Phi_\mu(0,0,\vartheta) = \frac{4U}{3}\cos\vartheta\sin\vartheta\,. \qquad (2,14,20)$$

For anti-symmetrical problems the function $g(\vartheta)$ is written in the form

$$g(\vartheta) = g^x(\vartheta)\cos\vartheta = \cos\vartheta\sum_{n=1}^{\infty}d_n\sin n\vartheta\,.$$

The coefficients d_n become here

$$d_1 = \frac{4U}{3}\,,\ d_2 = d_3 = \ldots\ldots = 0 \qquad (2,14,21)$$

The unknown weight function $k(\vartheta)$ be expressed by means of the series

$$k(\vartheta) = k^x(\vartheta)\cos\vartheta = \cos\vartheta\sum_{n=0}^{\infty}b_n\sin(2n+1)\vartheta\,. \qquad (2,14,22)$$

The coefficients b_n can now be solved from the system of linear equations (2,10,23). In this particular case the equations read

$$\sum_{n=0}^{j}\overset{a_n}{\tau}_k\,b_n = \frac{4U}{3}\,\overset{a_1}{\sigma}_k \qquad k=0,1,2,\ldots\ldots \qquad (2,14,23)$$

Substitution of the numerical values of $\overset{a_1}{\sigma}_k$ transforms (2,14,23) into

$$\sum_{n=0}^{j}\overset{a_n}{\tau}_0\,b_n = \frac{2U}{3}$$

$$\sum_{n=0}^{j}\overset{a_n}{\tau}_k\,b_n = 0 \quad \text{for } k=1,2,3,\ldots\ldots \qquad (2,14,24)$$

For $j=4$ we have the roots, apart from a factor U

$b_0 = -1{,}3730 \quad b_1 = +0{,}017714 \quad b_2 = -0{,}010305 \quad b_3 = +0{,}0068407 \quad b_4 = -0{,}0049177$

The function $k(\vartheta)$ is plotted in figure 5 for $j=2$ and $j=9$. The agreement between the successive approximations is again remarkable. The roll-moment M_x becomes

$$M_x = -0{,}3849\,\rho U^2\,.$$

The corresponding coefficient is thus

$$C_{m_x} = -0{,}1225$$

The pressure distribution on the wing sections $y = \frac{1}{2}$ and $y = \frac{1}{2}\sqrt{3}$ have been plotted in the figures 11a and 11b. In the middle section the pressure vanishes because of the anti-symmetrical character of the problem. The sectional center of pressure is also drawn in these figures. The spanwise lift distribution which is calculated by means of the formula (2,13,25) is given in the figure 17. The induced resistance appears to be

$$D = 0{,}1890 \ \rho U^2 .$$

14.4 The downwash distribution $w = Ux^2$

In order to calculate the regular velocity potential which corresponds with a downwash $w = Ux^2$ on the wing surface, it is necessary to develop the function $\mu w(\mu,\vartheta)$ in a series of surface harmonics. We find easily that

$$\mu w(\mu,\vartheta) = U\mu\,(1-\mu^2)\cos^2\vartheta = \frac{1}{2}\,U\mu\,(1-\mu^2)\,(1+\cos 2\vartheta) =$$
$$= \frac{U}{5}\left\{P_1(\mu) - P_3(\mu)\right\} + \frac{U}{30}\,P_3^2(\mu)\cos 2\vartheta . \tag{2,14,25}$$

According to formula (2,3,24), the velocity potential Φ becomes

$$\Phi(\eta,\mu,\vartheta) = U\int_0^{2\pi}\int_{-1}^{+1}\left[\frac{1}{5}\left\{P_1(\mu_1) - P_3(\mu_1)\right\} + \right.$$
$$\left. + \frac{1}{30}\,P_3^2(\mu_1)\cos 2\vartheta_1\right] G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,d\mu_1\,d\vartheta_1$$
$$= \frac{U}{5}\,P_1(\mu)\,\frac{Q_1(i\eta)}{Q_1'(i0)} - \frac{U}{5}\,P_3(\mu)\,\frac{Q_3(i\eta)}{Q_3'(i0)} + \frac{U}{30}\,P_3^2(\mu)\,\frac{Q_3^2(i\eta)}{Q_3^{2\,\prime}(i0)}\cos 2\vartheta$$
$$= -\frac{2U}{5\pi}\,\mu\left\{1 - \eta \arctan\frac{1}{\eta}\right\} + \frac{2U}{15\pi}\,(5\mu^3 - 3\mu)\left\{\frac{5}{2}\eta^2 + \frac{2}{3} + \right.$$
$$\left. - \frac{5\eta^3 + 3\eta}{2}\arctan\frac{1}{\eta}\right\} - \frac{U}{15\pi}\,\mu\,(1-\mu^2)\,(1+\eta^2)\left\{15 - \frac{5}{1+\eta^2} + \right.$$
$$\left. - \frac{2}{(1+\eta^2)^2} - 15\,\eta\arctan\frac{1}{\eta}\right\}\cos 2\vartheta . \tag{2,14,26}$$

The acceleration potential ψ becomes here

$$\psi(\eta,\mu,\vartheta) = U\iint_S \frac{\partial w}{\partial x_1}\,G\,dx_1\,dy_1 =$$
$$= 2U^2\int_0^{2\pi}\int_{-1}^{+1}\mu_1\sqrt{1-\mu_1^2}\,\cos\vartheta_1\,G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,d\mu_1\,d\vartheta_1 =$$
$$= -\frac{4U^2}{3\pi}\,\mu\sqrt{1-\mu^2}\cos\vartheta\,\sqrt{1+\eta^2}\left\{3 - \frac{1}{1+\eta^2} - 3\eta\arctan\frac{1}{\eta}\right\} . \tag{2,14,27}$$

The complete acceleration potential $\widetilde{\psi}$ can thus be written in the form

$$\widetilde{\Psi}(\eta,\mu,\vartheta) = -\frac{4U^2}{3\pi}\mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\left\{3-\frac{1}{1+\eta^2}-3\eta\arctan\frac{1}{\eta}\right\}\cos\vartheta +$$

$$+U\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k(\vartheta_1)\left[\frac{\partial G}{\partial\mu_1}\right]_{\mu_1=0} d\vartheta_1 \quad . \tag{2,14,28}$$

For the function $q(\vartheta)$ follows from (2,14,26)

$$q(\vartheta) = -\pi\cos\vartheta\,\phi_{\mu}(0,0,\vartheta) = -\pi\cos\vartheta\left(-\frac{2U}{3\pi}-\frac{4U}{15\pi}-\frac{8U}{15\pi}\cos 2\vartheta\right)$$

$$= \frac{U}{15}\cos\vartheta\,(10+8\cos 2\vartheta) \tag{2,14,29}$$

The Fourier coefficients c_n are thus

$$c_0 = \frac{10}{15}U\,,\ c_1 = 0\,,\ c_2 = \frac{8}{15}U\,,\ c_3 = c_4 = \ldots\ldots = 0 \quad . \tag{2,14,30}$$

The system of equations for the coefficients a_n reads now

$$\sum_{n=0}^{j} \tau_\ell^n a_n = \frac{10}{15}\delta_\ell^{0} + \frac{8}{15}\delta_\ell^{2} \qquad \ell = 0,1,2,\ \ldots . \tag{2,14,31}$$

Solution of this system for $j=4$, yields

$a_0 = -0{,}76019 \quad a_1 = -0{,}50206 \quad a_2 = -0{,}012630 \quad a_3 = +0{,}0068424 \quad a_4 = -0{,}0043061$

wherein the factor U has been omitted. The figure 6 shows the function $k(\vartheta)$ for $j=2$ and $j=9$. The corresponding points of the different approximations coincide nearly. The lift L and the moment M_y assume the values

$$L = -1{,}180\ \rho U^2 \qquad M_y = -0{,}03697\ \rho U^2$$

whereas the corresponding coefficients are

$$c_a = -0{,}3755 \qquad c_{m_y} = 0{,}0118$$

The center of pressure of the whole wing lies in the middle section $y=0$ on a distance 0,969 aft of the leading edge. In the figures 12a, 12b and 12c the pressure distributions on the wing sections $y=0$, $y=1/2$ and $y=1/2\sqrt{3}$ and the corresponding sectional centers of pressure have been plotted. The induced drag becomes in this case

$$D = 0{,}2568\ \rho U^2 \ .$$

14.5 The downwash distribution $w = Uy^2$

Similarly as in the preceding case, we must develop the function $\mu w(\mu,\vartheta)$ in a series of surface harmonics. We obtain easily

$$\mu w(\mu,\vartheta) = U\mu(1-\mu^2)\sin^2\vartheta = \frac{U}{2}\mu(1-\mu^2)(1-\cos 2\vartheta) =$$

$$= \frac{U}{5}\left\{P_1(\mu)-P_3(\mu)\right\} - \frac{U}{30}P_3^2(\mu)\cos 2\vartheta \quad . \tag{2,14,32}$$

The velocity potential can immediately be derived from (2,14,25) viz.

$$\Phi(\eta,\mu,\vartheta) = -\frac{2U}{5\pi}\mu\left\{1-\eta\arctan\frac{1}{\eta}\right\}+$$
$$+\frac{2U}{15\pi}\left(5\mu^3-3\mu\right)\left\{\frac{5}{2}\eta^2+\frac{2}{3}-\frac{5\eta^3+3\eta}{2}\arctan\frac{1}{\eta}\right\}+$$
$$+\frac{U}{15\pi}\mu\left(1-\mu^2\right)\left(1+\eta^2\right)\left\{15-\frac{5}{1+\eta^2}-\frac{2}{\left(1+\eta^2\right)^2}-15\eta\arctan\frac{1}{\eta}\right\}\cos 2\vartheta \,. \tag{2,14,33}$$

As the normal acceleration $U\frac{\partial w}{\partial x}$ is equal to zero, the regular acceleration potential ψ also vanishes. The complete acceleration potentials can thus be put in the form

$$\tilde{\psi}(\eta,\mu,\vartheta) = U\int_{\pi/2}^{3\pi/2} k(\vartheta_1)\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \quad , \tag{2,14,34}$$

The weight function $g(\vartheta)$ becomes in this case

$$g(\vartheta) = \frac{U}{15}\cos\vartheta\left(10-8\cos 2\vartheta\right) \tag{2,14,35}$$

and the coefficients c_n

$$c_0 = \frac{10}{15}U,\ c_1 = 0\ ,\ c_2 = -\frac{8}{15}U,\ c_3 = c_4 = \ldots\ldots = 0 \ .$$

The truncated system of equations can be written as

$$\sum_{n=0}^{j} \tau_l^{s_n} a_n = \frac{10}{15}\sigma_l^{s_0} - \frac{8}{15}\sigma_l^{s_2} \qquad l = 0,1,2,\ldots\ldots \tag{2,14,36}$$

For $j=4$ we find the coefficients

$$a_0 = -0{,}73534 \quad a_1 = +0{,}56399 \quad a_2 = -0{,}014598 \quad a_3 = +0{,}0088065 \quad a_4 = -0{,}0059843 \ .$$

Different approximations for the weight-function $k(\vartheta)$ viz. $j=2$ and $j=9$, are drawn in figure 7. The agreement between these approximations is again very good. Lift and moment take the values

$$L = -0{,}6952\ \rho U^2 \qquad M_y = 0{,}3022\ \rho U^2$$

and the corresponding coefficients are

$$c_a = -0{,}2213 \qquad c_{m_y} = 0{,}0962 \ .$$

The center of pressure of the whole wing lies in the section $y=0$ on a distance 0,565 aft of the leading edge. Pressure distributions and corresponding sectional centers of pressure for the sections $y=0$, $y=1/2$ and $y=1/2\sqrt{3}$ are plotted in the figures 13a, 13b and 13c. The spanwise lift distribution can be found in figure 19. The induced drag has the value

$$D = 0{,}1077\ \rho U^2 \ .$$

14.6 The downwash distribution $w = Uxy$

In contradistinction to the previous two cases, this problem is an example of the anti-symmetrical problems. The function $\mu w(\mu,\vartheta)$ can be written in the form

$$\mu w(\mu,\vartheta) = U\mu(1-\mu^2)\sin\vartheta\cos\vartheta = \frac{U}{2}\mu(1-\mu^2)\sin 2\vartheta = \frac{U}{30}P_3^2(\mu)\sin 2\vartheta .$$

It follows immediately from (2,14,26) that the regular velocity potential Φ reads

$$\Phi(\eta,\mu,\vartheta) = -\frac{U}{15\pi}\mu(1-\mu^2)(1+\eta^2)\left\{15 - \frac{5}{1+\eta^2} - \frac{2}{(1+\eta^2)^2} - 15\eta\arctan\frac{1}{\eta}\right\}\sin 2\vartheta .$$

The regular acceleration potential can easily be derived from the velocity potential Φ in the case 14.3:

$$\Psi(\eta,\mu,\vartheta) = -\frac{2U^2}{3\pi}\mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\left\{3 - \frac{1}{1+\eta^2} - 3\eta\arctan\frac{1}{\eta}\right\}\sin\vartheta$$

The complete acceleration potential $\tilde{\Psi}$ becomes thus

$$\tilde{\Psi}(\eta,\mu,\vartheta) = -\frac{2U^2}{3\pi}\mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\left\{3 - \frac{1}{1+\eta^2} - 3\eta\arctan\frac{1}{\eta}\right\}\sin\vartheta +$$

$$+ U\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} k(\vartheta_1)\left[\frac{\partial G}{\partial\mu_1}\right]_{\mu_1=0} d\vartheta_1 .$$

The weight function $g(\vartheta)$ is here

$$g(\vartheta) = -\pi\cos\vartheta\,\Phi_\mu(0,0,\vartheta) = \frac{8U}{15}\cos\vartheta\sin 2\vartheta$$

and thus

$$d_1 = 0,\ d_2 = \frac{8U}{15},\ d_3 = d_4 = \ldots\ldots = 0 .$$

The coefficients k_n of the unknown weight-function $k(\vartheta)$ must thus satisfy the equations

$$\sum_{n=0}^{j} \tau_k^{a_n} k_n = \frac{8U}{15}\sigma_k^{a_2} \qquad k = 0,1,2,\ldots.$$

For $j=4$ we find for the coefficients apart from the factor U

$$k_0 = +0{,}43812 \quad k_1 = +0{,}27750 \quad k_2 = -0{,}067837 \quad k_3 = +0{,}032157 \quad k_4 = -0{,}018989 .$$

This weight-function $k(\vartheta)$ is plotted in figure 8 for $j=2$ and $j=9$. The differences between these two approximations are negligible. The roll-moment M_x is

$$M_x = -0{,}1809\ \rho U^2$$

while the corresponding coefficient c_{m_x} becomes

$$c_{m_x} = -0{,}0576 .$$

The pressure distributions on the wing sections $y = 1/2$ and $y = 1/2\sqrt{3}$ are drawn in the figures 14a and 14b. The sectional centers of pressure are also indicated in the figures mentioned. The spanwise lift distribution is given in figure 20. The induced drag has the value

$$D = 0{,}0488\ \rho U^2 .$$

15 Comparison with other results.

In this section a comparison will be made between the numerical results found in the preceding paragraph and the values given by Kinner, Schade and Küssner in their papers. Furthermore our exact values are compared with results, derived by aid of an approximation method (ref.23). As has already been remarked in the introduction, Kinner's and Schade's solutions satisfy the Kutta condition only in a finite number of points of the trailing edge. This number of points is related with the number of equations by which an infinite system of linear equations is approximated. Kinner approximates this infinite system by 4 equations, whereas Schade applies an approximation by 6 equations. Although Kinner's and Schade's methods are completely the same, their numerical solutions display a greater discrepancy than between the approximations with 4 and 6 linear equations in our theory. From the papers of Kinner and Schade we can only find overall quantities as lift, moments, etc. for the downwash distributions $w = -\alpha U$, $w = Ux$ and $w = Uy$.

In the tables at the end of this section, some values found by Küssner are also mentioned. It immediately appears that these values disagree very much with the other values. The reason for this can be ascribed to the errors in Küssner's theory (ref.22).

Moreover we have treated all the six downwash distributions up to the second degree in x and y with the approximation method for lifting surface calculations, which the author has described in ref.23. For a complete survey of this method the reader is referred to the reference mentioned. It can be remarked that the approximate calculations have been performed with 4 pivotal points in chordwise direction and 5 points in spanwise direction. The results for lift, moments, induced resistance and center of pressure are given in the tables below, while the results for the spanwise lift distribution, the sectional centers of pressure and the pressure distribution on some sections are drawn in the same figures, which show the exact results. The agreement of the approximate values with the exact values is rather good. This result confirms the reliability of the approximation method.

Table IV. Lift coefficients c_a for symmetric downwash distributions.

	$w=-\alpha U$	$w=Ux$	$w=Ux^2$	$w=Uy^2$
Kinner	0,908	-0,468	-	-
Schade	0,8992	-0,4718	-	-
Küssner	0,8488	-0,4244	-	-
Approx.Theory	0,8927	-0,4520	-0,3632	-0,2215
This Theory	0,8951	-0,4663	-0,3755	-0,2213

Table V. Moment coefficients c_{m_y} for symmetric downwash distributions.

	$w=-\alpha U$	$w=Ux$	$w=Ux^2$	$w=Uy^2$
Kinner	-0,468	-0,219	-	-
Schade	-0,4659	-0,2191	-	-
Küssner	-0,4244	-0,2122	-	-
Approx.Theory	-0,4698	-0,2074	0,0070	0,1007
This Theory	-0,4663	-0,2194	0,0118	0,0962

Table VI. Position of the center of pressure aft of the leading edge for symmetric downwash distributions.

	$w=-\alpha U$	$w=Ux$	$w=Ux^2$	$w=Uy^2$
Kinner	0,485	1,468	-	-
Schade	0,482	1,464	-	-
Küssner	0,500	1,500	-	-
Approx.Theory	0,474	1,459	0,981	0,545
This Theory	0,479	1,471	0,969	0,565

Table VII. Roll-moment coefficients c_{m_x} for anti-symmetric downwash distributions.

	$w=Uy$	$w=Uxy$
Kinner	-0,127	-
Schade	-0,1276	-
Küssner	-0,1132	-0,0566
Approx.Theory	-0,1232	-0,0602
This Theory	-0,1225	-0,0576

Table VIII. Induced drag coefficients c_D for symmetric and anti-symmetric downwash distributions.

	$w=-\alpha U$	$w=Ux$	$w=Ux^2$	$w=Uy^2$	$w=Uy$	$w=Uxy$
Kinner	0,4122	0,1230	-	-	-	-
Approx.Theory	0,4017	0,1083	0,0738	0,0345	0,0607	0,0157
This Theory	0,4011	0,1186	0,0817	0,0343	0,0602	0,0155

Chapter III.

The circular wing in unsteady incompressible flow.

1 Formulation of the problem.

We consider a wing of circular planform which moves with constant velocity U in an incompressible and non-viscous fluid and having at the same time a motion of small amplitude in the transverse direction. Similarly as in the steady case the reference system is a right-hand system of Cartesian coordinates (x,y,z) which is fixed to the aerofoil, when the translational displacement of the wing in the direction of the vector U is considered only. The positive axis of x is again taken opposite to the direction of the vector U, the axis of y is taken in the spanwise direction.

According to chapter I the perturbation velocity potential Φ satisfies the equation of Laplace

$$\Phi_{xx}+\Phi_{yy}+\Phi_{zz}=0 \tag{3,1,1}$$

whereas the linearized boundary condition at the wing surface

$$z=z(x,y,t) \tag{3,1,2}$$

reads

$$w=\frac{\partial z}{\partial t}+U\frac{\partial z}{\partial x} \quad \text{for } z=0\ ,\ x^2+y^2\leq 1\ . \tag{3,1,3}$$

We shall now investigate in particular the case of harmonic oscillations of the circular wing. As already mentioned the amplitudes are assumed to be small, so that linearization of the boundary conditions is allowable. It is clear that the equation (3,1,2) of the wing surface can be written in the form

$$z=z_1(x,y)+z_2(x,y)e^{i\nu t}\ . \tag{3,1,4}$$

The term $z_1(x,y)$ represents the wing surface in its equilibrium state, while $z_2(x,y)$ denotes the amplitude of the harmonic oscillation in the point (x,y) of the wing surface.

Because of the linearity of the condition (3,1,3) the boundary value problem can be split into two problems associated with z_1 and z_2 respectively. The first problem is just the steady case, which has been described in the preceding chapter. The second problem deals with the pure unsteady part of the complete boundary value problem. This last problem shall be treated in detail in this chapter. It is clear from the foregoing considerations that the wing surface in its equilibrium state may be replaced by a flat plate, which of course can be deformed under influence of the harmonic oscillations.

In considering a circular flat plate which performs a harmonic oscillation indicated by

$$z(x,y,t)=\bar{z}(x,y)e^{i\nu t} \tag{3,1,5}$$

the boundary condition at the wing surface can be written in the form

$$w(x,y,t)=\bar{w}(x,y)e^{i\nu t}=\frac{\partial z}{\partial t}+U\frac{\partial z}{\partial x}=\left\{i\nu\bar{z}(x,y)+U\,\frac{\partial\bar{z}(x,y)}{\partial x}\right\}e^{i\nu t}$$

or

$$\bar{w}(x,y)=i\nu\bar{z}(x,y)+U\,\frac{\partial\bar{z}(x,y)}{\partial x} \quad \text{for } z=0\ ,\ x^2+y^2\leq 1\ .$$

In terms of the velocity potential $\Phi(x,y,z,t)=\bar{\Phi}(x,y,z)e^{i\nu t}$, this boundary condition can be expressed by the formula

$$\frac{\partial\bar{\Phi}}{\partial z}=i\nu\bar{z}(x,y)+U\,\frac{\partial\bar{z}(x,y)}{\partial x} \quad \text{for } z=0\ ,\ x^2+y^2\leq 1\ . \tag{3,1,7}$$

Equally as in the steady case the condition must be supplemented by the condition that no discontinuity of the pressure can exist across the xy - plane outside the wing. In order to express this condition in terms of the perturbation potential, use can be made of the unsteady Bernoulli equation (1,1,7) which can be written as

$$\frac{\partial \Phi}{\partial t}+\frac{1}{2}\left[\left(U+\frac{\partial \Phi}{\partial x}\right)^2+\left(\frac{\partial \Phi}{\partial y}\right)^2+\left(\frac{\partial \Phi}{\partial z}\right)^2\right]+\frac{p}{\rho}=\frac{1}{2}U^2+\frac{p_\infty}{\rho} \qquad (3,1,8)$$

or in linear approximation

$$\frac{\partial \Phi}{\partial t}+U\frac{\partial \Phi}{\partial x}=\frac{p_\infty-p}{\rho} \quad . \qquad (3,1,9)$$

The condition of no pressure discontinuity reads thus

$$\frac{\partial}{\partial t}\left\{\Phi(x,y,+0)-\Phi(x,y,-0)\right\}+U\frac{\partial}{\partial x}\left\{\Phi(x,y,+0)-\Phi(x,y,-0)\right\}=0 \quad \text{for } x^2+y^2>1 \quad . \qquad (3,1,10)$$

This condition can again be simplified by using the fact that the normal velocity w is a continuous and even function of the variable z . This property of the downwash implies that the velocity potential Φ is an odd function of z , or in formula

$$\Phi(x,y,+z)=-\Phi(x,y,-z) \quad . \qquad (3,1,11)$$

If Π represents the pressure jump over the xy -plane, it follows from (3,1,9) that the relation holds

$$\Pi=2\rho\left\{\frac{\partial \Phi}{\partial t}+U\frac{\partial \Phi}{\partial x}\right\} \quad . \qquad (3,1,12)$$

For the harmonic case the requirement (3,1,10) can thus be replaced by the simple expression

$$i\nu\bar{\Phi}+U\frac{\partial \bar{\Phi}}{\partial x}=0 \quad \text{for } z=0,\ x^2+y^2>1 \quad . \qquad (3,1,13)$$

The boundary value problem for the velocity potential $\bar{\Phi}$ can thus be formulated as follows: to find a solution of Laplace's equation, which fulfils the conditions

$$\begin{array}{lll} 1) & \bar{\Phi}(x,y,z)=0 & \text{for } (x,y,z) \text{ at infinity} \\ 2) & \dfrac{\partial \bar{\Phi}}{\partial z}=\bar{w}(x,y) & \text{for } z=0,\ x^2+y^2\leqq 1 \\ 3) & i\nu\bar{\Phi}+U\dfrac{\partial \bar{\Phi}}{\partial x}=0 & \text{for } z=0,\ x^2+y^2>1 \quad . \end{array}$$

2 Determination of the complete acceleration potential.

The theory developed in paragraph 3 of chapter II, enables us to write down at once an expression for the regular velocity potential $\bar{\Phi}(x,y,z)$, which fulfils the three conditions, mentioned in the preceding paragraph. The function of Green for the unsteady boundary value problem is completely the same as the function of Green, used in the steady case. According to formula (2,3,25) we can write

$$\bar{\Phi}(x,y,z)=\iint_S \bar{w}(x_1,y_1)\,G(x,y,z;x_1,y_1)\,dx_1\,dy_1 \quad , \qquad (3,2,1)$$

wherein $\overline{w}(x,y)$ represents the amplitude of the downwash distribution over the wing surface, which is determined by means of the relation (3.1.6). Similarly as in the steady case, we can introduce an acceleration potential $\overline{\psi^x}$ using the formula (1,3,7) , which simplifies to

$$\overline{\psi^x} = i\nu\overline{\Phi} + U\frac{\partial\overline{\Phi}}{\partial x} \tag{3,2,2}$$

for harmonic oscillations. Hence

$$\overline{\psi^x} = i\nu\iint_S \overline{w}(x_1,y_1)G(x,y,z;x_1,y_1)\,dx_1\,dy_1 + \\ + U\frac{\partial}{\partial x}\iint_S \overline{w}(x_1,y_1)G(x,y,z;x_1,y_1)\,dx_1\,dy_1 . \tag{3,2,3}$$

Moreover it is possible to obtain an expression for the acceleration potential by considering the normal acceleration at the wing surface as the given boundary condition. This acceleration potential, denoted by $\overline{\psi}$, can thus be written in the form

$$\overline{\psi} = \iint_S \overline{a}(x_1,y_1)G(x,y,z;x_1,y_1)\,dx_1\,dy_1 \tag{3,2,4}$$

It would now be possible to derive an exact formula for this normal acceleration $\overline{a}(x,y)$ along the same lines as is done in paragraph 6 of chapter II. However, we shall follow here a somewhat different way for finding the ultimate expression for the real acceleration potential. To this purpose we assume that the normal acceleration $\overline{a}(x,y)$ be given by the formula

$$a(x,y,t) = \overline{a}(x,y)e^{i\nu t} = \frac{dw}{dt} = \left\{ i\nu\overline{w}(x,y) + U\frac{\partial\overline{w}(x,y)}{\partial x}\right\} e^{i\nu t}$$

or

$$\overline{a}(x,y) = i\nu\overline{w}(x,y) + \frac{\partial\overline{w}(x,y)}{\partial x} . \tag{3,2,5}$$

The corresponding acceleration potential $\overline{\psi}$ reads then

$$\overline{\psi} = i\nu\iint_S \overline{w}(x_1,y_1)G(x,y,z;x_1,y_1)\,dx_1\,dy_1 + U\iint_S \frac{\partial\overline{w}}{\partial x_1}G(x,y,z;x_1,y_1)\,dx_1\,dy_1 \tag{3,2,6}$$

Henceforth we consider the difference of the potentials $\overline{\psi^x}$ and $\overline{\psi}$. We obtain the formula

$$\overline{\psi^x} - \overline{\psi} = U\frac{\partial}{\partial x}\iint_S \overline{w}(x_1,y_1)G(x,y,z;x_1,y_1)\,dx_1\,dy_1 + \\ - U\iint_S \frac{\partial\overline{w}}{\partial x_1}G(x,y,z;x_1,y_1)\,dx_1\,dy_1 . \tag{3,2,7}$$

The right-hand member of (3.2.7) contains exactly the same terms as in the corresponding expression in the steady case. Thus we can immediately conclude that the identity (2,7,7) retains its validity. This means, that the following formula holds

$$\overline{\psi^x} - \overline{\psi} = \pi U\int_0^{2\pi} \cos\vartheta_1\,\overline{\Phi}_{\mu_1}(0,0,\vartheta_1)G_{\mu_1}(\eta,\mu,\vartheta;0,\vartheta_1)\,d\vartheta_1 . \tag{3,2,8}$$

After quite the same reasoning as in the steady case, we come to the normal expression for the actual acceleration potential, viz.

$$\bar{\Psi}(\eta,\mu,\vartheta)+U\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1)\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \ . \tag{3,2,9}$$

This expression (3,2,9) satisfies the Kutta-condition, while at the leading edge of the wing the square root singularity is guaranteed. The unknown weight-function $h(\vartheta)$ can again be determined by the requirement that the normal velocity at the wing surface which corresponds with this acceleration potential must coincide with the prescribed downwash. Using the relations (3,2,8) and (3,2,9), we can rewrite the complete acceleration potential in the form

$$\psi(\eta,\mu,\vartheta)+Ue^{i\nu t}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1)\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 =$$

$$= \psi^{x}+Ue^{i\nu t}\int_{0}^{2\pi} g(\vartheta_1)\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1+Ue^{i\nu t}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1)\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1$$

wherein $g(\vartheta)=-\pi\cos\vartheta\,\Phi_{\mu}(0,0,\vartheta)$ (3,2,10)

Remembering that ψ^{x} is the acceleration potential which has been derived from the velocity potential Φ , we can state that the normal velocity at the wing surface which corresponds with this potential ψ^{x} is equal to the prescribed downwash. Hence the above mentioned requirement of coinciding downwashes can be replaced by the condition that the normal velocity at the wing surface, which belongs to the acceleration potential Ω , where

$$\Omega(\eta,\mu,\vartheta)=Ue^{i\nu t}\int_{0}^{2\pi} g(\vartheta_1)\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1+Ue^{i\nu t}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1)\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \ , \tag{3,2,11}$$

vanishes. In order to express this condition in terms of an equation use must be made of the relationship between velocity potential and acceleration potential. This connection has been derived in Chapter I and reads

$$\varphi(x,y,z,t)=\frac{1}{U}\int_{-\infty}^{x}\psi\left(x',y,z,t-\frac{x-x'}{U}\right)dx' \ . \tag{3,2,12}$$

Under the assumption that the time dependency is represented by the factor $e^{i\nu t}$, this relation can be transformed into

$$\bar{\varphi}(x,y,z)=\frac{1}{U}e^{-i\omega x}\int_{-\infty}^{x}\bar{\psi}(x',y,z)e^{+i\omega x'}dx' \ , \tag{3,2,13}$$

where $\omega=\frac{\nu}{U}$ is the so-called reduced frequency.

The equation for the unknown weight-function $h(\vartheta)$ reads thus

$$\lim_{x\to 0} \frac{\partial}{\partial x} \frac{1}{U} e^{-i\omega x} \int_{-\infty}^{x} \Omega\,(x', y, z)\, e^{+i\omega x'}\, dx' = 0 \tag{3,2,14}$$

or written out

$$\begin{aligned} &\lim_{z\to 0} \frac{\partial}{\partial x} \int_{-\infty}^{x} e^{+i\omega x'} dx' \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 + \\ &+ \lim_{z\to 0} \frac{\partial}{\partial x} \int_{-\infty}^{x} e^{+i\omega x'} dx' \int_{0}^{2\pi} g(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 = 0 \quad . \end{aligned} \tag{3,2,15}$$

Quite analogous to the steady case we now introduce the Fourier expansions for the weight-functions $g(\vartheta)$ and $h(\vartheta)$, viz.

$$g(\vartheta) = g^{*}(\vartheta) \cos\vartheta = \cos\vartheta \sum_{n=0}^{\infty} c_n \cos n\vartheta$$

$$h(\vartheta) = h^{*}(\vartheta) \cos\vartheta = \cos\vartheta \sum_{n=0}^{\infty} a_n \cos 2n\vartheta \quad ,$$

if the boundary value problem is symmetric with respect to the variable y and

$$g(\vartheta) = g^{*}(\vartheta) \cos\vartheta = \cos\vartheta \sum_{n=1}^{\infty} d_n \sin n\vartheta$$

$$h(\vartheta) = h^{*}(\vartheta) \cos\vartheta = \cos\vartheta \sum_{n=0}^{\infty} b_n \sin (2n+1)\vartheta \; ,$$

if the boundary value problem is anti-symmetric in the variable y. We shall also use the Fourier expansion (2,10,11), which reads

$$U \cos\vartheta_1 \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} = \sum_{m=0}^{\infty} \overset{s}{\psi}_m \cos m\vartheta_1 + \sum_{m=1}^{\infty} \overset{a}{\psi}_m \sin m\vartheta_1$$

where $\overset{s}{\psi}_m$ and $\overset{a}{\psi}_m$ have the same meanings as in (2,10,10).

For the symmetric problems we can then write for the acceleration potential Ω

$$\Omega\;(\eta,\mu,\vartheta) = \pi \sum_{n=0}^{\infty} \varepsilon_n c_n \overset{s}{\psi}_n + \sum_{n=0}^{\infty} a_n \left\{ p_{2n}^{2n} \overset{s}{\psi}_{2n} + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \overset{s}{\psi}_{2m+1} \right\} \tag{3,2,16}$$

and for the anti-symmetric problems

$$\Omega\;(\eta,\mu,\vartheta) = \pi \sum_{n=1}^{\infty} d_n \overset{a}{\psi}_n + \sum_{n=0}^{\infty} b_n \left\{ q_{2n+1}^{2n+1} \overset{a}{\psi}_{2n+1} + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \overset{a}{\psi}_{2m} \right\} . \tag{3,2,17}$$

The quantities p_n^m and q_n^m are again defined by the formulas (2,10,13) and (2,10,20) respectively.

If we denote the normal velocity at the wing surface which corresponds to the acceleration potential $\overset{s}{\psi}_n$ by $\overset{s}{w}_n$ and the normal velocity corresponding to $\overset{a}{\psi}_n$ by $\overset{a}{w}_n$, the equations for the unknown coefficients a_n and b_n read respectively

$$\pi \sum_{n=0}^{\infty} \varepsilon_n c_n \overset{s}{w}_n + \sum_{n=0}^{\infty} a_n \left\{ p_{2n}^{2n} \overset{s}{w}_{2n} + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \overset{s}{w}_{2m+1} \right\} = 0 \qquad (3,2,18)$$

and

$$\pi \sum_{n=1}^{\infty} d_n \overset{a}{w}_n + \sum_{n=0}^{\infty} b_n \left\{ q_{2n+1}^{2n+1} \overset{a}{w}_{2n+1} + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \overset{a}{w}_{2m} \right\} = 0 \quad . \qquad (3,2,19)$$

The only new problem for the unsteady case is the evaluation of the downwashes $\overset{s}{w}_n$ and $\overset{a}{w}_n$, which will be performed in the next paragraph.

3 Evaluation of the downwashes $\overset{s}{w}_n$ and $\overset{a}{w}_n$ for the unsteady problem.

In this section the normal velocities at the wing surface, which correspond to the acceleration potentials $\overset{s}{\psi}_n$ and $\overset{a}{\psi}_n$ will be determined by means of the relation (3.2.13). In fact we must reduce the two expressions

$$\overset{s}{u}_m = e^{-i\omega x} \lim_{z \to 0} \frac{\partial}{\partial x} \int_{-\infty}^{x} e^{i\omega x} \frac{\mu}{\mu^2+\eta^2} \frac{\sqrt{1-\mu^2}^{\,m}}{\sqrt{1+\eta^2}^{\,m}} \cos m\vartheta \, dx \qquad (3,3,1)$$

and

$$\overset{a}{u}_m = e^{-i\omega x} \lim_{z \to 0} \frac{\partial}{\partial x} \int_{-\infty}^{x} e^{i\omega x} \frac{\mu}{\mu^2+\eta^2} \frac{\sqrt{1-\mu^2}^{\,m}}{\sqrt{1+\eta^2}^{\,m}} \sin m\vartheta \, dx \quad . \qquad (3,3,2)$$

The integration in both formulas extends along a straight line parallel to the x -axis from $-\infty$ to x . For the line element dx on this line we can derive with the aid of the transformation formulas (2,3,1) the three relations:

$$dx = -\frac{\mu}{\sqrt{1-\mu^2}} \sqrt{1+\eta^2} \cos\vartheta \, d\mu + \frac{\eta}{\sqrt{1+\eta^2}} \sqrt{1-\mu^2} \cos\vartheta \, d\eta - \sqrt{1-\mu^2}\sqrt{1+\eta^2} \sin\vartheta \, d\vartheta$$

$$0 = -\frac{\mu}{\sqrt{1-\mu^2}} \sqrt{1+\eta^2} \sin\vartheta \, d\mu + \frac{\eta}{\sqrt{1+\eta^2}} \sqrt{1-\mu^2} \sin\vartheta \, d\eta + \sqrt{1-\mu^2}\sqrt{1+\eta^2} \cos\vartheta \, d\vartheta$$

$$0 = \eta \, d\mu + \mu \, d\eta \qquad (3,3,3)$$

Solving these equations for $d\eta$, we obtain the formula

$$dx = \frac{\mu^2+\eta^2}{\eta} \frac{1}{\sqrt{1-\mu^2}\sqrt{1+\eta^2} \cos\vartheta} \, d\eta \quad . \qquad (3,3,4)$$

Substitution of this expression into the formulas (3,3,1) and (3,3,2) yields respectively

$$\overset{s}{u}_m = e^{-i\omega x} \lim_{z \to 0} \frac{\partial}{\partial x} \int_{\infty}^{\eta} \frac{\mu}{\eta} \frac{\sqrt{1-\mu^2}^{\,m}}{\sqrt{1+\eta^2}^{\,m}} \cos m\vartheta \, \frac{e^{i\omega x}}{x} \, d\eta \qquad (3,3,5)$$

and

$$\overset{a}{u}_m = e^{-i\omega x} \lim_{z\to 0} \frac{\partial}{\partial z} \int_\infty^\eta \frac{\mu}{\eta} \frac{\sqrt{1-\mu^2}^m}{\sqrt{1+\eta^2}^m} \sin m\vartheta \, \frac{e^{i\omega x}}{x} \, d\eta \quad . \tag{3,3,6}$$

It is clear that (3,3,5) is an even function of the variable y and (3,3,6) an odd function of y . Hence it lies at hand to consider the combination

$$\overset{s}{u}_m + i\overset{a}{u}_m = e^{-i\omega x} \lim_{z\to 0} \frac{\partial}{\partial z} \int_\infty^\eta \frac{\mu}{\eta} \frac{\sqrt{1-\mu^2}^m}{\sqrt{1+\eta^2}^m} \cos m\vartheta \, \frac{e^{i\omega x}}{x} \, dy +$$

$$+ ie^{-i\omega x} \lim_{z\to 0} \frac{\partial}{\partial z} \int_\infty^\eta \frac{\mu}{\eta} \frac{\sqrt{1-\mu^2}^m}{\sqrt{1+\eta^2}^m} \sin m\vartheta \, \frac{e^{i\omega x}}{x} \, d\eta \quad . \tag{3,3,7}$$

This expression (3,3,7) can be rewritten into the form

$$\overset{s}{u}_m + i\overset{a}{u}_m = e^{-i\omega x} \lim_{z\to 0} \frac{\partial}{\partial z} \int_\infty^\eta \frac{\mu}{\eta} \frac{\sqrt{1-\mu^2}^m}{\sqrt{1+\eta^2}^m} e^{im\vartheta} \, \frac{e^{i\omega x}}{x} \, d\eta$$

or

$$\overset{s}{u}_m + i\overset{a}{u}_m = e^{-i\omega x} \lim_{z\to 0} \frac{\partial}{\partial z} \int_\infty^\eta \frac{\mu}{\eta} \frac{1}{(1+\eta^2)^m} (x+iy)^m \, \frac{e^{i\omega x}}{x} \, dy \quad . \tag{3,3,8}$$

We introduce now the new variable t , defined by

$$1+\eta^2 = t^2 \quad , \quad t \geq 1 \quad .$$

On the wing surface holds $\eta = 0$ or $t = 1$, whereas $\eta = \infty$ corresponds with $t = \infty$. The line element $d\eta$ becomes

$$d\eta = \frac{t}{\sqrt{t^2-1}} \, dt \quad .$$

Furthermore we have

$$\frac{\mu}{\eta} = \frac{z}{\eta^2} = \frac{z}{t^2-1}$$

and

$$x^2+y^2 = (1-\mu^2)(1+\eta^2) = t^2 \left(1 - \frac{z^2}{\eta^2}\right) = t^2 \left(1 - \frac{z^2}{t^2-1}\right) \quad .$$

With respect to the integration interval in the formulas (3,3,1) and (3,3,2) we can remark that x is always less than zero. Consequently we can write

$$x = -\sqrt{-y^2 + t^2 \left(1 - \frac{z^2}{t^2-1}\right)} \quad .$$

The expression (3,3,8) can finally be written into the form

$$\overset{s}{u}_m + i\overset{a}{u}_m = -e^{-i\omega x} \lim_{z\to 0} \frac{\partial}{\partial z} \int_\infty^\eta \frac{z t^{1-2m}}{\sqrt{t^2-1}^3} \left\{ iy - \sqrt{-y^2 + t^2\left(1-\frac{z^2}{t^2-1}\right)} \right\}^m \frac{e^{-i\omega\sqrt{-y^2+t^2\left(1-\frac{z^2}{t^2-1}\right)}}}{\sqrt{-y^2+t^2\left(1-\frac{z^2}{t^2-1}\right)}} \, dt \quad . \tag{3,3,9}$$

Performing the differentiation under the integral sign and taking the limit $z \to 0$, the expression (3,3,9) transforms into an infinite integral. However, improper integrals of this kind can be treated with the theory of the finite part of an infinite integral, which has been developed by Hadamard (ref.5). Mostly the common integral sign is replaced by the sign $⨍$, which indicates that the finite part of the infinite integral is meant. Instead of (3,3,9) we may thus write

$$\overset{s}{u}_m + i\overset{a}{u}_m = e^{-i\omega x} ⨍_1^\infty \frac{t^{1-2m}}{\sqrt{t^2-1}^3} \left\{ iy - \sqrt{t^2-y^2} \right\}^m \frac{e^{-i\omega\sqrt{t^2-y^2}}}{\sqrt{t^2-y^2}} dt \, . \qquad (3,3,10)$$

Henceforth we substitute into (3,3,10)

$$t^2 - y^2 = s^2 (1-y^2) \quad with \quad s > 0$$

thus obtaining the expression in the more convenient form

$$\overset{s}{u}_m + i\overset{a}{u}_m = e^{-i\omega x} \frac{(-1)^m}{1-y^2} ⨍_1^\infty \frac{1}{\sqrt{s^2-1}^3} \frac{e^{-i\omega s\sqrt{1-y^2}}}{\left\{ s\sqrt{1-y^2} + iy \right\}^m} ds \, . \qquad (3,3,11)$$

In order to be able to evaluate the expression (3,3,11) we consider at first the general form

$$⨍_1^\infty \frac{f(s)}{\sqrt{s^2-1}^3} ds \quad , \qquad (3,3,12)$$

where $f(s)$ is assumed to be a differentiable function of the variable s . The formula (3,3,12) can easily be reduced as follows:

$$⨍_1^\infty \frac{f(s)}{\sqrt{s^2-1}^3} ds = ⨍_1^\infty \frac{f(s)-f(1)}{\sqrt{s^2-1}^3} ds + f(1) ⨍_1^\infty \frac{ds}{\sqrt{s^2-1}^3} = \int_1^\infty \frac{f(s)-f(1)}{\sqrt{s^2-1}^3} ds + f(1) ⨍ \frac{ds}{\sqrt{s^2-1}^3} \, .$$
$$(3,3,13)$$

The second equality in (3,3,13) is justified by the fact that the integrand concerned possesses an integrable singularity. Thus it appears that the main problem becomes the evaluation of the expression

$$⨍_1^\infty \frac{ds}{\sqrt{s^2-1}^3} \qquad (3,3,14)$$

In fact the calculation can be performed using the concept "finite part of an improper integral" , this being defined as a contour integral in the complex domain. Replacing the segment $(1,\infty)$ on the real axis by a circuit C around it, we can write

$$⨍_1^\infty \frac{ds}{\sqrt{s^2-1}^3} = \frac{1}{2} \int_C \frac{ds}{\sqrt{s^2-1}^3} \, . \qquad (3,3,15)$$

By application of Cauchy's theorem we easily see that the contour integral can be transformed into an integral along the imaginary axis, viz.

$$\int_C \frac{ds}{\sqrt{s^2-1}^{\,3}} = \int_{-i\infty}^{+i\infty} \frac{ds}{\sqrt{s^2-1}^{\,3}} = -\int_{-\infty}^{+\infty} \frac{dy}{\sqrt{y^2+1}^{\,3}} = -2 \quad . \tag{3,3,16}$$

Hence

$$\not\int_1^{\infty} \frac{ds}{\sqrt{s^2-1}^{\,3}} = -1 \quad . \tag{3,3,17}$$

Equation (3,3,13) thus becomes

$$\not\int_1^{\infty} \frac{f(s)}{\sqrt{s^2-1}^{\,3}}\, ds = \int_1^{\infty} \frac{f(s)-f(1)}{\sqrt{s^2-1}^{\,3}} - f(1)$$

or

$$\not\int_1^{\infty} \frac{f(s)}{\sqrt{s^2-1}^{\,3}}\, ds = -\int_1^{\infty} \left\{f(s)-f(1)\right\} \frac{d}{ds} \frac{s}{\sqrt{s^2-1}}\, ds - f(1) =$$

$$= -\left\{f(s)-f(1)\right\} \frac{s}{\sqrt{s^2-1}} \Bigg|_{s=1}^{s=\infty} + \int_1^{\infty} \frac{s}{\sqrt{s^2-1}} f'(s)\, ds - f(1) =$$

$$= -f(\infty) + \int_1^{\infty} \frac{s}{\sqrt{s^2-1}} f'(s)\, ds \quad . \tag{3,3,18}$$

The assumption that $f(s)$ vanishes at infinity, simplifies the relation (3,3,18) to

$$\not\int_1^{\infty} \frac{f(s)}{\sqrt{s^2-1}^{\,3}}\, ds = \not\int_1^{\infty} \frac{s}{\sqrt{s^2-1}} f'(s)\, ds \quad , \tag{3,3,19}$$

In our case the function $f(s)$ takes the form

$$f(s) = \frac{e^{-i\omega s\sqrt{1-y^2}}}{\left\{s\sqrt{1-y^2}+iy\right\}^m} \quad . \tag{3,3,20}$$

If $\omega = \frac{\nu}{U}$ is a pure real number, the condition that $f(\infty)$ is equal to zero is only fulfilled for $m > 0$. However, in unsteady aerofoil theory the assumption is mostly made that the imaginary part of ω is less than zero, thus ascertaining that integrals concerning the wake become convergent. In this way we still satisfy the requirement that $f(\infty)$ vanishes in the case $m=0$. According to (3,3,19) the expression (3,3,11) can be replaced by

$$\overset{s}{u}_m + i\overset{a}{u}_m = e^{-i\omega x} \frac{(-1)^m}{1-y^2} \int_1^{\infty} \frac{s}{\sqrt{s^2-1}} \frac{d}{ds} \left\{ \frac{e^{-i\omega s\sqrt{1-y^2}}}{\left(s\sqrt{1-y^2}+iy\right)^m} \right\} ds \quad . \tag{3,3,21}$$

This formula can now be rewritten into the form

$$\overset{s}{u}_m + i\overset{a}{u}_m = e^{-i\omega x} \frac{(-1)^{m+1}}{\sqrt{1-y^2}} \int_1^\infty \frac{s}{\sqrt{s^2-1}} \left\{ \frac{i\omega}{\left(s\sqrt{1-y^2}+iy\right)^m} + \right.$$

$$\left. + \frac{m}{\left(s\sqrt{1-y^2}+iy\right)^{m+1}} \right\} e^{-i\omega s\sqrt{1-y^2}}\, ds \quad . \qquad (3,3,22)$$

The fundamental integral, which has to be evaluated, is

$$I_m(\omega) = \int_1^\infty \frac{s}{\sqrt{s^2-1}} \frac{e^{-i\omega s\sqrt{1-y^2}}}{\left(s\sqrt{1-y^2}+iy\right)^m}\, ds \; . \qquad (3,3,23)$$

Putting $s = \cosh t$, we obtain the formula

$$I_m(\omega) = \int_0^\infty \cosh t \, \frac{e^{-i\omega\sqrt{1-y^2}\cosh t}}{\left(\sqrt{1-y^2}\cosh t + iy\right)^m}\, dt \; . \qquad (3,3,24)$$

The following reduction can now be made

$$I_{m+1}(\omega) = \int_0^\infty \cosh t \, \frac{e^{-i\omega\sqrt{1-y^2}\cosh t}}{\left(\sqrt{1-y^2}\cosh t + iy\right)^{m+1}}\, dt =$$

$$= e^{-\omega y} \int_0^\infty \cosh t \, \frac{e^{-i\omega\left(\sqrt{1-y^2}\cosh t + iy\right)}}{\left(\sqrt{1-y^2}\cosh t + iy\right)^{m+1}}\, dt \; .$$

Furthermore the relation holds

$$\frac{d}{d\omega}\left\{ e^{\omega y} I_{m+1}(\omega) \right\} = -i \int_0^\infty \cosh t \, \frac{e^{-i\omega\left(\sqrt{1-y^2}\cosh t + iy\right)}}{\left(\sqrt{1-y^2}\cosh t + iy\right)^m}\, dt = -ie^{\omega y} I_m(\omega) \; .$$

Hence

$$e^{\omega y} I_{m+1}(\omega) = -i \int_0^\omega e^{\omega y} I_m(\omega)\, d\omega + I_{m+1}(0) \; . \qquad (3,3,25)$$

Applying this relation (3,3,25) repeatedly we finally obtain the formula

$$I_{m+1}(\omega) = (-i)^{m+1} e^{-\omega y} \underbrace{\int_0^\omega d\omega \int_0^\omega d\omega -- \int_0^\omega}_{(m+1)\text{ integrals}} e^{\omega y} I_0(\omega)\, d\omega + e^{-\omega y} \sum_{j=0}^m (-i)^j \frac{\omega^j}{j!} I_{m+1-j}(0) \; . \qquad (3,3,26)$$

By partial integration we find

$$I_{m+1}(\omega) = (-i)^{m+1} \frac{e^{-\omega y}}{m!} \int_0^\omega (\omega-\omega_1)^m e^{\omega_1 y} I_0(\omega_1)\, d\omega_1 +$$

$$+ e^{-\omega y} \sum_{j=0}^m (-i)^j \frac{\omega^j}{j!} I_{m+1-j}(0) \; . \qquad (3,3,27)$$

Using the well-known relation (ref.26)

$$I_0(\omega)=\int_0^\infty \cosh t \, e^{-i\omega\sqrt{1-y^2}\cosh t}\,dt=-\frac{\pi}{2}H_1^{(2)}\left(\omega\sqrt{1-y^2}\right), \qquad (3,3,28)$$

in which $H_1^{(2)}$ represents the Hankel-function of the second kind, we obtain the formula

$$I_{m+1}(\omega)=(-i)^{m-1}\frac{\pi}{2}\frac{e^{-\omega y}}{m!}\int_0^\omega(\omega-\omega_1)^m e^{\omega_1 y}H_1^{(2)}\left(\omega_1\sqrt{1-y^2}\right)d\omega_1+$$

$$+e^{-\omega y}\sum_{j=0}^{m}(-i)^j\frac{\omega^j}{j!}I_{m+1-j}(0). \qquad (3,3,29)$$

The only problem which remains is to determine the value of $I_m(0)$ for $m \geqq 1$. We can write

$$I_m(0)=\int_0^\infty\frac{\cosh t}{\left(\sqrt{1-y^2}\cosh t+iy\right)^m}dt \qquad (3,3,30)$$

In order to obtain an expression for this integral in terms of known functions, use will be made of the integral representation of the associated Legendre function of the second kind, viz.

$$Q_\nu^\mu(x)=e^{i\mu\pi}\frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\int_0^\infty\frac{\cosh\mu t}{\left\{x+(x^2-1)^{1/2}\cosh t\right\}^{\nu+1}}dt \qquad (3,3,31)$$

wherein $\mathrm{Re}(\nu\pm\mu)>-1, \nu\neq-1,-2,-3,-\ldots\ldots$ whilst x is not situated on the real axis between $+1$ and $-\infty$ (ref.2). In particular we put $x=y-io$, with $-1\leqq y\leqq+1$, into the formula (3,3,31). Then we obtain

$$Q_\nu^\mu(y-io)=e^{i\mu\pi}\frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\int_0^\infty\frac{\cosh\mu t}{\left\{y-i\sqrt{1-y^2}\cosh t\right\}^{\nu+1}}dt. \qquad (3,3,32)$$

Moreover use is made of the important relation (ref.2)

$$Q_\nu^\mu(y-io)=e^{1/2\,i\mu\pi}\left\{Q_\nu^\mu(x)+\frac{\pi i}{2}P_\nu^\mu(x)\right\}. \qquad (3,3,33)$$

Thus

$$Q_\nu^\mu(y)+\frac{\pi i}{2}P_\nu^\mu(y)=e^{1/2\,i\mu\pi}\frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\int_0^\infty\frac{\cosh\mu t}{\left\{y-i\sqrt{1-y^2}\cosh t\right\}^{\nu+1}}dt. \qquad (3,3,34)$$

For $\mu=1$ and $\nu=m-1$ we thus find

$$Q'_{m-1}(y)+\frac{\pi i}{2}P'_{m-1}(y)=i(m-1)\int_0^\infty\frac{\cosh t}{\left\{y-i\sqrt{1-y^2}\cosh t\right\}^m}dt=$$

$$= i^{m+1}(m-1)\int_0^{\infty} \frac{\cosh t}{\left(\sqrt{1-y^2}\cosh t + iy\right)^m}\, dt \ . \tag{3,3,35}$$

The two formulas (3,3,30) and (3,3,35) together yield

$$I_m(0) = \frac{(-i)^{m+1}}{m-1}\left\{ Q'_{m-1}(y) + \frac{\pi i}{2} P'_{m-1}(y) \right\} \ . \tag{3,3,36}$$

By aid of the definitions of $Q_n^m(x)$ and $P_n^m(x)$, we can write

$$I_m(0) = \frac{(-i)^{m-1}}{m-1}\sqrt{1-y^2}\left\{ \frac{dQ_{m-1}}{dy} + \frac{\pi i}{2}\frac{dP_{m-1}}{dy} \right\} \ . \tag{3,3,37}$$

Let us now return to the expression (3,3,7). Using the foregoing considerations, we can easily derive that we have for $m \geq 1$

$$\overset{s}{u}_m + i\overset{a}{u}_m = e^{-i\omega x} \lim_{z\to 0} \frac{\partial}{\partial z}\int_{\infty}^{\eta} \frac{\mu}{\eta}\frac{\sqrt{1-\mu^2}^{\,m}}{\sqrt{1+\eta^2}^{\,m}}\cos m\vartheta\,\frac{e^{i\omega x}}{x}\, d\eta +$$

$$+ ie^{-i\omega x} \lim_{z\to 0} \frac{\partial}{\partial z}\int_{\infty}^{\eta} \frac{\mu}{\eta}\frac{\sqrt{1-\mu^2}^{\,m}}{\sqrt{1+\eta^2}^{\,m}}\sin m\vartheta\,\frac{e^{i\omega x}}{x}\, d\eta =$$

$$= e^{-i\omega x}\frac{(-1)^{m+1}}{\sqrt{1-y^2}}\left\{ i\omega I_m(\omega) + m I_{m+1}(\omega) \right\} =$$

$$= e^{-i\omega x}\frac{(-1)^{m+1}}{\sqrt{1-y^2}}\Bigg[i\omega (-i)^{m-2}\frac{\pi}{2}\frac{e^{-\omega y}}{(m-1)!}\int_0^{\omega}(\omega-\omega_1)^{m-1} e^{\omega_1 y} H_1^{(2)}\left(\omega_1\sqrt{1-y^2}\right) d\omega_1 +$$

$$+ m(-i)^{m-1}\frac{\pi}{2}\frac{e^{-\omega y}}{m!}\int_0^{\omega}(\omega-\omega_1)^m e^{\omega_1 y} H_1^{(2)}\left(\omega_1\sqrt{1-y^2}\right) d\omega_1 +$$

$$+ i\omega e^{-\omega y}\sum_{j=0}^{m-1}(-i)^j\frac{\omega^j}{j!} I_{m-j}(0) + m e^{-\omega y}\sum_{j=0}^{m}(-i)^j\frac{\omega^j}{j!} I_{m+1-j}(0) \Bigg] =$$

$$= e^{-i\omega x}\frac{(-1)^{m+1}}{\sqrt{1-y^2}}\Bigg[-(-i)^{m-1}\frac{\pi}{2}\frac{e^{-\omega y}}{(m-1)!}\int_0^{\omega}\omega_1(\omega-\omega_1)^{m-1} e^{\omega_1 y} H_1^{(2)}\left(\omega_1\sqrt{1-y^2}\right) d\omega_1 +$$

$$+ i\omega e^{-\omega y}\sum_{j=0}^{m-2}(-i)^j\ \frac{\omega^j}{j!}\ I_{m-j}(0)+me^{-\omega y}\sum_{j=0}^{m-1}(-i)^j\ \frac{\omega^j}{j!}\ I_{m+1-j}(0)\Bigg] =$$

$$= e^{-i\omega x}\frac{(-1)^{m+1}}{\sqrt{1-y^2}}\Bigg[-(-i)^{m-1}\ \frac{\pi}{2}\ \frac{e^{-\omega y}}{(m-1)!}\int_0^\omega \omega_1\ (\omega-\omega_1)^{m-1}e^{\omega_1 y}H_1^{(2)}\left(\omega_1\sqrt{1-y^2}\right)d\omega_1 +$$

$$+ e^{-\omega y}\sum_{j=0}^{m-1}(-i)^j\ \frac{\omega^j}{j!}\ (m-j)\ I_{m+1-j}(0)\Bigg] =$$

$$= e^{-i\omega x}\ \frac{(-1)^{m+1}}{\sqrt{1-y^2}}\Bigg[(-1)^{m+1}\ \frac{\pi}{2}\ \frac{e^{-\omega y}}{(m-1)!}\int_0^\omega \omega_1\,(\omega-\omega_1)^{m-1}e^{\omega_1 y}H_1^{(2)}\left(\omega_1\sqrt{1-y^2}\right)d\omega_1 +$$

$$+ e^{-\omega y}\sum_{j=0}^{m-1}\ (-i)^m\ \frac{\omega^j}{j!}\sqrt{1-y^2}\left\{\frac{dQ_{m-j}}{dy}+\frac{\pi i}{2}\ \frac{dP_{m-j}}{dy}\right\}\Bigg] =$$

$$= e^{-i\omega x}\frac{(-1)^{m-1}}{\sqrt{1-y^2}}\Bigg[(-i)^{m+1}\ \frac{\pi}{2}\ \frac{1}{(m-1)!}\int_0^\omega \omega_1\ (\omega-\omega_1)^{m-1}\ \Big\{\cosh\ (\omega_1-\omega)\,y +$$

$$\sinh\,(\omega_1-\omega)\,y\Big\}\ H_1^{(2)}\left(\omega_1\sqrt{1-y^2}\right)d\omega_1 +$$

$$+ \left(\cosh\ \omega y-\sinh\,\omega y\right)\ (-i)^m\sqrt{1-y^2}\sum_{j=0}^{m-1}\ \frac{\omega^j}{j!}\ \left\{\frac{dQ_{m-j}}{dy}+\frac{\pi i}{2}\ \frac{dP_{m-j}}{dy}\right\}\Bigg] =$$

$$= e^{-i\omega x}\ \frac{(-1)^{m-1}}{\sqrt{1-y^2}}\Bigg[(-i)^{m+1}\ \frac{\pi}{2}\ \frac{1}{(m-1)!}\int_0^\omega \omega_1\ (\omega-\omega_1)^{m-1}\ \Big\{\cosh\,(\omega_1-\omega)\ y +$$

$$+ \sinh\,(\omega_1-\omega)\,y\Big\}\ H_1^{(2)}\ \left(\omega_1\sqrt{1-y^2}\right)d\omega_1 +$$

$$+ \left(\cosh\ \omega y-\sinh\ \omega y\right)\ (-i)^m\sqrt{1-y^2}\sum_{j=1}^{m}\ \frac{\omega^{m-j}}{(m-j)!}\left\{\frac{dQ_j}{dy}+\frac{\pi i}{2}\ \frac{dP_j}{dy}\right\}\Bigg] .$$

$$(3,3,38)$$

It is rather easy to split expression (3,3,38) into a symmetric part and an anti-symmetric part in connection to the coordinate y . After some elementary calculations we obtain the relations

$$\overset{s}{u}_m = e^{-i\omega x}\ \lim_{z\to 0}\ \frac{\partial}{\partial z}\int_{-\infty}^{x} e^{\,i\omega x}\ \frac{\mu}{\mu^2+\eta^2}\ \frac{\sqrt{1-\mu^2}^{\,m}}{\sqrt{1+\eta^2}^{\,m}}\cos m\vartheta\ dx =$$

$$e^{-i\omega x}\ \frac{1}{\sqrt{1-y^2}}\Bigg[\ i^{m+1}\ \frac{\pi}{2}\ \frac{1}{(m-1)!}\int_0^\omega\ \omega_1\ (\omega-\omega_1)^{m-1}\cosh\,(\omega_1-\omega)y\ H_1^{(2)}\left(\omega_1\sqrt{1-y^2}\right)d\omega_1 +$$

$$-i^m\sqrt{1-y^2}\sum_{j=1}^{\left[\frac{m}{2}\right]}\frac{\omega^{m-2j}}{(m-2j)!}\left\{\frac{dQ_{2j}}{dy}\cosh\omega y-\frac{\pi i}{2}\frac{dP_{2j}}{dy}\sinh\omega y\right\}+$$

$$+i^m\sqrt{1-y^2}\sum_{j=0}^{\left[\frac{m-1}{2}\right]}\frac{\omega^{m-2j-1}}{(m-2j-1)!}\left\{\frac{dQ_{2j+1}}{dy}\sinh\omega y-\frac{\pi i}{2}\frac{dP_{2j+1}}{dy}\cosh\omega y\right\}\Bigg]\quad \text{for } m\geqq 1 \tag{3,3,39}$$

and

$$\overset{a}{u}_m=e^{-i\omega x}\lim_{z\to 0}\frac{\partial}{\partial z}\int_{-\infty}^{x}e^{i\omega x}\frac{\mu}{\mu^2+\eta^2}\frac{\sqrt{1-\mu^2}^{\,m}}{\sqrt{1+\eta^2}^{\,m}}\sin m\vartheta\, dx=$$

$$e^{-i\omega x}\frac{1}{\sqrt{1-y^2}}\Bigg[i^m\frac{\pi}{2}\frac{1}{(m-1)!}\int_0^{\omega}\omega_1(\omega-\omega_1)^{m-1}\sinh(\omega_1-\omega)y\; H_1^{(2)}\left(\omega_1\sqrt{1-y^2}\right)d\omega_1+$$

$$-i^{m+1}\sqrt{1-y^2}\sum_{j=1}^{\left[\frac{m}{2}\right]}\frac{\omega^{m-2j}}{(m-2j)!}\left\{\frac{dQ_{2j}}{dy}\sinh\omega y-\frac{\pi i}{2}\frac{dP_{2j}}{dy}\cosh\omega y\right\}+$$

$$+i^{m+1}\sqrt{1-y^2}\sum_{j=0}^{\frac{m-1}{2}}\frac{\omega^{m-2j-1}}{(m-2j-1)!}\left[\frac{dQ_{2j+1}}{dy}\cosh\omega y-\frac{\pi i}{2}\frac{dP_{2j+1}}{dy}\sinh\omega y\right]\Bigg]\quad\text{for } m\geqq 1 \tag{3,3,40}$$

The symbol $[n]$ denotes the largest integer less than or equal to n. For $m=0$ we have

$$\overset{s}{u}_0+i\overset{a}{u}_0=\overset{s}{u}_0=e^{-i\omega x}\lim_{z\to 0}\frac{\partial}{\partial z}\int_{\infty}^{\eta}\frac{\mu}{\eta}\frac{e^{i\omega x}}{x}d\eta=-i\omega e^{-i\omega x}\frac{1}{\sqrt{1-y^2}}I_0(\omega)=$$

$$=\frac{1}{2}i\omega\pi e^{-i\omega x}\frac{1}{\sqrt{1-y^2}}H_1^{(2)}\left(\omega\sqrt{1-y^2}\right) \tag{3,3,41}$$

The formulas (3,3,39), (3,3,40) and (3,3,41) enable us to determine the downwashes $\overset{s}{w}_m$ and $\overset{a}{w}_m$ by means of the relations

$$\begin{aligned}\overset{s}{w}_m&=-\frac{1}{2\pi^2}\left(\overset{s}{u}_{m-1}+\overset{s}{u}_{m+1}\right)\quad\text{for } m\geqq 1\\ \overset{s}{w}_0&=-\frac{1}{2\pi^2}\overset{s}{u}_1\\ \overset{a}{w}_m&=-\frac{1}{2\pi^2}\left(\overset{a}{u}_{m-1}+\overset{a}{u}_{m+1}\right)\end{aligned} \tag{3,3,42}$$

which follow from the definitions (2,10,10).

4 Approximation for low frequencies.

Because of the complicated nature of the downwash distributions, given in the formulas (3,3,39) and (3,3,40), it can be expected that the necessary calculations for the determination of the forces and moments, which occur for some prescribed harmonic oscillations, will be rather cumbersome. In order to limit the required numerical work, we restrict ourselves to the case of harmonic oscillations of low frequencies. In fact we still take into account terms of order ω , but neglect terms of higher order in ω . This means that the results of our calculations will be of interest for dynamic stability investigations.

The first thing to be done is to expand the downwashes $\overset{s}{u}_m$ and $\overset{a}{u}_m$ to the parameter ω . We easily find the following approximations to the first order of ω :

$$
\begin{aligned}
e^{i\omega x}\,\overset{s}{u}_{2m} &= (-1)^{m+1}\frac{dQ_{2m}}{dy} + i\omega\frac{\pi}{2}(-1)^m 2m P_{2m} \qquad \text{for } m \geq 1 \\
e^{i\omega x}\,\overset{s}{u}_{0} &= -\frac{dQ_0}{dy} \\
e^{i\omega x}\,\overset{s}{u}_{2m+1} &= (-1)^m\frac{\pi}{2}\frac{dP_{2m+1}}{dy} + i\omega(-1)^m(2m+1)Q_{2m+1} \\
e^{i\omega x}\,\overset{a}{u}_{2m} &= (-1)^{m+1}\frac{\pi}{2}\frac{dP_{2m}}{dy} + i\omega(-1)^{m+1}2mQ_{2m} \\
e^{i\omega x}\,\overset{a}{u}_{2m+1} &= (-1)^{m+1}\frac{dQ_{2m+1}}{dy} + i\omega\frac{\pi}{2}(-1)^m(2m+1)P_{2m+1} \quad .
\end{aligned}
\tag{3,4,1}
$$

where use has been made of the well-known formula

$$
y\frac{dQ_n}{dy} - Q_{n-1} = nQ_n \quad . \tag{3,4,2}
$$

With the aid of (3,3,42) we obtain the approximations

$$
\begin{aligned}
e^{i\omega x}\,\overset{s}{w}_{2m} &= \frac{(-1)^{m+1}}{4\pi}(4m+1)P_{2m} + \frac{(-1)^m}{2\pi^2}\,i\omega\left\{(2m-1)Q_{2m-1} - (2m+1)Q_{2m+1}\right\} \\
e^{i\omega x}\,\overset{s}{w}_{0} &= -\frac{1}{4\pi}P_0 - \frac{i\omega}{2\pi^2}Q_1 \\
e^{i\omega x}\,\overset{s}{w}_{2m+1} &= \frac{(-1)^{m+1}}{2\pi^2}(4m+3)Q_{2m+1}\ \frac{(-1)^{m+1}}{4\pi}\,i\omega\left\{2mP_{2m} - (2m+2)P_{2m+2}\right\} \\
e^{i\omega x}\,\overset{a}{w}_{2m} &= \frac{(-1)^m}{2\pi^2}(4m+1)\,Q_{2m} + \frac{(-1)^m}{4\pi}\,i\omega\left\{(2m-1)P_{2m-1} - (2m+1)P_{2m+1}\right\} \\
e^{i\omega x}\,\overset{a}{w}_{2m+1} &= \frac{(-1)^{m+1}}{4\pi}(4m+3)P_{2m+1} + \frac{(-1)^m}{2\pi^2}\,i\omega\left\{2mQ_{2m} - (2m+2)Q_{2m+2}\right\} \quad .
\end{aligned}
\tag{3,4,3}
$$

The zeroth order terms in ω are just the downwashes in the steady case; they agree completely with the formulas, derived in section 11 of chapter II. These approximations for the downwash distributions $\overset{s}{w}_n$ and $\overset{a}{w}_n$ enable us to find approximate values for the unknown coefficients a_n and b_n in the equations (3,2,18) and (3,2,19) resp., up to the first order in the reduced frequency ω . Therefore we assume for the coefficients concerned

$$
\begin{aligned}
c_n &= c_n^{(0)} + i\omega c_n^{(1)} \quad , & d_n &= d_n^{(0)} + i\omega d_n^{(1)} \\
a_n &= a_n^{(0)} + i\omega a_n^{(1)} \quad , & b_n &= b_n^{(0)} + i\omega b_n^{(1)} \quad .
\end{aligned}
\tag{3,4,4}
$$

whereas the downwashes can be represented by

$$\overset{s}{w}_n = \overset{s}{w}{}_n^{(0)} + i\omega \overset{s}{w}{}_n^{(1)} \quad , \qquad \overset{a}{w}_n = \overset{a}{w}{}_n^{(0)} + i\omega \overset{a}{w}{}_n^{(1)} \quad . \tag{3,4,5}$$

Substitution of the expressions (3,4,4) and (3,4,5) into (3,2,18) and (3,2,19) yields

$$\pi \sum_{n=0}^{\infty} \varepsilon_n \left(c_n^{(0)} + i\omega c_n^{(1)}\right)\left(\overset{s}{w}{}_n^{(0)} + i\omega \overset{s}{w}{}_n^{(1)}\right) + \sum_{n=0}^{\infty}\left(a_n^{(0)} + i\omega a_n^{(1)}\right)\left\{p_{2n}^{2n}\left(\overset{s}{w}{}_{2n}^{(0)} + i\omega \overset{s}{w}{}_{2n}^{(1)}\right) + \right.$$

$$\left. + \sum_{m=0}^{\infty} p_{2n}^{2m+1}\left(\overset{s}{w}{}_{2m+1}^{(0)} + i\omega \overset{s}{w}{}_{2m+1}^{(1)}\right)\right\} = 0 \tag{3,4,6}$$

and

$$\pi \sum_{n=1}^{\infty}\left(d_n^{(0)} + i\omega d_n^{(1)}\right)\left(\overset{a}{w}{}_n^{(0)} + i\omega \overset{a}{w}{}_n^{(1)}\right) + \sum_{n=0}^{\infty}\left(b_n^{(0)} + i\omega b_n^{(1)}\right)\left\{q_{2n+1}^{2n+1}\left(\overset{a}{w}{}_{2n+1}^{(0)} + i\omega \overset{a}{w}{}_{2n+1}^{(1)}\right) + \right.$$

$$\left. + \sum_{m=1}^{\infty} q_{2n+1}^{2m}\left(\overset{a}{w}{}_{2m}^{(0)} + i\omega \overset{a}{w}{}_{2m}^{(1)}\right)\right\} = 0 \ . \tag{3,4,7}$$

Equating the terms, which are constant in the parameter ω , and thereupon the terms, which contain $i\omega$ only, we obtain the system of equations

$$\pi \sum_{n=0}^{\infty} \varepsilon_n c_n^{(0)} \overset{s}{w}{}_n^{(0)} + \sum_{n=0}^{\infty} a_n^{(0)} \left\{p_{2n}^{2n} \overset{s}{w}{}_{2n}^{(0)} + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \overset{s}{w}{}_{2m+1}^{(0)}\right\} = 0$$

$$\pi \sum_{n=0}^{\infty} \varepsilon_n c_n^{(0)} \overset{s}{w}{}_n^{(1)} + \pi \sum_{n=0}^{\infty} \varepsilon_n c_n^{(1)} \overset{s}{w}{}_n^{(0)} + \sum_{n=0}^{\infty} a_n^{(0)} \left\{p_{2n}^{2n} \overset{s}{w}{}_{2n}^{(1)} + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \overset{s}{w}{}_{2m+1}^{(1)}\right\} +$$

$$+ \sum_{n=0}^{\infty} a_n^{(1)} \left\{p_{2n}^{2n} \overset{s}{w}{}_{2n}^{(0)} + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \overset{s}{w}{}_{2m+1}^{(0)}\right\} = 0 \tag{3,4,8}$$

and

$$\pi \sum_{n=1}^{\infty} d_n^{(0)} \overset{a}{w}{}_n^{(0)} + \sum_{n=0}^{\infty} b_n^{(0)} \left\{q_{2n+1}^{2n+1} \overset{a}{w}{}_{2n+1}^{(0)} + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \overset{a}{w}{}_{2m}^{(0)}\right\} = 0$$

$$\pi \sum_{n=1}^{\infty} d_n^{(0)} \overset{a}{w}{}_n^{(1)} + \pi \sum_{n=1}^{\infty} d_n^{(1)} \overset{a}{w}{}_n^{(0)} + \sum_{n=0}^{\infty} b_n^{(0)} \left\{q_{2n+1}^{2n+1} \overset{a}{w}{}_{2n+1}^{(1)} + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \overset{a}{w}{}_{2m}^{(1)}\right\} +$$

$$+ \sum_{n=0}^{\infty} b_n^{(1)} \left\{q_{2n+1}^{2n+1} \overset{a}{w}{}_{2n+1}^{(0)} + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \overset{a}{w}{}_{2m}^{(0)}\right\} = 0 \ . \tag{3,4,9}$$

The first equation in either of the systems is exactly equal to the corresponding equation in the steady case. This means that the coefficients $a_n^{(0)}$ and $b_n^{(0)}$ can be immediately derived from the steady theory. If these zeroth order coefficients have been evaluated, they can be substituted into the second equation of the systems (3,4,8) and (3,4,9). In this way these second equations become equations for the unknowns $c_n^{(1)}$ and $d_n^{(1)}$ respectively, which can be solved in a analogous way as has been followed in the steady case. The equation for the symmetric problems can be written in the form

$$\sum_{n=0}^{\infty} a_n^{(1)} \left\{ p_{2n}^{2n} \overset{s}{w}{}_{2n}^{(0)} + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \overset{s}{w}{}_{2m+1}^{(0)} \right\} = -\pi \sum_{n=0}^{\infty} \varepsilon_n c_n^{(1)} \overset{s}{w}{}_n^{(0)} - \pi \sum_{n=0}^{\infty} \varepsilon_n c_n^{(0)} \overset{s}{w}{}_n^{(1)} +$$

$$- \sum_{n=0}^{\infty} a_n^{(0)} \left\{ p_{2n}^{2n} \overset{s}{w}{}_{2n}^{(1)} + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \overset{s}{w}{}_{2m+1}^{(1)} \right\} \qquad (3,4,10)$$

and for the anti-symmetrical problems as

$$\sum_{n=0}^{\infty} b_n^{(1)} \left\{ q_{2n+1}^{2n+1} \overset{a}{w}{}_{2n+1}^{(0)} + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \overset{a}{w}{}_{2m}^{(0)} \right\} = -\pi \sum_{n=1}^{\infty} d_n^{(1)} \overset{a}{w}{}_n^{(0)} - \pi \sum_{n=1}^{\infty} d_n^{(0)} \overset{a}{w}{}_n^{(1)} +$$

$$- \sum_{n=0}^{\infty} b_n^{(0)} \left\{ q_{2n+1}^{2n+1} \overset{a}{w}{}_{2n+1}^{(1)} + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \overset{a}{w}{}_{2m}^{(1)} \right\} . \qquad (3,4,11)$$

In order to transform the identity (3,4,10) into an infinite system of linear equations we multiply both sides of (3,4,10) with the Legendre polynomial $P_{2n}(y)$ $(n=0,1,2,\ldots)$ and integrate thereupon over the variable y from -1 to $+1$. We obtain in this manner

$$\sum_{n=0}^{\infty} a_n^{(1)} \left\{ p_{2n}^{2n} \int_{-1}^{+1} \overset{s}{w}{}_{2n}^{(0)} P_{2\ell}(y)\,dy + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \int_{-1}^{+1} \overset{s}{w}{}_{2m+1}^{(0)} P_{2\ell}(y)\,dy \right\} =$$

$$= -\pi \sum_{n=0}^{\infty} \varepsilon_n c_n^{(1)} \int_{-1}^{+1} \overset{s}{w}{}_n^{(0)} P_{2\ell}(y)\,dy - \pi \sum_{n=0}^{\infty} \varepsilon_n c_n^{(0)} \int_{-1}^{+1} \overset{s}{w}{}_n^{(1)} P_{2\ell}(y)\,dy +$$

$$- \sum_{n=0}^{\infty} a_n^{(0)} \left\{ p_{2n}^{2n} \int_{-1}^{+1} \overset{s}{w}{}_{2n}^{(1)} P_{2\ell}(y)\,dy + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \int_{-1}^{+1} \overset{s}{w}{}_{2m+1}^{(1)} P_{2\ell}(y)\,dy \right\} \quad \text{for } \ell = 0,1,2,\ldots \qquad (3,4,12)$$

In (2,10,16) of section 10 of chapter II we have already introduced the abbreviations

$$p_{2n}^{2n} \int_{-1}^{+1} \overset{s}{w}{}_{2n}^{(0)} P_{2\ell}(y)\,dy + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \int_{-1}^{+1} \overset{s}{w}{}_{2m+1}^{(0)} P_{2\ell}(y)\,dy = \overset{s}{\tau}{}_\ell^n$$

$$-\pi \varepsilon_n \int_{-1}^{+1} \overset{s}{w}{}_n^{(0)} P_{2\ell}(y)\,dy = \overset{s}{\sigma}{}_\ell^n \quad .$$

Furthermore we put

$$-\pi \varepsilon_n \int_{-1}^{+1} \overset{s}{w}{}_n^{(1)} P_{2\ell}(y)\,dy = \overset{s}{\mu}{}_\ell^n$$

and

$$p_{2n}^{2n} \int_{-1}^{+1} \overset{s}{w}{}_{2n}^{(1)} P_{2\ell}(y)\,dy + \sum_{m=0}^{\infty} p_{2n}^{2m+1} \int_{-1}^{+1} \overset{s}{w}{}_{2m+1}^{(1)} P_{2\ell}(y)\,dy = \overset{s}{\nu}{}_\ell^n \quad . \qquad (3,4,13)$$

The system of equations for the symmetric problems can thus be written as

$$\sum_{n=0}^{\infty} \overset{s}{\tau}{}_{\ell}^{n} a_n^{(1)} = \sum_{n=0}^{\infty} \overset{s}{\sigma}{}_{\ell}^{n} c_n^{(1)} + \sum_{n=0}^{\infty} \overset{s}{\mu}{}_{\ell}^{n} c_n^{(0)} + \sum_{n=0}^{\infty} \overset{s}{\nu}{}_{\ell}^{n} a_n^{(0)} \qquad \ell = 0, 1, 2, \ldots \tag{3,4,14}$$

For the anti-symmetric case we multiply the identity (3,4,12) with the Legendre polynomials P_{2k+1} $(k = 0, 1, 2, \ldots)$ and introduce again the quantities

$$\overset{a}{\tau}{}_{k}^{n} = q_{2n+1}^{2n+1} \int_{-1}^{+1} \overset{a}{w}{}_{2n+1}^{(0)} P_{2k+1}(y)\, dy + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \int_{-1}^{+1} \overset{a}{w}{}_{2m}^{(0)} P_{2k+1}(y)\, dy$$

$$\overset{a}{\sigma}{}_{k}^{n} = -\pi \int_{-1}^{+1} \overset{a}{w}{}_{n}^{(0)} P_{2k+1}(y)\, dy$$

$$\overset{a}{\mu}{}_{k}^{n} = -\pi \int_{-1}^{+1} \overset{a}{w}{}_{n}^{(1)} P_{2k+1}(y)\, dy \tag{3,4,15}$$

$$\overset{a}{\nu}{}_{k}^{n} = q_{2n+1}^{2n+1} \int_{-1}^{+1} \overset{a}{w}{}_{2n+1}^{(1)} P_{2k+1}(y)\, dy + \sum_{m=1}^{\infty} q_{2n+1}^{2m} \int_{-1}^{+1} \overset{a}{w}{}_{2m}^{(1)} P_{2k+1}(y)\, dy \quad .$$

The system of linear equations reads now

$$\sum_{n=0}^{\infty} \overset{a}{\tau}{}_{k}^{n} b_n^{(1)} = \sum_{n=1}^{\infty} \overset{a}{\sigma}{}_{k}^{n} d_n^{(1)} + \sum_{n=1}^{\infty} \overset{a}{\mu}{}_{k}^{n} d_n^{(0)} + \sum_{n=0}^{\infty} \overset{a}{\nu}{}_{k}^{n} b_n^{(0)} \qquad k = 0, 1, 2, \ldots \tag{3,4,16}$$

In order to find approximate values of the unknown coefficients $a_n^{(1)}$ and $b_n^{(1)}$ it is necessary to truncate the infinite systems (3,4,14) and (3,4,16) respectively, thus obtaining finite systems of linear algebraic equations, which can be solved.

5 Calculation of the coefficients.

For the evaluation of the coefficients $\overset{s}{\mu}{}_{\ell}^{n}$, $\overset{s}{\nu}{}_{\ell}^{n}$, $\overset{a}{\mu}{}_{k}^{n}$ and $\overset{a}{\nu}{}_{k}^{n}$, use will be made of some relations, which have been derived in section 12 of chapter II. We begin with the simplest coefficients, viz. $\overset{s}{\mu}{}_{\ell}^{n}$ and $\overset{a}{\mu}{}_{k}^{n}$. According to their definitions and the relations (3,4,3) we can write

$$\overset{s}{\mu}{}_{\ell}^{2n} = -\pi \int_{-1}^{+1} \overset{s}{w}{}_{2n}^{(1)} P_{2\ell}(y)\, dy = \frac{(-1)^{n+1}}{2\pi} \int_{-1}^{+1} \left\{ (2n-1)\, Q_{2n-1} - (2n+1)\, Q_{2n+1} \right\} P_{2\ell}\, dy =$$

$$= \frac{(-1)^{n+1}}{2\pi} \left\{ (2n-1) \frac{2}{(2\ell-2n+1)(2\ell+2n)} - (2n+1) \frac{2}{(2\ell-2n-1)(2\ell+2n+2)} \right\}$$

$$= \frac{(-1)^{n+1}}{\pi} \left\{ \frac{2n-1}{(2\ell-2n+1)(2\ell+2n)} - \frac{2n+1}{(2\ell-2n-1)(2\ell+2n+2)} \right\} \qquad \text{for } n \geq 1$$

$$\overset{s}{\mu}{}_{\ell}^{0} = -2\pi \int_{-1}^{+1} \overset{s}{w}{}_{0}^{(1)} P_{2\ell}(y)\, dy = \frac{1}{\pi} \int_{-1}^{+1} Q_1 P_{2\ell}\, dy = \frac{1}{\pi} \frac{2}{(2\ell-1)(2\ell+2)}$$

$$\overset{s}{\mu}{}_{\ell}^{2n+1} = -\pi \int_{-1}^{+1} \overset{s}{w}{}^{(1)}_{2n+1} P_{2\ell}(y)\,dy = \frac{(-1)^n}{4} \int_{-1}^{+1} \left\{ 2n P_{2n} - (2n+2) P_{2n+2} \right\} P_{2\ell}\, dy = 0$$

for $\ell \neq n$ and $\ell \neq n+1$

$$\overset{s}{\mu}{}_{\ell}^{2\ell+1} = -\pi \int_{-1}^{+1} \overset{s}{w}{}^{(1)}_{2\ell+1} P_{2\ell}(y)\,dy = \frac{(-1)^\ell}{4} \int_{-1}^{+1} \left\{ 2\ell P_{2\ell} - (2\ell+2) P_{2\ell+2} \right\} P_{2\ell}\, dy = \frac{(-1)^\ell \ell}{4\ell+1}$$

$$\overset{s}{\mu}{}_{\ell}^{2\ell-1} = -\pi \int_{-1}^{+1} \overset{s}{w}{}^{(1)}_{2\ell-1} P_{2\ell}(y)\,dy = \frac{(-1)^{\ell-1}}{4} \int_{-1}^{+1} \left\{ (2\ell-2) P_{2\ell-2} - 2\ell P_{2\ell} \right\} P_{2\ell}\, dy = \frac{(-1)^\ell \ell}{4\ell+1} \tag{3,5,1}$$

$$\overset{a}{\mu}{}_{k}^{2n} = -\pi \int_{-1}^{+1} \overset{a}{w}{}^{(1)}_{2n} P_{2k+1}(y)\,dy = \frac{(-1)^{n+1}}{4} \int_{-1}^{+1} \left\{ (2n-1) P_{2n-1} - (2n+1) P_{2n+1} \right\} P_{2k+1}\, dy = 0$$

for $k \neq n-1$ and $k \neq n$

$$\overset{a}{\mu}{}_{k}^{2k+2} = -\pi \int_{-1}^{+1} \overset{a}{w}{}^{(1)}_{2k+2} P_{2k+1}(y)\,dy = \frac{(-1)^k}{4} \int_{-1}^{+1} \left\{ (2k+1) P_{2k+1} - (2k+3) P_{2k+3} \right\} P_{2k+1}\, dy =$$

$$= \frac{(-1)^k (2k+1)}{2(4k+3)}$$

$$\overset{a}{\mu}{}_{k}^{2k} = -\pi \int_{-1}^{+1} \overset{a}{w}{}^{(1)}_{2k} P_{2k+1}(y)\,dy = \frac{(-1)^{k+1}}{4} \int_{-1}^{+1} \left\{ (2k-1) P_{2k-1} - (2k+1) P_{2k+1} \right\} P_{2k+1}\, dy =$$

$$= \frac{(-1)^k (2k+1)}{2(4k+3)}$$

$$\overset{a}{\mu}{}_{k}^{2n+1} = -\pi \int_{-1}^{+1} \overset{a}{w}{}^{(1)}_{2n+1} P_{2k+1}(y)\,dy = \frac{(-1)^{n+1}}{2\pi} \int_{-1}^{+1} \left\{ 2n Q_{2n} - (2n+2) Q_{2n+2} \right\} P_{2k+1}\, dy =$$

$$= \frac{(-1)^{n+1}}{2\pi} \left\{ 2n \frac{2}{(2k-2n+1)(2k+2n+2)} - (2n+2) \frac{2}{(2k-2n-1)(2k+2n+4)} \right\} =$$

$$= \frac{(-1)^{n+1}}{\pi} \left\{ \frac{2n}{(2k-2n+1)(2k+2n+2)} - \frac{2n+2}{(2k-2n-1)(2k+2n+4)} \right\} . \tag{3,5,2}$$

For the coefficients $\overset{s}{\nu}{}_{\ell}^{n}$ we can derive the following relations

$$\overset{s_n}{\nu}_\ell = p_{2n}^{2n}\int_{-1}^{+1}\overset{s}{w}{}^{(1)}_{2n}\,P_{2\ell}(y)\,dy + \sum_{m=0}^{\infty} p_{2n}^{2m+1}\int_{-1}^{+1}\overset{s}{w}{}^{(1)}_{2m+1}\,P_{2\ell}(y)\,dy =$$

$$= p_{2n}^{2n}\,\frac{(-1)^n}{2\pi^2}\int_{-1}^{+1}\Big[(2n-1)\,Q_{2n-1} - (2n+1)\,Q_{2n+1}\Big]\,P_{2\ell}(y)\,dy +$$

$$+\sum_{m=0}^{\infty} p_{2n}^{2m+1}\,\frac{(-1)^{m+1}}{4\pi}\int_{-1}^{+1}\Big\{2m\,P_{2m} - (2m+2)\,P_{2m+2}\Big\}\,P_{2\ell}\,dy$$

$$= p_{2n}^{2n}\,\frac{(-1)^n}{2\pi^2}\left[(2n-1)\,\frac{2}{(2\ell-2n+1)\,(2\ell+2n)} - (2n+1)\,\frac{2}{(2\ell-2n-1)\,(2\ell+2n+2)}\right] +$$

$$+ p_{2n}^{2\ell+1}\,\frac{(-1)^{\ell+1}}{4\pi}\,\frac{4\ell}{4\ell+1} + p_{2n}^{2\ell-1}\,\frac{(-1)^{\ell+1}}{4\pi}\,\frac{4\ell}{4\ell+1}$$

$$= \frac{(-1)^n}{2\pi}\left\{\frac{2n-1}{(2\ell-2n+1)\,(2\ell+2n)} - \frac{2n+1}{(2\ell-2n-1)\,(2\ell+2n+2)}\right\} +$$

$$+ \frac{(-1)^n\,\ell\,(4\ell+2)}{\pi\,(4\ell+1)\left\{(2\ell+1)^2-4n^2\right\}} + \frac{(-1)^{n+1}\,\ell\,(4\ell-2)}{\pi\,(4\ell+1)\left\{(2\ell-1)^2-4n^2\right\}} \qquad \text{for } n \geqq 1$$

$$\overset{s_0}{\nu}_\ell = p_0^0\int_{-1}^{+1}\overset{s}{w}{}^{(1)}_{0}\,P_{2\ell}(y)\,dy + \sum_{m=0}^{\infty} p_0^{2m+1}\int_{-1}^{+1}\overset{s}{w}{}^{(1)}_{2m+1}\,P_{2\ell}(y)\,dy =$$

$$= p_0^0\,\frac{-1}{2\pi^2}\int_{-1}^{+1} Q_1\,P_{2\ell}\,dy + \sum_{m=0}^{\infty} p_0^{2m+1}\,\frac{(-1)^{m+1}}{4\pi}\int_{-1}^{+1}\Big\{2m\,P_{2m} - (2m+2)\,P_{2m+2}\Big\}\,P_{2\ell}\,dy$$

$$= p_0^0\,\frac{-1}{2\pi^2}\,\frac{2}{(2\ell-1)\,(2\ell+2)} + p_0^{2\ell+1}\,\frac{(-1)^{\ell+1}}{4\pi}\,\frac{4\ell}{4\ell+1} + p_0^{2\ell-1}\,\frac{(-1)^{\ell+1}}{4\pi}\,\frac{4\ell}{4\ell+1}$$

$$= -\frac{1}{\pi\,(2\ell-1)\,(2\ell+2)} + \frac{2\ell}{\pi\,(4\ell+1)\,(2\ell+1)} - \frac{2\ell}{\pi\,(4\ell+1)\,(2\ell-1)}\;. \qquad (3,5,3)$$

Numerical values for the coefficients $\overset{s_n}{\nu}_\ell$ are given in table IX. The anti-symmetric coefficients $\overset{a_n}{\nu}_k$ can be reduced in a similar way. We find

$$\overset{a_n}{\nu}_k = \frac{(-1)^n}{2\pi}\left\{\frac{2n}{(2k-2n+1)\,(2k+2n+2)} - \frac{2n+2}{(2k-2n-1)\,(2k+2n+4)}\right\} +$$

$$+ \frac{(-1)^n\,(2k+1)\,(2k+2)}{\pi\,(4k+3)\left\{4\,(k+1)^2-(2n+1)^2\right\}} - \frac{(-1)^n\,2k\,(2k+1)}{\pi\,(4k+3)\left\{4k^2-(2n+1)^2\right\}}\;. \qquad (3,5,4)$$

The numerical values of $\overset{a_n}{\nu}_k$ are given in table X.

6 Examples.

6.1 Introduction.

In this section several examples of oscillating circular wings will be treated. As our results are mainly of interest for stability investigations, it is sufficient to evaluate the lift-force and the moments. A detailed insight in the pressure distribution over the wing surface is not of so much importance as in the steady case. The complete acceleration potential once being known, the pressure jump Π over the aerofoil is found by multiplying this potential with 2ρ , where ρ denotes the air density. According to formula (3,2,9) we can write

$$\Pi\,(0,\mu,\vartheta) = 2\rho\psi\,(0,\mu,\vartheta) + 2\rho U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\substack{\mu_1=0\\ \eta=0}} d\vartheta_1 \;. \qquad (3,6,1)$$

In quite the same way as in the steady case formulas for lift and moments can be derived. Therefore we can immediately conclude that the expressions (2,13,8) (2,13,10), and (2,13,13), which give approximations for lift, moment about the y -axis and moment about the x -axis respectively, remain valid in the unsteady case.

6.2 The vertical translation.

The vertical translation can be represented by the equation

$$z\,(x,y,t) = Ae^{i\nu t} \qquad . \qquad (3,6,2)$$

The downwash distribution w on the wing surface is then

$$w\,(x,y,t) = \frac{dz}{dt} = i\nu Ae^{i\nu t} = i\omega UAe^{i\nu t} \qquad ,$$

whereas the normal acceleration becomes

$$a\,(x,y,t) = \frac{dw}{dt} = (i\nu)^2 Ae^{i\nu t} = -\nu^2 Ae^{i\nu t} = -\omega^2 U^2 Ae^{i\nu t} \quad .$$

In our approximation up to the first order in ω , we obtain for the downwash

$$\bar{w} = i\omega UA \qquad (3,6,3)$$

and for the normal acceleration $\bar{a}$ we find the value zero. The regular velocity potential is found by means of (3,2,1), viz.

$$\bar{\Phi}\,(x,y,z) = \int_0^{2\pi}\int_{-1}^{+1} i\omega UAG\,(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,\mu_1\,d\mu_1\,d\vartheta_1 = i\omega UAP_1(\mu)\,\frac{Q_1(i\eta)}{Q_1'(i0)} =$$
$$= -\frac{2}{\pi}\, i\omega UA\mu\left(1-\eta\,\arctan\frac{1}{\eta}\right) \quad . \qquad (3,6,4)$$

The weight-function $g(\vartheta)$ is thus

$$g(\vartheta) = -\pi\cos\vartheta\,\bar{\Phi}_\mu(0,0,\vartheta) = 2\,i\omega UA\cos\vartheta \quad . \qquad (3,6,5)$$

For the Fourier-coefficients of $g^*(\vartheta)$ we find

$$c_n^{(0)} = 0 \quad \text{for every } n$$

$$c_0^{(1)} = 2\,UA\,, \; c_n^{(1)} = 0 \quad \text{for } n \geq 1 \quad . \tag{3,6,6}$$

The regular acceleration potential vanishes in this case. The complete acceleration potential assumes the form

$$U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \quad . \tag{3,6,7}$$

The function $h(\vartheta)$ is written in the form

$$h(\vartheta) = h^{*}(\vartheta)\cos\vartheta = \cos\vartheta \sum_{n=0}^{\infty} \left\{a_n^{(0)} + i\omega a_n^{(1)}\right\} \cos 2n\vartheta \,,$$

where the coefficients $a_n^{(0)}$ and $a_n^{(1)}$ must be solved from the identities (3,4,8). In this special case the coefficients $a_n^{(0)}$ are all equal to zero as follows from the first identity in (3,4,8). The second identity in (3,4,8) is equivalent to the infinite system (3,4,14) of linear equations. Here we can write

$$\sum_{n=0}^{\infty} t_\ell^{n} \, a_n^{(1)} = 2\,UA\,\sigma_\ell^{0} \qquad \ell = 0, 1, 2, \ldots \tag{3,6,8}$$

This system is exactly equal to the system (2,14,6), if the quantity A is replaced by $-\alpha$. Thus we can immediately write down the coefficients $a_n^{(1)}$. Apart from a factor UA they are in an approximation of five terms in the Fourier series.

$$a_0^{(1)} = -2{,}2433 \quad a_1^{(1)} = +0{,}092870 \quad a_2^{(1)} = -0{,}040842 \quad a_3^{(1)} = +0{,}023473 \quad a_4^{(1)} = -0{,}015436 \;.$$

For the lift K we obtain

$$K = -2{,}812 \; i\omega\rho A U^2 e^{i\nu t}$$

and for the moment M about the axis of y

$$M = 1{,}465 \; i\omega\rho A U^2 e^{i\nu t} \quad .$$

Mostly the values of lift and moment are given in dimensionless coefficients according to the definitions:

$$K = \pi\rho U^2 \left(k_A' + i k_A''\right) A e^{i\nu t}$$

$$M = \pi\rho U^2 \left(m_A' + i m_A''\right) A e^{i\nu t} \quad .$$

Hence

$$k_A' = 0 \;; \quad k_A'' = -0{,}8951\,\omega \quad \text{or} \quad \left[\frac{dk_A''}{d\omega}\right]_{\omega=0} = -0{,}8951$$

$$m_A' = 0 \;; \quad m_A'' = 0{,}4663\,\omega \quad \text{or} \quad \left[\frac{dm_A''}{d\omega}\right]_{\omega=0} = 0{,}4663 \quad .$$

6.3 Rotation about the axis of y .

This harmonic oscillation of the wing can be expressed by the equation

$$z(x,y,t) = Bxe^{i\nu t} \quad . \tag{3,6,9}$$

Hence the downwash distribution on the wing surface is given by

$$w(x,y,t) = \frac{dz}{dt} = \left\{ i\nu Bx + BU \right\} e^{i\nu t} = \left\{ 1 + i\omega x \right\} BUe^{i\nu t}$$

and the normal acceleration by

$$a(x,y,t) = \frac{dw}{dt} = \left\{ 2i\omega - \omega^2 x \right\} BU^2 e^{i\nu t} \quad .$$

In our approximation we can write for downwash and normal acceleration respectively

$$\bar{w} = \left\{ 1 + i\omega x \right\} BU \tag{3,6,10}$$

$$\bar{a} = 2\, i\omega BU^2 \tag{3,6,11}$$

The regular velocity potential $\bar{\Phi}$ corresponding to the normal velocity (3,6,10) becomes now

$$\bar{\Phi} = UB \int_0^{2\pi} \int_{-1}^{+1} \left[1 + i\omega \sqrt{1-\mu_1^2} \cos\vartheta_1 \right] \mu_1 G(\eta, \mu, \vartheta; \mu_1, \vartheta_1)\, d\mu_1\, d\vartheta_1$$

$$= UBP_1(\mu) \frac{Q_1(i\eta)}{Q_1'(i0)} - \frac{1}{3} i\omega\, UBP_2^1(\mu) \frac{Q_2^1(i\eta)}{Q_2^{1\prime}(i0)}$$

$$= -\frac{2}{\pi} UB\mu \left(1 - \eta \arctan\frac{1}{\eta}\right) - \frac{2}{3\pi} i\omega UB\mu \sqrt{1-\mu^2} \cos\vartheta \sqrt{1+\eta^2} \left(3 - \frac{1}{1+\eta^2} - 3\eta \arctan\frac{1}{\eta}\right) \tag{3,6,12}$$

whereas the regular acceleration potential corresponding to the normal acceleration (3,6,11) is

$$\bar{\Psi} = 2\, i\omega BU^2 \int_0^{2\pi} \int_{-1}^{+1} G(\eta, \mu, \vartheta; \mu_1, \vartheta_1) \mu_1\, d\mu_1\, d\vartheta_1 =$$

$$= -\frac{4i}{\pi} \omega BU^2 \mu \left(1 - \eta \arctan\frac{1}{\eta}\right) \quad . \tag{3,6,13}$$

The complete acceleration potential can thus be written in the form

$$-\frac{4i}{\pi} \omega BU^2 \mu \left(1 - \eta \arctan\frac{1}{\eta}\right) + U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1) \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1 = 0} d\vartheta_1 \,. \tag{3,6,14}$$

For the weight-function $g(\vartheta)$ we find

$$g(\vartheta) = -\pi \cos\vartheta\, \bar{\Phi}_\mu(0,0,\vartheta) = -\pi \cos\vartheta \left\{ -\frac{2}{\pi} - \frac{4i\omega}{3\pi} \cos\vartheta \right\} UB = \cos\vartheta \left\{ 2 + \frac{4i\omega}{3} \cos\vartheta \right\} UB \,.$$

Thus

$$g^{x}(\vartheta)=\left\{2+\frac{4}{3}\,i\omega\cos\vartheta\right\}\,UB \qquad (3,6,15)$$

and consequently

$$c_0^{(0)}=2\,UB\,,\quad c_n^{(0)}=0 \quad \text{for} \quad n\geq 1$$

$$c_0^{(1)}=0\,,\quad c_1^{(1)}=\frac{4}{3}UB\,,\quad c_n^{(1)}=0 \quad \text{for}\ n\geq 2 \qquad (3,6,16)$$

The function $h(\vartheta)$ is written as

$$h(\vartheta)=\cos\vartheta\, h^{x}(\vartheta)=\cos\vartheta\sum_{n=0}^{\infty}\left\{a_n^{(0)}+i\omega a_n^{(1)}\right\}\cos 2n\vartheta\ .$$

The coefficients $a_n^{(0)}$ follow from the system of equations

$$\sum_{n=0}^{\infty}\overset{s}{\tau}{}_{\ell}^{n}\,a_n^{(0)}=\sum_{n=0}^{\infty}\overset{s}{\sigma}{}_{\ell}^{n}\,c_n^{(0)}=2\,UB\,\overset{s}{\sigma}{}_{\ell}^{0} \qquad \ell=0,1,2,\ldots \qquad (3,6,17)$$

The solution can immediately be found in example 14.1 of section 14 of chapter II. We have apart from the factor UB

$a_0^{(0)}=-2,2433$ $a_1^{(0)}=+0,092890$ $a_2^{(0)}=-0,040842$ $a_3^{(0)}=+0,023473$ $a_4^{(0)}=-0,015436$.

The coefficients $a_n^{(1)}$ must satisfy the system of equations

$$\sum_{n=0}^{\infty}\overset{s}{\tau}{}_{\ell}^{n}\,a_n^{(1)}=\sum_{n=0}^{\infty}\overset{s}{\sigma}{}_{\ell}^{n}\,c_n^{(1)}+\sum_{n=0}^{\infty}\overset{s}{\mu}{}_{\ell}^{n}\,c_n^{(0)}+\sum_{n=0}^{\infty}\overset{s}{\nu}{}_{\ell}^{n}\,a_n^{(0)} \qquad \ell=0,1,2,\ldots$$

or in our special case

$$\sum_{n=0}^{\infty}\overset{s}{\tau}{}_{\ell}^{n}\,a_n^{(1)}=\frac{4}{3}\,UB\,\overset{s}{\sigma}{}_{\ell}^{1}+2UB\,\overset{s}{\mu}{}_{\ell}^{0}+\sum_{n=0}^{\infty}\overset{s}{\nu}{}_{\ell}^{n}\,a_n^{(0)} \qquad \ell=0,1,2,\ldots \qquad (3,6,18)$$

A good approximation appears to be

$a_0^{(1)}=+1,0474$ $a_1^{(1)}=+0,52725$ $a_2^{(1)}=-0,099754$ $a_3^{(1)}=+0,042165$ $a_4^{(1)}=-0,025405$

wherein the factor UB has been omitted. For the lift K we find

$$K=\left\{-2,812-3,766\ i\omega\right\}\rho U^2 B$$

and for the moment about the axis of y

$$M=\left\{+1,465-0,8470\ i\omega\right\}\rho U^2 B$$

Introducing the coefficients k_b and m_b according to the definitions

$$K = \pi \rho U^2 \, (k_b' + i k_b'') B e^{i\nu t}$$

$$M = \pi \rho U^2 \, (m_b' + i m_b'') B e^{i\nu t}$$

we can write

$$k_b' = -0{,}8951 \quad ; \quad k_b'' = -1{,}199\,\omega \quad or \quad \left[\frac{dk_b''}{d\omega}\right]_{\omega=0} = -1{,}199$$

$$m_b' = +0{,}4663 \quad ; \quad m_b'' = -0{,}2696\,\omega \quad or \quad \left[\frac{dm_b''}{d\omega}\right]_{\omega=0} = -0{,}2696 \; .$$

6.4 Rotation about the axis of x .

The roll oscillation of the wing is given by the equation

$$z = C y e^{i\nu t} \tag{3,6,19}$$

The downwash distribution on the wing surface takes the form

$$w\,(x,y,t) = \frac{dz}{dt} = i\nu C y e^{i\nu t} = i\omega y U C e^{i\nu t} \quad ,$$

while the normal acceleration $a(x,y,t)$ is

$$a\,(x,y,t) = \frac{dw}{dt} = -\omega^2 y U^2 C e^{i\nu t} \quad .$$

Taking into account only terms up to the first order in ω , we can write

$$\bar{w}\,(x,y) = i\omega y C U \tag{3,6,20}$$

$$\bar{a}\,(x,y) = 0 \quad . \tag{3,6,21}$$

The regular velocity potential is thus

$$\bar{\Phi}\,(x,y,z) = UC \int_0^{2\pi}\int_{-1}^{+1} i\omega \sqrt{1-\mu_1^2}\, \sin\vartheta_1\, G\,(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,\mu_1\, d\mu_1\, d\vartheta_1 =$$

$$= -\frac{1}{3}\, i\omega U C P_2^1\,(\mu)\, \frac{Q_2^1\,(i\eta)}{Q_2^{1\prime}(i0)}\, \sin\vartheta =$$

$$= -\frac{2}{3\pi}\, i\omega U C\, \mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\, \left\{3 - \frac{1}{1+\eta^2} - 3\eta\, \arctan\frac{1}{\eta}\right\} \sin\vartheta . \tag{3,6,22}$$

The regular acceleration potential is equal to zero. Consequently the complete acceleration potential has the form

$$U \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h\,(\vartheta_1)\, \left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \quad . \tag{3,6,23}$$

The weight-function $g(\vartheta)$ reads

$$g\,(\vartheta) = -\pi \cos\vartheta\, \bar{\Phi}_\mu\,(0,0,\vartheta) = \frac{4}{3}\, i\omega U C \sin\vartheta \cos\vartheta \; .$$

Thus

$$g^*(\vartheta) = \sum_{n=1}^{\infty} \left\{d_n^{(0)} + i\omega d_n^{(1)}\right\} \sin n\vartheta = \frac{4}{3}\, i\omega U C \sin\vartheta \; , \tag{3,6,24}$$

where

$$d_n^{(0)} = 0 \quad \text{for every } n$$
$$d_1^{(1)} = \frac{4}{3} UC \;,\; d_n^{(1)} = 0 \quad \text{for } n \geq 2 \quad . \qquad (3,6,25)$$

The weight-function $k(\vartheta)$ is written in this anti-symmetric problem as

$$k(\vartheta) = \cos\vartheta\, k^{*}(\vartheta) = \cos\vartheta \sum_{n=0}^{\infty} \left\{ b_n^{(0)} + i\omega b_n^{(1)} \right\} \sin(2n+1)\vartheta \;.$$

The fact that all coefficients $d_n^{(0)}$ vanish also implies that $b_n^{(0)} = 0$ for every n. According to (3,4,16) the system of equations for $b_n^{(1)}$ reads

$$\sum_{n=0}^{\infty} \overset{a}{\tau}{}_k^n\, b_n^{(1)} = \frac{4}{3} UC\, \overset{a}{\sigma}{}_k^1 \qquad k = 0,1,2,\ldots \qquad (3,6,26)$$

This system is equivalent to the system (2,14,23), which had to be solved in one of the applications of the steady theory. The solution can immediately be written down, viz.

$$b_0^{(1)} = -1,3730 \quad b_1^{(1)} = +0,017714 \quad b_2^{(1)} = -0,010305 \quad b_3^{(1)} = +0,0068407 \quad b_4^{(1)} = -0,0049177$$

where the factor UC has been left out. The moment about the axis of x appears to be

$$M = -0,3849\; i\omega\rho\, U^2 C$$

or expressed in terms of the coefficient m_c, defined by

$$M = \pi\rho U^2 \left(m_c' + i m_c''\right) C e^{i\nu t}$$

we have

$$m_c' = 0 \qquad ; \quad m_c'' = -0,1225\,\omega \qquad \text{or} \quad \left[\frac{dm_c''}{d\omega}\right]_{\omega=0} = -0,1225$$

6.5 The oscillation $z = Dx^2 e^{i\nu t}$.

The normal velocity on the wing surface is given by

$$w(x,y,t) = \frac{dz}{dt} = \left\{ i\nu Dx^2 + 2UDx \right\} e^{i\nu t} = \left\{ i\omega x^2 + 2x \right\} UDe^{i\nu t}$$

and the normal acceleration by

$$a(x,y,t) = \frac{dw}{dt} = \left\{ -\omega^2 x^2 + 4\,i\omega x + 2 \right\} U^2 De^{i\nu t} \quad .$$

In an approximation to the first order of ω , we obtain the formulas for downwash and normal acceleration respectively

$$\bar{w}(x,y) = \left\{ i\omega x^2 + 2x \right\} UD \qquad (3,6,27)$$
$$\bar{a}(x,y) = \left\{ 4\,i\omega x + 2 \right\} U^2 D \quad . \qquad (3,6,28)$$

The corresponding velocity potential is

$$\bar{\Phi}(x,y,z) = UD\int_0^{2\pi}\int_{-1}^{+1}\left\{i\omega x_1^2+2x_1\right\} G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\mu_1\, d\mu_1\, d\vartheta_1 =$$

$$= \frac{1}{5}\, i\omega UDP_1(\mu)\,\frac{Q_1(i\eta)}{Q_1'(io)} - \frac{1}{5}\, i\omega UDP_3(\mu)\,\frac{Q_3(i\eta)}{Q_3'(io)} +$$

$$+ \frac{1}{30}\, i\omega UDP_3^2(\mu)\,\frac{Q_3^2(i\eta)}{Q_3^{2'}(io)}\cos 2\vartheta - \frac{2}{3}\, UDP_2^1(\mu)\,\frac{Q_2^1(i\eta)}{Q_2^{1'}(io)}\cos\vartheta =$$

$$- \frac{2}{5\pi} i\omega UD\mu\left\{1-\eta\arctan\frac{1}{\eta}\right\} + \frac{2}{15\pi}\, i\omega UD\,(5\mu^3-3\mu)\left\{\frac{5}{2}\eta^2 + \frac{2}{3} +\right.$$

$$\left. - \frac{5\eta^3+3\eta}{2}\arctan\frac{1}{\eta}\right\} - \frac{1}{15\pi}\, i\omega UD\mu\,(1-\mu^2)(1+\eta^2)\left\{15 - \frac{5}{1+\eta^2} - \frac{2}{(1+\eta^2)^2} +\right.$$

$$\left. -15\eta\arctan\frac{1}{\eta}\right\}\cos 2\vartheta - \frac{4}{3\pi}\, UD\mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\left\{3 - \frac{1}{1+\eta^2} - 3\eta\arctan\frac{1}{\eta}\right\}\cos\vartheta \tag{3,6,29}$$

and the regular acceleration potential

$$\tilde{\Psi}(x,y,z) = U^2D\int_0^{2\pi}\int_{-1}^{+1}\left\{4\, i\omega x_1 + 2\right\} G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\mu_1\, d\mu_1\, d\vartheta_1 =$$

$$= -\frac{4}{3} i\omega U^2DP_2^1(\mu)\,\frac{Q_2^1(i\eta)}{Q_2^{1'}(io)}\cos\vartheta + 2U^2DP_1(\mu)\,\frac{Q_1(i\eta)}{Q_1'(io)}$$

$$= -\frac{8}{3\pi} i\omega U^2D\mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\left\{3 - \frac{1}{1+\eta^2} - 3\eta\arctan\frac{1}{\eta}\right\}\cos\vartheta - \frac{4}{\pi}U^2D\mu\left\{1-\eta\arctan\frac{1}{\eta}\right\}. \tag{3,6,30}$$

The complete acceleration potential can be written in the form

$$\tilde{\Psi} + U\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1)\left[\frac{\partial G}{\partial\mu_1}\right]_{\mu_1=0} d\vartheta_1 \quad . \tag{3,6,31}$$

The weight-function $g(\vartheta)$ is in this case

$$g(\vartheta) = -\pi\cos\vartheta\,\Phi_\mu(0,0,\vartheta) = -\pi\cos\vartheta\left\{-\frac{2}{5\pi}\, i\omega UD - \frac{4}{15\pi}\, i\omega UD - \frac{8}{15\pi}\, i\omega UD\cos 2\vartheta - \frac{8}{3\pi}\, UD\cos\vartheta\right\}$$

or

$$g^{\times}(\vartheta) = \frac{2}{3}\, i\omega UD + \frac{8}{15}\, i\omega UD\cos 2\vartheta + \frac{8}{3}\, UD\cos\vartheta \quad . \tag{3,6,32}$$

Thus

$$c_0^{(0)}=0\ ,\quad c_1^{(0)}=\frac{8}{3}\,UD\ ,\quad c_n^{(0)}=0\quad \text{for } n>1$$

$$c_0^{(1)}=\frac{2}{3}\,UD\ ,\quad c_1^{(1)}=0\ ,\quad c_2^{(1)}=\frac{8}{15}\,UD\ ,\quad c_n^{(1)}=0\quad \text{for } n>2\ . \tag{3,6,33}$$

The systems of equations become now

$$\sum_{n=0}^{\infty} \overset{s}{\tau}{}_{\ell}^{n}\, a_n^{(0)} = \frac{8}{3}\,UD\,\overset{s}{\sigma}{}_{\ell}^{1} \qquad \ell=0,1,2,\ \ldots$$

$$\sum_{n=0}^{\infty} \overset{s}{\tau}{}_{\ell}^{n}\, a_n^{(1)} = \frac{2}{3}\,UD\,\overset{s}{\sigma}{}_{\ell}^{0}+\frac{8}{15}\,UD\,\overset{s}{\sigma}{}_{\ell}^{2}+\frac{8}{3}\,UD\,\overset{s}{\mu}{}_{\ell}^{1}+\sum_{n=0}^{\infty}\overset{s}{\nu}{}_{\ell}^{n}\,a_n^{(0)}\ . \tag{3,6,34}$$

For the coefficients $a_n^{(0)}$ and $a_n^{(1)}$ we find apart from the factor UD

$a_0^{(0)}=+1,4615\quad a_1^{(0)}=+1,2128\quad a_2^{(0)}=-0,25916\quad a_3^{(0)}=+0,11483\quad a_4^{(0)}=-0,065112$

$a_0^{(1)}=-0,44188\quad a_1^{(1)}=-0,57780\quad a_2^{(1)}=+0,014592\quad a_3^{(1)}=-0,0053921\quad a_4^{(1)}=-0,0055189$

Lift and moment about the y -axis obtain the values

$$K=\left\{-2,931-0,8091\ i\omega\right\}\rho U^2 D$$

$$M=\left\{-1,379-0,9350\ i\omega\right\}\rho U^2 D$$

If we write

$$K=\pi\rho U^2\left(k_d'+ik_d''\right)De^{i\nu t}$$

$$M=\pi\rho U^2\left(m_d'+im_d''\right)De^{i\nu t}$$

we have

$$k_d'=-0,9329\qquad k_d''=-0,2575\ \omega\qquad \text{or}\qquad \left[\frac{dk_d''}{d\omega}\right]_{\omega=0}=-0,2575$$

$$m_d'=-0,4389\qquad m_d''=-0,2976\ \omega\qquad \text{or}\qquad \left[\frac{dm_d''}{d\omega}\right]_{\omega=0}=-0,2976\ .$$

6.6 The oscillation $z=Exye^{i\nu t}$.

The downwash on the wing surface is represented by the equation

$$w(x,y,t)=\frac{dz}{dt}=\left\{i\nu Exy+UEy\right\}e^{i\nu t}=\left\{i\omega xy+y\right\}UEe^{i\nu t}$$

and the normal acceleration by the equation

$$a(x,y,t)=\frac{dw}{dt}=\left\{-\omega^2 xy+2\ i\omega y\right\}U^2Ee^{i\nu t}\ .$$

The approximations for downwash and normal acceleration can thus be written

$$\bar{w}(x,y)=\left\{i\omega xy+y\right\}UE \tag{3,6,35}$$

$$\bar{a}(x,y)=2\ i\omega y\ U^2E\ . \tag{3,6,36}$$

The expression for the regular velocity potential becomes here

$$\bar{\Phi}(x,y,z) = UE \int_0^{2\pi}\int_{-1}^{+1} \left\{i\omega x_1 y_1 + y_1\right\} G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\,\mu_1\, d\mu_1\, d\vartheta_1$$

$$= \frac{1}{30}\, i\omega\, UE\, P_3^2(\mu)\, \frac{Q_3^2(i\eta)}{Q_3^{2'}(i0)} \sin 2\vartheta - \frac{1}{3}\, UE\, P_2^1(\mu)\, \frac{Q_2^1(i\eta)}{Q_2^{1'}(i0)} \sin\vartheta$$

$$= -\frac{1}{15\pi}\, i\omega UE\, \mu\,(1-\mu^2)(1+\eta^2)\left\{15 - \frac{5}{1+\eta^2} - \frac{2}{(1+\eta^2)^2} - 15\eta \arctan\frac{1}{\eta}\right\}\sin 2\vartheta +$$

$$- \frac{2}{3\pi}\, UE\mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\left\{3 - \frac{1}{1+\eta^2} - 3\eta\arctan\frac{1}{\eta}\right\}\sin\vartheta \qquad (3,6,37)$$

whereas the regular acceleration potential obtains the form

$$\bar{\Psi}(x,y,z) = -\frac{4}{3\pi}\, i\omega U^2 E\mu\sqrt{1-\mu^2}\sqrt{1+\eta^2}\left\{3 - \frac{1}{1+\eta^2} - 3\eta\arctan\frac{1}{\eta}\right\}\sin\vartheta\,. \qquad (3,6,38)$$

The complete acceleration potential assumes the form

$$\bar{\Psi}(x,y,z) + U\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(\vartheta_1)\left[\frac{\partial G}{\partial\mu_1}\right]_{\mu_1=0} d\vartheta_1 \,. \qquad (3,6,39)$$

The weight-function $g(\vartheta)$ is easily found by

$$g(\vartheta) = -\pi\cos\vartheta\,\bar{\Phi}_\mu(0,0,\vartheta) = -\pi\cos\vartheta\left\{-\frac{8}{15\pi}\, i\omega UE\sin 2\vartheta - \frac{4}{3\pi}\, UE\sin\vartheta\right\} =$$
$$= \cos\vartheta\left\{\frac{8}{15}\, i\omega\sin 2\vartheta + \frac{4}{3}\sin\vartheta\right\}UE\,.$$

In this anti-symmetric case we write

$$g(\vartheta) = \cos\vartheta\, g^x(\vartheta) = \cos\vartheta\sum_{n=1}^{\infty}\left\{d_n^{(0)} + i\omega d_n^{(1)}\right\}\sin n\vartheta\,. \qquad (3,6,40)$$

Thus we have

$$d_1^{(0)} = \frac{4}{3}UE\,, \quad d_n^{(0)} = 0 \quad \text{for } n>1$$

$$d_1^{(1)} = 0\,, \quad d_2^{(1)} = \frac{8}{15}UE\,, \quad d_n^{(1)} = 0 \quad \text{for } n>2 \qquad (3,6,41)$$

The coefficients $b_n^{(0)}$ and $b_n^{(1)}$ of the unknown function $h^x(\vartheta)$ must now satisfy the equations

$$\sum_{n=0}^{\infty} \overset{a}{\tau}{}_k^n\, b_n^{(0)} = \frac{4}{3}UE\,\overset{a}{\sigma}{}_k^1 \qquad k = 0,1,2,\ldots\ldots$$

and (3,6,42)

$$\sum_{n=0}^{\infty} \overset{a}{\tau}{}_k^n\, b_n^{(1)} = \frac{8}{15}UE\,\overset{a}{\sigma}{}_k^2 + \frac{4}{3}UE\,\overset{a}{\mu}{}_k^1 + \sum_{n=0}^{\infty}\overset{a}{\vartheta}{}_k^n\, b_n^{(0)} \qquad k=0,1,2,\ldots\ldots$$

The solution appears to be apart from a factor UE

$b_0^{(0)} = -1{,}3730 \quad b_1^{(0)} = +0{,}017714 \quad b_2^{(0)} = -0{,}010305 \quad b_3^{(0)} = +0{,}0068407 \quad b_4^{(0)} = -0{,}0049177$

and

$b_0^{(1)} = +1,2986 \quad b_1^{(1)} = +0,83824 \quad b_2^{(1)} = -0,20639 \quad b_3^{(1)} = +0,098046 \quad b_4^{(1)} = -0,057240$

The moment about the x -axis is

$$M = \left\{-0,3849 - 0,1906\ i\omega\right\} \rho U^2 E$$

If we write

$$M = \pi \rho U^2 (m_e' + i m_e'') E e^{i\nu t}$$

the following values are obtained

$$m_e' = -0,1225 \quad ; \quad m_e'' = -0,0607\ \omega \quad \text{or} \quad \left[\frac{dm_e''}{d\omega}\right]_{\omega=0} = -0,0607 \quad .$$

6.7 The oscillation: $z = Fy^2 e^{i\nu t}$.

The downwash distribution over the wing surface is expressed by

$$w(x,y,t) = \frac{dz}{dt} = i\nu F y^2 e^{i\nu t} = i\omega U F y^2 e^{i\nu t}$$

while the normal acceleration becomes

$$a(x,y,t) = \frac{dw}{dt} = -\omega^2 U^2 F y^2 e^{i\nu t} \quad .$$

In our approximation we may write

$$\bar{w}(x,y) = i\omega U F y^2 \tag{3,6,43}$$

$$\bar{a}(x,y) = 0 \quad . \tag{3,6,44}$$

Hence the regular velocity potential is

$$\bar{\Phi}(x,y,z) = UF \int_0^{2\pi} \int_{-1}^{+1} i\omega y_1^2\, G(\eta,\mu,\vartheta;\mu_1,\vartheta_1)\, \mu_1\, d\mu_1\, d\vartheta_1 =$$

$$= \frac{1}{5} i\omega U F P_1(\mu)\ \frac{Q_1(i\eta)}{Q_1'(i0)} - \frac{1}{5} i\omega U F P_3(\mu)\ \frac{Q_3(i\eta)}{Q_3'(i0)} +$$

$$- \frac{1}{30} i\omega U F P_3^2(\mu)\ \frac{Q_3^2(i\eta)}{Q_3^{2\prime}(i0)} \cos 2\vartheta = -\frac{2}{5\pi} i\omega U F \mu \left\{1 - \eta \arctan\frac{1}{\eta}\right\} +$$

$$+ \frac{2}{15\pi} i\omega U F (5\mu^3 - 3\mu) \quad \left\{\frac{5}{2}\eta^2 + \frac{2}{3} - \frac{5\eta^3 + 3\eta}{2} \arctan\frac{1}{\eta}\right\} +$$

$$+ \frac{1}{15\pi} i\omega U F \mu\ (1-\mu^2)\ (1+\eta^2) \left[\ 15 - \frac{5}{1+\eta^2} - \frac{2}{(1+\eta^2)^2} - 15\eta \arctan\frac{1}{\eta}\right\} \cos 2\vartheta \,. \tag{3,6,45}$$

Because the regular acceleration potential vanishes, the complete acceleration potential can be written as

$$U\int_{\pi/2}^{3\pi/2} k(\vartheta_1)\left[\frac{\partial G}{\partial \mu_1}\right]_{\mu_1=0} d\vartheta_1 \quad . \tag{3,6,46}$$

The weight-function $g(\vartheta)$ is

$$g(\vartheta)=-\pi\cos\vartheta\,\bar{\Phi}_{\mu}(0,0,\vartheta)=-\pi\cos\vartheta\left\{-\frac{2}{5\pi}i\omega UF-\frac{4}{15\pi}i\omega UF+\frac{8}{15\pi}i\omega UF\cos 2\vartheta\right\}=$$

$$=\cos\vartheta\left\{\frac{2}{3}i\omega-\frac{8}{15}i\omega\cos 2\vartheta\right\}UF \quad . \tag{3,6,47}$$

For the coefficients $c_n^{(0)}$ and $c_n^{(1)}$ we find

$$c_n^{(0)}=0 \quad \text{for every } n$$

$$c_0^{(1)}=\frac{2}{3}UF\,,\quad c_1^{(1)}=0\,,\quad c_2^{(1)}=-\frac{8}{15}UF\,,\quad c_n^{(1)}=0 \quad \text{for } n>2 \tag{3,6,48}$$

It can immediately be concluded that

$$a_n^{(0)}=0 \quad \text{for every } n \quad .$$

According to (3,4,14) the coefficients $a_n^{(1)}$ satisfy the equations

$$\sum_{n=0}^{\infty} \overset{s}{\tau}{}_{\ell}^{n}\, a_n^{(1)}=\frac{2}{3}UF\,\overset{s}{\sigma}{}_{\ell}^{0}-\frac{8}{15}UF\,\overset{s}{\sigma}{}_{\ell}^{2} \qquad \ell=0,1,2,\ldots \tag{3,6,49}$$

The solution of this system is

$$a_0^{(1)}=-0{,}73534 \quad a_1^{(1)}=+0{,}56399 \quad a_2^{(1)}=-0{,}014598 \quad a_3^{(1)}=+0{,}0088065 \quad a_4^{(1)}=-0{,}0059843$$

wherein the factor UF has been omitted. Lift and moment about the y -axis obtain the values

$$K=-0{,}6951\; i\omega\,\rho U^2F$$

$$M=+0{,}3022\; i\omega\,\rho U^2F$$

Putting

$$K=\pi\rho U^2\left(k_f'+ik_f''\right)Fe^{i\nu t}$$

$$M=\pi\rho U^2\left(m_f'+im_f''\right)Fe^{i\nu t} \quad ,$$

we can write

$$k_f'=0 \qquad ; \quad k_f''=-0{,}2213\,\omega \qquad \text{or} \quad \left[\frac{dk_f''}{d\omega}\right]_{\omega=0}=-0{,}2213$$

$$m_f'=0 \qquad ; \quad m_f''=+0{,}0962\,\omega \qquad \text{or} \quad \left[\frac{dm_f''}{d\omega}\right]_{\omega=0}=+0{,}0962 \quad .$$

7 Comparison with other results.

As already mentioned in the introduction Krienes and Schade have evaluated forces and moments for the six harmonic oscillations with a downwash distribution over the wing surface up to the second degree in x and y .

Their numerical results also contain the derivatives with respect to the reduced frequency ω of the imaginary parts of the force- and moment coefficients in the point $\omega = 0$. These values can be compared with the corresponding quantities obtained with our theory.

From table XI below we see that the agreement between some values of Krienes and Schade and our corresponding values is very good, whereas other values differ very strongly. From a close examination of Schade's theory it appears that there are some mistakes in it. In fact in the singular part of the solution Schade uses the Hankel function of the first kind, while the Hankel function of the second kind is required. Furthermore some errors in signs have been made.

Some of our results, viz. the damping derivatives for the pitching oscillation $z = Bxe^{i\nu t}$ can be compared with results found by Garner who applies an approximate method which is closely related to Multhopp's lifting surface theory. It can be remarked that it is possible to extend the approximate method developed in ref. 23 to slowly oscillating aerofoils.

Table XI Aerodynamic derivatives for six harmonic oscillations.

	k'_a	$\left(\frac{dk''_a}{d\omega}\right)_{\omega=0}$	m'_a	$\left(\frac{dm''_a}{d\omega}\right)_{\omega=0}$	k'_b	$\left(\frac{dk''_b}{d\omega}\right)_{\omega=0}$	m'_b	$\left(\frac{dm''_b}{d\omega}\right)_{\omega=0}$	m'_c	$\left(\frac{dm''_c}{d\omega}\right)_{\omega=0}$
Krienes and Schade	0	-0,8992	0	+0,4659	-0,8992	-0,6880	+0,4659	-0,5981	0	-0,1276
Garner	—	—	—	—	-0,8940	-1,219	+0,4689	-0,2440	—	—
This theory	0	-0,8951	0	+0,4663	-0,8951	-1,199	+0,4663	-0,2696	0	-0,1225

	k'_d	$\left(\frac{dk''_d}{d\omega}\right)_{\omega=0}$	m'_d	$\left(\frac{dm''_d}{d\omega}\right)_{\omega=0}$	m'_e	$\left(\frac{dm''_e}{d\omega}\right)_{\omega=0}$	k'_f	$\left(\frac{dk''_f}{d\omega}\right)_{\omega=0}$	m'_f	$\left(\frac{dm''_f}{d\omega}\right)_{\omega=0}$
Krienes and Schade	-0,9435	-0,8033	-0,4381	+0,0596	-0,1276	-0,0652	0	-0,2235	0	+0,0960
Garner	—	—	—	—	—	—	—	—	—	—
This theory	-0,9329	-0,2575	-0,4389	-0,2976	-0,1225	-0,0607	0	-0,2213	0	+0,0962

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Appendix:

Derivation of a closed expression for Green's function.

In order to obtain a closed expression for the function of Green of the boundary value problem we can apply Sommerfeld's theory concerning the Riemann spaces (ref.21). As the underlying theory of this method is rather unknown, we shall give a short description of it.

The classical potential theoretical method of images results in a boundary value problem for a certain bounded region to be carried over in a problem for the whole space. This method is special adapted to problems with plane or spherical boundaries. However, it is possible to formulate such problems in a surveyable way by means of functions of Green. One of the characteristic features of Green's functions is that they possess a singular point P, a simple pole, in the interior of the considered region, where the functions behave as $1/R$, if R denotes the distance of an arbitrary point Q to the point P . It is obvious that application of the method of images results in the fact that a pole in the interior of the region is carried over in poles in the exterior of the region. A potential is called rational, when it is uniquely defined in the entire space and when it possesses a finite number of poles.

In his paper above-mentioned Sommerfeld introduced algebraic potentials which are also defined in the entire space and possess a finite number of poles without fulfilling the condition of uniqueness. In fact algebraic potentials have branch lines in a similar way as two-dimensional potentials can have branch points. In order to distinguish the different branches of the potential, Sommerfeld introduces the concept of Riemann space, thus achieving that the potential is uniquely defined in the entire Riemann space. Any section of a Riemann space gives a Riemann surface, where the points of intersection with the branch lines are exactly the branch points of the Riemann surface.

One of the simplest examples is found by the region between two planes, which intersect at an angle $\frac{n\pi}{m}$, where n and m are positive integers. The function of Green for such a region becomes a unique potential in a n -sheet Riemann space, which has the sharp side of the wedge as branch line. Moreover the function of Green has $2m$ poles in this space. For $m=1$, $n=2$ the region between the two planes degenerates to the entire space and the boundary becomes a half-plane. A 2-sheet Riemann space corresponds with this configuration.

It will be our first purpose to find an elementary solution, the so-called source solution, of the potential equation in a n -sheet Riemann space. The point of departure is the ordinary source solution $1/R$ for the normal three-dimensional space, where R denotes the distance between the pole $P(x_1, y_1, z_1)$ and a point $Q(x,y,z)$:

$$R^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 \tag{A,1}$$

It is immediately clear that the integral expression

$$U = \int \frac{1}{R} f(\alpha)\, d\alpha \ , \tag{A,2}$$

wherein the integration extends over an arbitrary path in the complex α -plane, is also a solution of the potential equation. By a suitable choice of the parameter α and the function $f(\alpha)$, it is possible to obtain an integral representation for the ordinary source solution $1/R$.

For that purpose we introduce polar coordinates in the plane $x=0$, according to the formulas

$$y+iz=re^{i\varphi} \quad , \quad y_1+ix_1=r_1e^{i\varphi_1} \tag{A,3}$$

We can write

$$R^2=r^2+r_1^2-2rr_1\cos(\varphi-\varphi_1)+(x-x_1)^2 \quad . \tag{A,4}$$

In this expression for R^2 we replace the parameter φ_1 by α and call the new expression R'^2 :

$$R'^2=r^2+r_1^2-2rr_1\cos(\varphi-\alpha)+(x-x_1)^2 \quad . \tag{A,5}$$

If α is a real number, the sign of R' is taken positive. The function $f(\alpha)$ is chosen in such a way that it has a single pole with a residu $=1$ in the point $\alpha=\varphi_1$ and that it is periodic in α and φ_1 with a period 2π. Such a function is for instance

$$f(\alpha)=\frac{ie^{i\alpha}}{e^{i\alpha}-e^{i\varphi_1}} \quad . \tag{A,6}$$

Further the path of integration is taken as a contour C_1 around the point $\alpha=\varphi_1$, which is passed through in a positive sense. Hence we can write

$$\frac{1}{R}=\frac{1}{2\pi}\int_C \frac{1}{R'}\;\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\varphi_1}}\,d\alpha \quad . \tag{A,7}$$

The branch points of the integrand are given by the equations

$$R'^2=0 \quad \text{and} \quad R'^2=\infty \quad .$$

The second equation is satisfied by the root $\alpha=i\infty$. The first equation means

$$\cos(\varphi-\alpha)=\frac{r^2+r_1^2+(x-x_1)^2}{2rr_1} \quad . \tag{A,8}$$

If we put

$$\cosh a_1=\cos ia_1=\frac{r^2+r_1^2+(x-x_1)^2}{2rr_1} \quad ,$$

then equation (A,8) can be written in the form

$$\cos(\varphi-\alpha)=\cos ia_1 \quad . \tag{A,9}$$

Consequently the other branch points are given by

$$\varphi-\alpha=\pm ia_1+2k\pi \quad \text{or} \quad \alpha=\varphi+2k\pi\pm ia_1$$

where k is an integer. The poles of the integrand are found in the points

$$\alpha=\varphi_1+2k\pi \; .$$

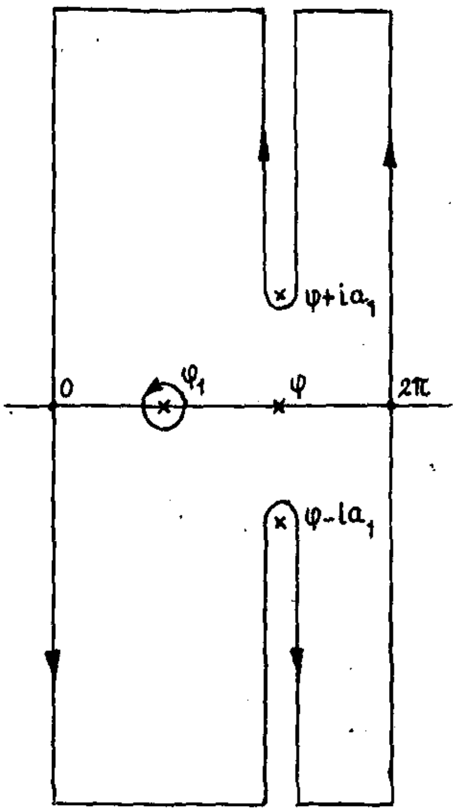


Fig.22

From this the conclusion can be drawn that an infinite number of aequidistant poles and an infinite number of pairs of aequidistant branch points lie in the α -plane. The path of integration is now transformed as shown in figure 22. The length of the vertical sides of the rectangle is chosen in an arbitrary way. Because the integrand in the expression (A,7) is periodic with period 2π , the integrals over these vertical sides cancel each other. If the length of the vertical sides tends to infinity, the integrals over the horizontal sides of the rectangle vanish. Hence it may be written

$$\frac{1}{R}=\frac{1}{2\pi}\int_D \frac{1}{R'}\,\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\varphi_1}}\,d\alpha , \qquad (A,10)$$

where D denotes the path of integration, which exists of the two contours around the cuts ($\varphi+ia_1 \rightarrow \varphi\pm i\infty$). The right-hand side of (A,10) gives an integral representation of the source solution in the ordinary three-dimensional space. For the transition to the n -sheet Riemann space we take a function $f(\alpha)$, which has a single pole with residu 1 for $\alpha=\varphi_1$, but which has a period $2\pi n$ in α and φ_1 . Such a function is given by

$$f(\alpha)=\frac{i}{n}\,\frac{e^{i\frac{\alpha}{n}}}{e^{i\frac{\alpha}{n}}-e^{i\frac{\varphi_1}{n}}} . \qquad (A,11)$$

We consider now the function

$$u=\frac{1}{2\pi i}\int_D \frac{1}{R'}\,f(\alpha)\,d\alpha=\frac{1}{2\pi n}\int_D \frac{1}{R'}\,\frac{e^{i\frac{\alpha}{n}}}{e^{i\frac{\alpha}{n}}-e^{i\frac{\varphi_1}{n}}}\,d\alpha . \qquad (A,12)$$

The branch points are again the same points, while the poles are given by $\alpha=\varphi_1+2kn\pi$. It can now be proved that this function u has all the properties of the source-solution for a n -sheet Riemann space (ref. 26). If we put in the upper half-plane $\alpha=\varphi+i\beta$ and in the lower half-plane $\alpha=\varphi-i\beta$, we can reduce the expression (A,12) after some elementary calculations to

$$u=\frac{1}{\pi n}\,\frac{1}{\sqrt{2rr_1}}\int_{a_1}^{\infty}\frac{1}{\sqrt{\cos ia_1-\cos i\beta}}\;\frac{\sin i\frac{\beta}{n}}{\cos\frac{\varphi-\varphi_1}{n}-\cos\frac{i\beta}{n}}\,d\beta . \qquad (A,13)$$

Of particular interest for us is the case $n=2$. In this case the expression for u simplifies to

$$u=\frac{1}{\pi}\,\frac{1}{R}\left\{\arctan\frac{\tau}{\sqrt{\sigma^2-\tau^2}}+\frac{\pi}{2}\right\} , \qquad (A,14)$$

wherein $\tau = \cos \frac{\varphi - \varphi_1}{2}$ and $\sigma = \cos \frac{i a_1}{2}$. In rectangular coordinates the formula (A,14) for u can be written in the form

$$u = \frac{1}{\pi}\frac{1}{R}\left\{\arctan \frac{\sqrt{2}\sqrt{yy_1 + zz_1 + rr_1}}{R} + \frac{\pi}{2}\right\}. \qquad (A,15)$$

We shall now apply this theory to a practical problem. Let us consider the half plane $z=0$, $y<0$ to be a wedge of which the sharp side coincides with the x-axis, the wedge angle being 2π. If the point P has the coordinates r_1, x_1, φ_1, then the image point P' has the coordinates $r_1, x_1, -\varphi_1$. As already remarked a 2-sheet Riemann space belongs to this problem. It is obvious that Green's function of the second kind is given by the difference $u_P - u_{P'}$, when u_P represents the elementary source solution of the two-sheet Riemann space with the pole P. We now apply an inversion with respect to the sphere $x^2+(y-1)^2+z^2 = 2$, which has the point $(0,1,0)$ as center of inversion. If $P(x,y,z)$ and $P(\bar{x},\bar{y},\bar{z})$ be any two points which are inverse with respect to the sphere, at distances ρ and $\bar{\rho}$ from the center of inversion, we have thus

$$\rho\bar{\rho} = 2$$

whereas the transformation formulas read

$$\bar{x} = \frac{2}{\rho^2}x, \quad \bar{y} - 1 = \frac{2}{\rho^2}(y-1), \quad \bar{z} = \frac{2}{\rho^2}z. \qquad (A,16)$$

It is easily seen that by the inversion the axis of x is transformed into the circle $\bar{x}^2+\bar{y}^2=1$, $\bar{z}=0$, while the half plane $z=0$, $y\le 0$ is transformed into the region $\bar{x}^2+\bar{y}^2 \leqq 1$, $\bar{z}=0$. Now the following well-known theorem concerning the inverse transformation (ref.8) will be used.

If $U(x,y,z)$ is a harmonic function of x, y and z in a domain T, then

$$V(\bar{x},\bar{y},\bar{z}) = \rho U(x,y,z)$$

is harmonic in $\bar{x}$, $\bar{y}$ and $\bar{z}$ in the domain $\bar{T}$ into which T is carried by the inversion.

Application of this theorem to our elementary solution (A,15) yields the harmonic function for the transformed problem

$$V_p(\bar{x},\bar{y},\bar{z};\bar{x}_1,\bar{y}_1,\bar{z}_1) = \frac{\rho\rho_1}{\pi R}\left\{\arctan \frac{\sqrt{2}\sqrt{yy_1 + zz_1 + rr_1}}{R} + \frac{\pi}{2}\right\},$$

wherein

$$x = \frac{\rho^2}{2}\bar{x}, \quad y - 1 = \frac{\rho^2}{2}(\bar{y}-1), \quad z = \frac{\rho^2}{2}\bar{z} \quad \text{with } \rho^2 = x^2 + (y-1)^2 + z^2$$

$$x_1 = \frac{\rho_1^2}{2}\bar{x}_1, \quad y_1 - 1 = \frac{\rho_1^2}{2}(\bar{y}_1-1), \quad z_1 = \frac{\rho_1^2}{2}\bar{z}_1 \quad \text{with } \rho_1^2 = x_1^2 + (y_1-1)^2 + z_1^2. \qquad (A,17)$$

Furthermore it can be derived that

$$R^2 = (x-x_1)^2+(y-y_1)^2+(z-z_1)^2=\left\{\frac{\rho^2}{2}\bar{x}-\frac{\rho_1^2}{2}\bar{x}_1\right\}^2+\left\{\frac{\rho^2}{2}(\bar{y}-1)-\frac{\rho_1^2}{2}(\bar{y}_1-1)\right\}^2+\left\{\frac{\rho^2}{2}\bar{z}-\frac{\rho_1^2}{2}\bar{z}_1\right\}^2=$$

$$=\frac{\rho^4}{4}\left\{\bar{x}^2+(\bar{y}-1)^2+\bar{z}^2\right\}+\frac{\rho_1^4}{4}\left\{\bar{x}_1^2+(\bar{y}_1-1)^2+\bar{z}_1^2\right\}-\frac{\rho^2\rho_1^2}{2}\left\{\bar{x}\bar{x}_1+(\bar{y}-1)(\bar{y}_1-1)+\bar{z}\bar{z}_1\right\}=$$

$$=\frac{\rho^4\bar{\rho}^2}{4}+\frac{\rho_1^4\bar{\rho}_1^2}{4}-\frac{\rho^2\rho_1^2}{2}\left\{\bar{x}\bar{x}_1+(\bar{y}-1)(\bar{y}_1-1)+\bar{z}\bar{z}_1\right\}=\rho^2+\rho_1^2-\frac{\rho^2\rho_1^2}{2}\left\{\bar{x}\bar{x}_1+(\bar{y}-1)(\bar{y}_1-1)+\bar{z}\bar{z}_1\right\}=$$

$$=\frac{\rho^2\rho_1^2}{4}\left\{\frac{4}{\rho_1^2}+\frac{4}{\rho^2}-2\,\bar{x}\bar{x}_1-2\,(\bar{y}-1)(\bar{y}_1-1)-2\,\bar{z}\bar{z}_1\right\}=$$

$$=\frac{\rho^2\rho_1^2}{4}\left\{\bar{\rho}_1^2+\bar{\rho}^2-2\,\bar{x}\bar{x}_1-2\,(\bar{y}-1)(\bar{y}_1-1)-2\,\bar{z}\bar{z}_1\right\}=\frac{\rho^2\rho_1^2}{4}\left\{(\bar{x}-\bar{x}_1)^2+(\bar{y}-\bar{y}_1)^2+(\bar{z}-\bar{z}_1)^2\right\}. \tag{A,18}$$

$$r^2=y^2+z^2=\left\{1+\frac{\rho^2}{2}(\bar{y}-1)\right\}^2+\frac{\rho^4}{4}\bar{z}^2=\left\{1+\frac{2}{\bar{\rho}^2}(\bar{y}-1)\right\}^2+\frac{4}{\bar{\rho}^4}\bar{z}^2=\frac{1}{\bar{\rho}^4}\left[\left\{\bar{\rho}^2+2(\bar{y}-1)\right\}^2+4\bar{z}^2\right] \tag{A,19}$$

and similarly

$$r_1^2=\frac{1}{\bar{\rho}_1^4}\left[\left\{\bar{\rho}_1^2+2(\bar{y}_1-1)\right\}^2+4\bar{z}_1^2\right]. \tag{A,20}$$

Let us now introduce the ellipsoidal coordinates

$$\begin{aligned}
\bar{x}&=\sqrt{1-\mu^2}\sqrt{1+\eta^2}\cos\vartheta & \bar{x}_1&=\sqrt{1-\mu_1^2}\sqrt{1+\eta_1^2}\cos\vartheta_1\\
\bar{y}&=\sqrt{1-\mu^2}\sqrt{1+\eta^2}\sin\vartheta & \bar{y}_1&=\sqrt{1-\mu_1^2}\sqrt{1+\eta_1^2}\sin\vartheta_1\\
\bar{z}&=\mu\eta & \bar{z}_1&=\mu_1\eta_1 .
\end{aligned} \tag{A,21}$$

Substitution of the relations (A,18), (A,19), (A,20) and (A,21) into the formula (A,17) yields after elementary calculations

$$V_P(\eta,\mu,\vartheta;\eta_1,\mu_1,\vartheta_1)=\frac{2}{\pi}\frac{1}{R}\left\{\arctan\frac{\mu\mu_1+\eta\eta_1}{R}+\frac{\pi}{2}\right\}. \tag{A,22}$$

As to our circular wing problem Green's function of the second kind is found by

$$V_p(\eta,\mu,\vartheta;\eta_1,\mu_1,\vartheta_1)-V_{p'}(\eta,\mu,\vartheta;\eta_1,-\mu_1,\vartheta_1)=$$

$$=\frac{2}{\pi}\frac{1}{R}\left\{\arctan\frac{\mu\mu_1+\eta\eta_1}{R}+\frac{\pi}{2}\right\}-\frac{2}{\pi}\frac{1}{R'}\left\{\arctan\frac{-\mu\mu_1+\eta\eta_1}{R'}+\frac{\pi}{2}\right\} \quad \text{(A.23)}$$

On the wing-surface, i.e. $\eta_1=0$, the following expression (apart from a constant factor) is obtained

$$G(\eta,\mu,\vartheta;0,\mu_1,\vartheta_1)=\frac{2}{\pi}\frac{1}{R}\left\{\arctan\frac{\mu\mu_1}{R}+\frac{\pi}{2}\right\}-\frac{2}{\pi}\frac{1}{R}\left\{\arctan\frac{-\mu\mu_1}{R}+\frac{\pi}{2}\right\}$$

or

$$G(\eta,\mu,\vartheta;0,\mu_1,\vartheta_1)=\frac{4}{\pi}\frac{1}{R}\arctan\frac{\mu\mu_1}{R} . \quad \text{(A,24)}$$

In our theory the quantity $\left[\frac{\partial G}{\partial\mu_1}\right]_{\substack{\eta_1=0\\ \mu_1=0}}$ plays an important role. We easily derive

$$\left[\frac{\partial G}{\partial\mu_1}\right]_{\substack{\eta_1=0\\ \mu_1=0}}=\frac{4\mu}{\pi}\left[\frac{1}{R^2}\right]_{\substack{\eta_1=0\\ \mu_1=0}}=\frac{4\mu}{\pi}\frac{1}{(1-\mu^2)-2\sqrt{1-\mu^2}\sqrt{1+\eta^2}\cos(\vartheta-\vartheta_1)+(1+\eta^2)} . \quad \text{(A,25)}$$

This formula agrees completely with the relation (2,8,47) ,apart from the constant factor $\frac{-1}{8\pi}$.

Table I Numerical values of the coefficients $\overset{s}{\tau}{}^{n}_{\ell}$.

$\ell \backslash n$	0	1	2	3	4	5	6	7	8	9
0	-0.89046	+0.020727	-0.0086486	-0,0048645	-0.0031620	+0.0022396	-0.0016793	+0.0013114	-0.0010558	+0.00087047
1	+0.020727	+0.49981	-0.00091680	+0.00090482	-0.00077169	+0.00064550	-0.00054263	+0.00046103	-0.00039618	+0.00034414
2	+0,0090589	-0,0011348	-0:49994	+0.00020137	-0.00026463	+0.00026872	-0.00025382	+0.00023356	-0.00021288	+0.00019354
3	+0.0051840	-0.0010416	+0.00027410	+0.49998	-0.000073664	+0.00011177	-0.00012498	+0.00012691	-0.00012363	+0.00011801
4	+0.0033976	0.00086557	+0.00031475	-0.00010542	-0.49999	+0.000034704	-0.000057397	+0.000068245	-0.000072727	+0.000073696
5	+0.0024173	-0.00071407	+0.00030554	-0.00013525	+0.000051202	+0.49999	-0.000019018	+0.000033315	-0.000041332	+0.000045597
6	+0.0018173	-0.00059503	+0.00028212	-0.00014313	+0.000070213	-0.000028637	-0.50000	+0.000011524	-0.000021034	+0.000026924
7	+0.0014213	-0.00050243	+0.00025604	-0.00014141	+0.000078525	-0.000041060	+0.000017609	+0.50000	-0.0000075030	+0.000014122
8	+0.0011454	-0.00042977	+0.00023121	-0.00013550	+0.000081180	-0.000047722	+0.000026068	-0,000011591	-0.50000	+0.0000051545
9	+0.00094477	-0.00037196	+0.00020879	-0.00012793	+0.000080783	-0.000050970	+0.000031168	-0.000017576	+0.0000080308	+0.50000

Table II Numerical values of the coefficients $\overset{a}{\tau}{}^{n}_{k}$

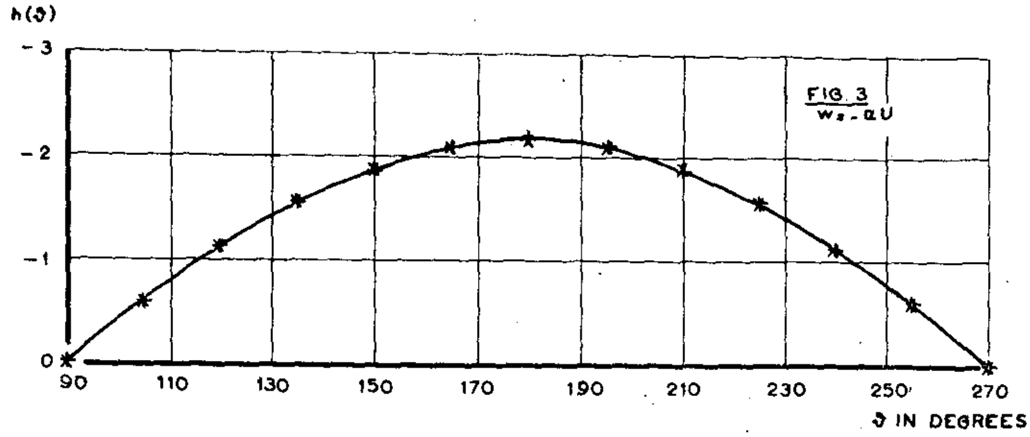
$k \backslash n$	0	1	2	3	4	5	6	7	8	9
0	-0.48553	+0.0012309	-0.0015797	+0.0012947	-0.0010307	+0.00083097	-0.00068244	+0.00057043	-0.00048426	+0.00041666
1	+0.0064496	+0.49949	-0.00024192	+0.00038179	-0.00038649	+0.00035715	-0.00032121	+0.00028678	-0.00025606	+0.00022934
2	+0.0037441	-0.00068386	-0.49990	+0.000084216	-0.00014686	+0.00016539	-0.00016617	+0.00015966	-0.00015036	+0.00014028
3	+0.0024815	-0.00063184	+0.00019826	+0.49997	-0.000038564	+0.000071352	-0.000085720	+0.000090880	-0.000091285	+0.000089220
4	+0.0017817	-0.00054848	+0.00022192	-0.000083080	-0.49999	+0.000020748	-0.000039921	+0.000050092	-0.000055138	+0.000057184
5	+0.0013497	-0.00047113	+0.00021887	-0.00010337	+0.000042414	+0.49999	-0.000012412	+0.000024555	-0.000031791	+0.000035977
6	+0.0010626	-0.00040601	+0.00020638	-0.00010998	+0.000056436	-0.000024508	-0.50000	+0.0000080033	-0.000016167	+0.000021432
7	+0.00086118	-0.00035246	+0.00019107	-0.00010996	+0.000063150	-0.000034161	+0.000015420	+0.50000	-0.0000054576	+0.000011205
8	+0.00071396	-0.00030851	+0.00017563	-0.00010672	+0.000065752	-0.000039628	+0.000022238	-0.000010325	-0.50000	+0.0000038868
9	+0.00060277	-0.00027223	+0.00016109	-0.00010199	+0.000066001	-0.000042509	+0.000026506	-0.000015282	+0.0000072485	+0.50000

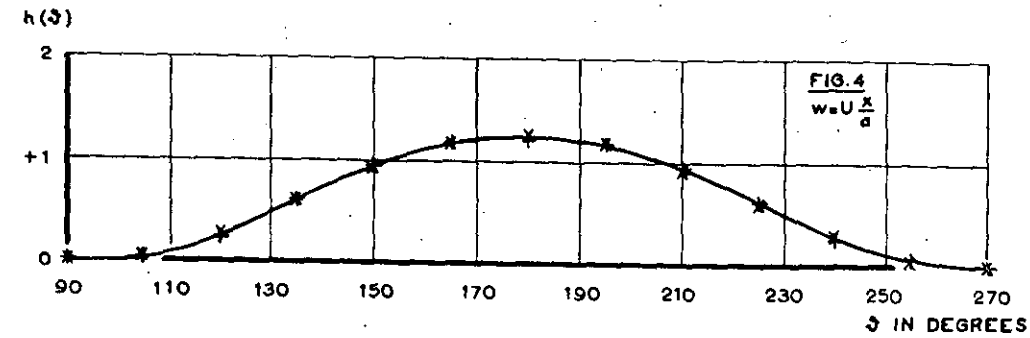
Table IX Numerical values of the coefficients $\vartheta_{\ell}^{s_n}$

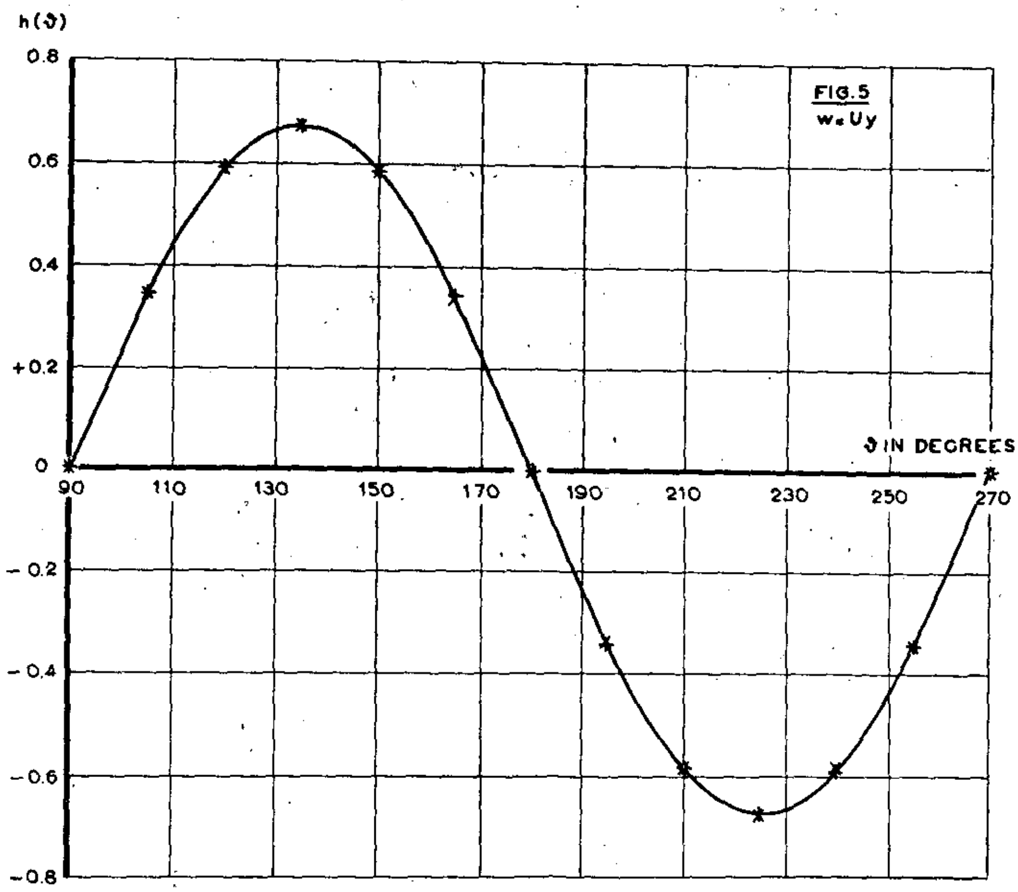
$\ell \backslash n$	0	1	2	3	4	5	6	7	8	9
0	-0,15915	-0,039789	+0,013263	-0,0066315	+0,0039789	-0,0026526	+0,0018947	-0,0014210	+0,0011052	-0,00088419
1	+0,16446	+0,23820	+0,092499	-0,021385	+0,010153	-0,0060692	+0,0040778	-0,0029432	+0,0022306	-0,0017518
2	+0,036547	-0,10204	-0,27849	-0,097217	+0,020905	-0,0095990	+0,0056600	-0,0037872	+0,0027351	-0,0020788
3	+0,016353	-0,024063	+0,10085	+0,29178	+0,099460	-0,020760	+0,0093259	-0,0054216	+0,0035967	-0,0025849
4	+0,0093026	-0,011477	+0,022023	-0,10138	-0,29841	-0,10079	+0,020731	-0,0091889	+0,0052659	-0,0034795
5	+0,0060095	-0,0068644	+0,010183	-0,021384	+0,10198	+0,30239	+0,10168	-0,020741	+0,0091151	-0,0052040
6	+0,0042039	-0,046091	+0,0060243	-0,0096636	+0,021131	-0,10249	-0,30505	-0,10231	+0,020764	-0,0090732
7	+0,0031064	-0,0033235	+0,0040375	-0,0056376	+0,0094109	-0,021020	+0,10290	+0,30694	+0,10279	-0,020790
8	+0,0023894	-0,0025162	+0,0029180	-0,0037479	+0,0054306	-0,0092729	+0,020970	-0,10323	-0,30836	-0,10316
9	+0,0018951	-0,0019741	+0,0022184	-0,0026970	+0,0035823	-0,0053083	+0,0091913	-0,020949	+0,10350	+0,30947

Table X Numerical values of the coefficients $\vartheta_{\ell}^{c_n}$.

$\ell \backslash n$	0	1	2	3	4	5	6	7	8	9
0	+0,15031	+0,086651	-0,021600	+0,010400	-0,0062085	+0,0041486	-0,0029754	+0,0022412	-0,0017501	+0,0014052
1	-0,10762	-0,26515	-0,095385	+0,021083	-0,0098320	+0,0058423	-0,0039225	+0,0028356	-0,0021545	+0,0016969
2	-0,027043	+0,10092	+0,28646	+0,098503	-0,020809	+0,0094380	-0,0055234	+0,0036802	-0,0026520	+0,0020138
3	-0,012968	+0,022721	-0,10108	-0,29557	-0,10020	+0,020737	-0,0092464	+0,0053448	-0,0035313	+0,0025308
4	-0,0077057	+0,010668	-0,021625	+0,10169	+0,30062	+0,10128	-0,020733	+0,0091466	-0,0052401	+0,0034380
5	-0,0051286	+0,0063550	-0,0098728	+0,021231	-0,10225	-0,30384	-0,10202	+0,020752	-0,0090913	+0,0051752
6	-0,0036663	+0,0042703	-0,0057984	+0,0095170	-0,021065	+0,10270	+0,30607	+0,10257	-0,020777	+0,0090592
7	-0,0027542	+0,0030873	-0,0038709	+0,0055196	-0,0093323	+0,020990	-0,10307	-0,30770	-0,10298	+0,020804
8	-0,0021460	+0,0023453	-0,0027925	+0,0036546	-0,0053621	+0,0092271	-0,020957	+0,10337	+0,30895	+0,10331
9	-0,0017199	+0,0018465	-0,0021218	+0,0026228	-0,0035252	+0,0052655	-0,0091630	+0,020944	-0,10362	-0,30993







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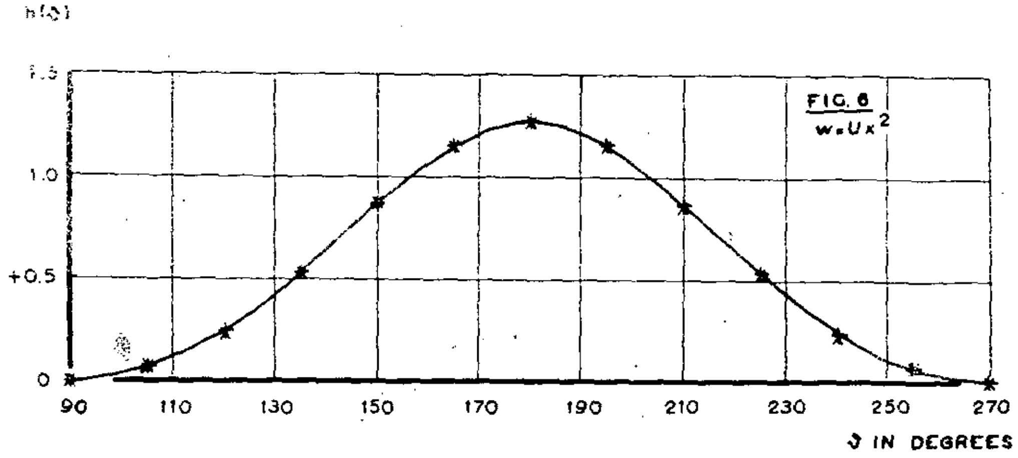
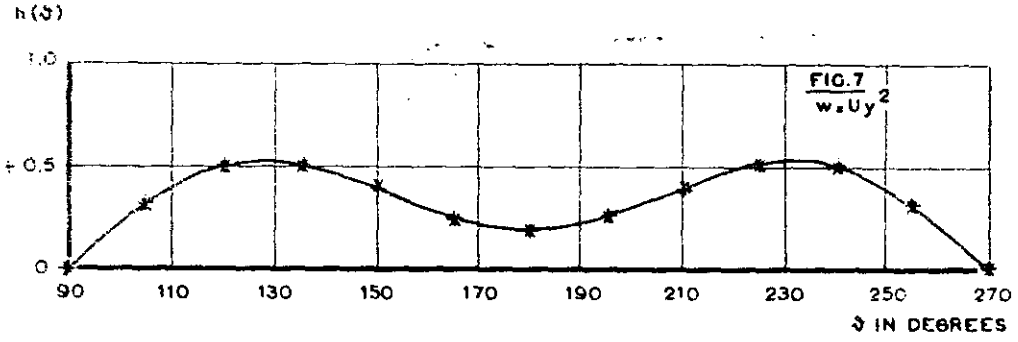
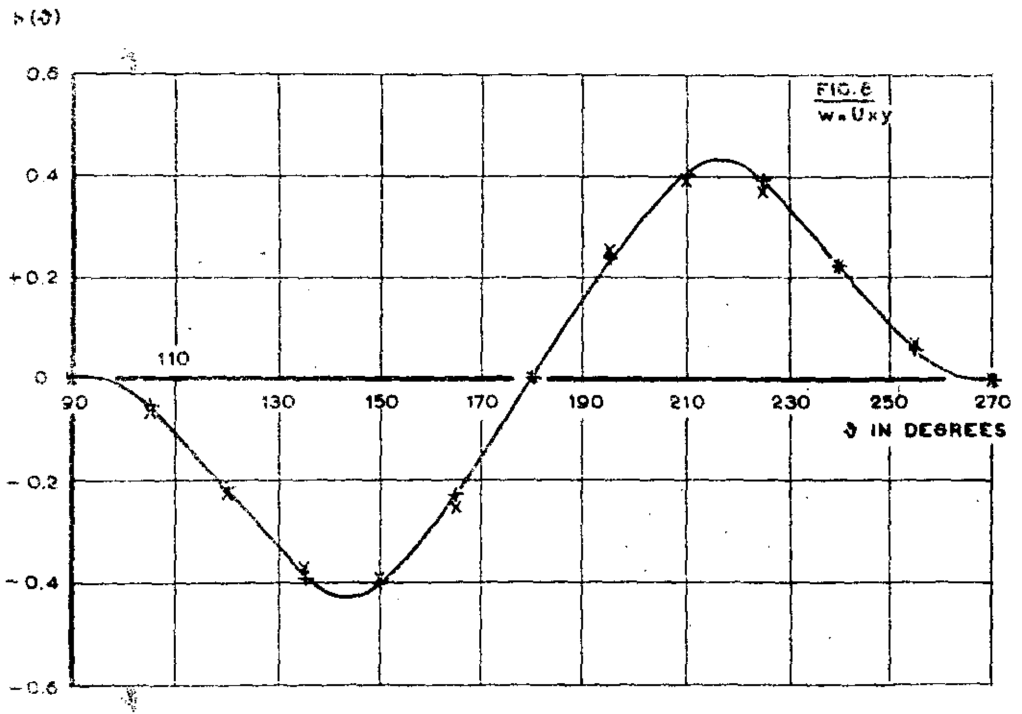
X THE FUNCTIONS h(ϑ) APPROX. BY 10 TERMS OF THE FOURIER SERIES







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X THE FUNCTION h(ϑ) APPROX. BY 10 TERMS OF THE FOURIER SERIES

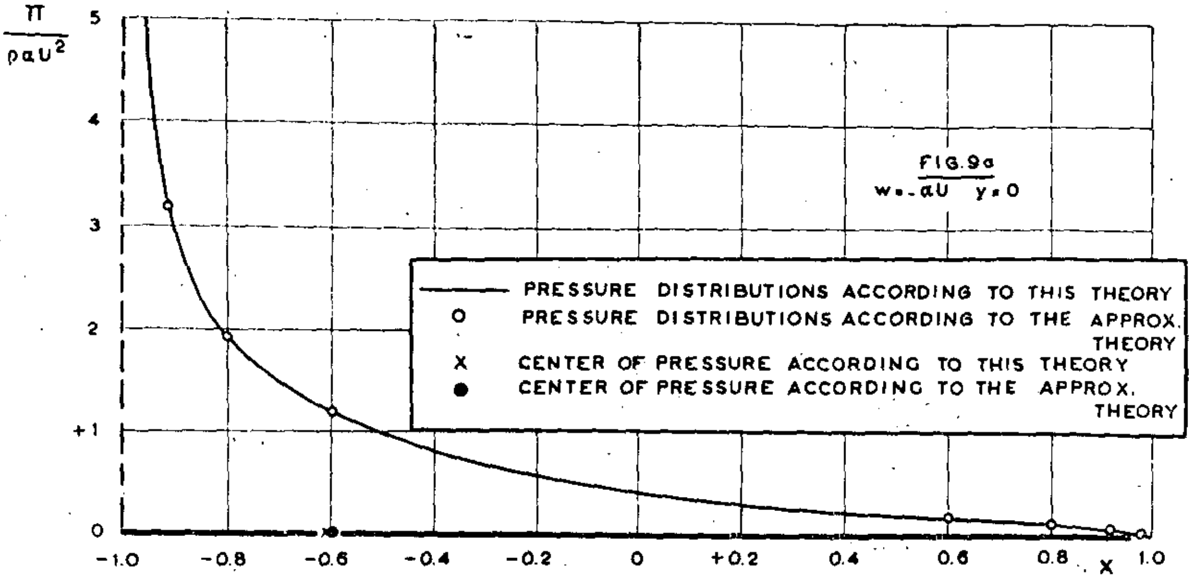


FIG. 9a
w = -αU y = 0
PRESSURE DISTRIBUTIONS ACCORDING TO THIS THEORY
O PRESSURE DISTRIBUTIONS ACCORDING TO THE APPROX. THEORY
X CENTER OF PRESSURE ACCORDING TO THIS THEORY
● CENTER OF PRESSURE ACCORDING TO THE APPROX. THEORY
π / ραU²
5
4
3
2
+1
0
-1.0
-0.8
-0.6
-0.4
-0.2
0
+0.2
0.4
0.6
0.8
X
1.0

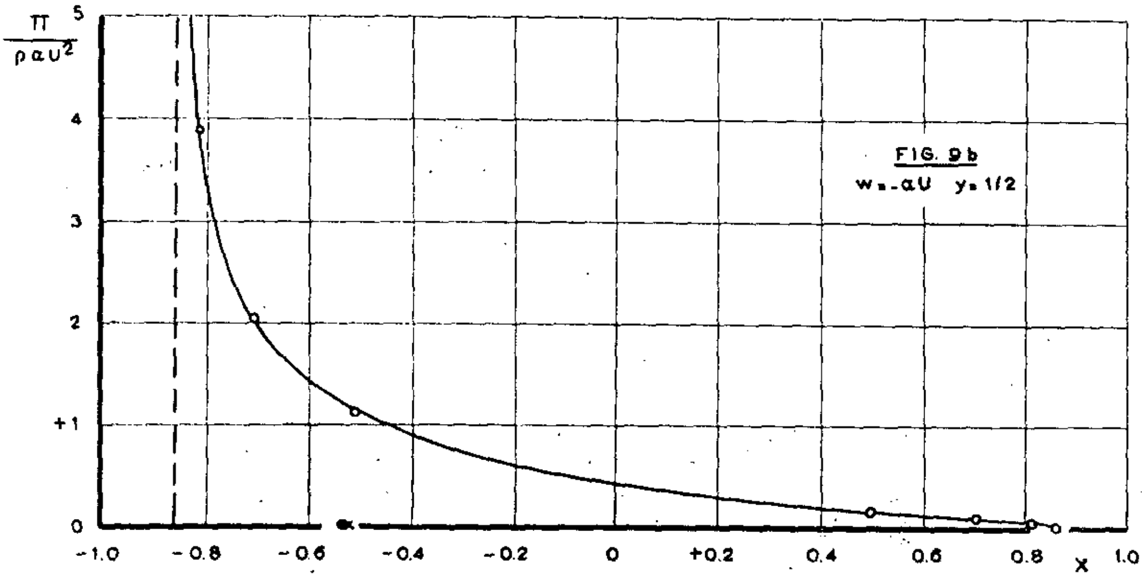


FIG. 9b
w = -αU y = 1/2
π / ραU²
5
4
3
2
+1
0
-1.0
-0.8
-0.6
-0.4
-0.2
0
+0.2
0.4
0.6
0.8
X
1.0

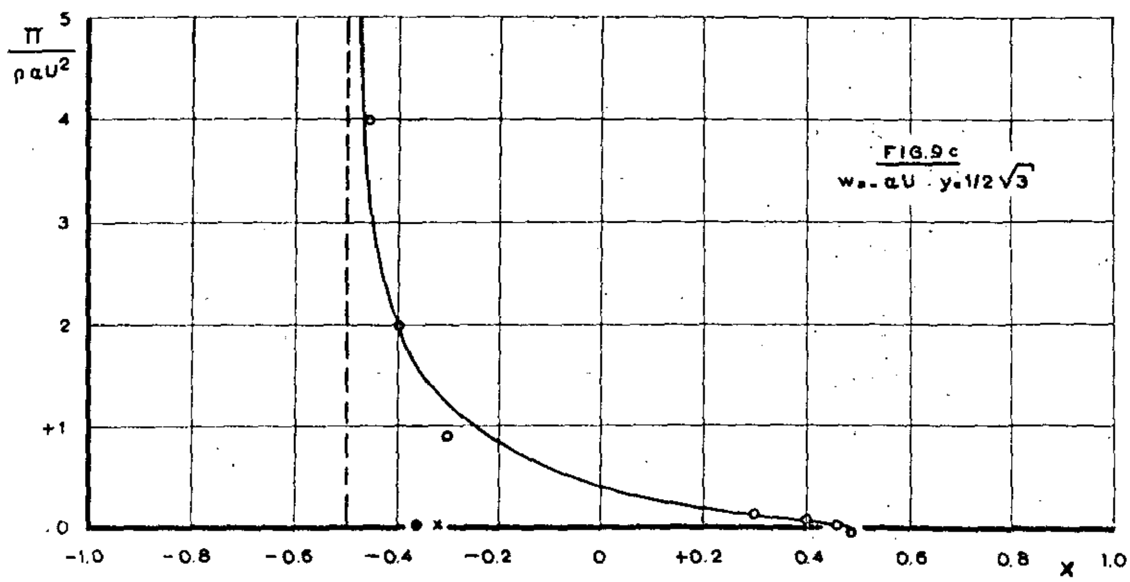


FIG. 9c
w = -αU y = 1/2√3
π / ραU²
5
4
3
2
+1
0
-1.0
-0.8
-0.6
-0.4
-0.2
0
+0.2
0.4
0.6
0.8
X
1.0

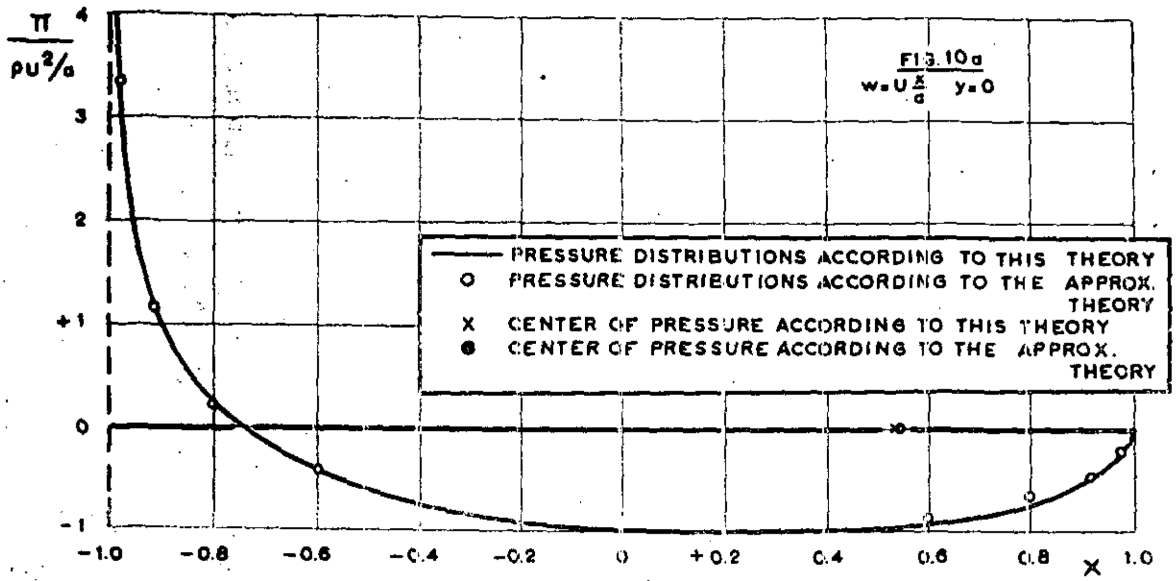


FIG. 10a
w = U x/a y = 0
π / ρU²/a
PRESSURE DISTRIBUTIONS ACCORDING TO THIS THEORY
PRESSURE DISTRIBUTIONS ACCORDING TO THE APPROX. THEORY
CENTER OF PRESSURE ACCORDING TO THIS THEORY
CENTER OF PRESSURE ACCORDING TO THE APPROX. THEORY
x

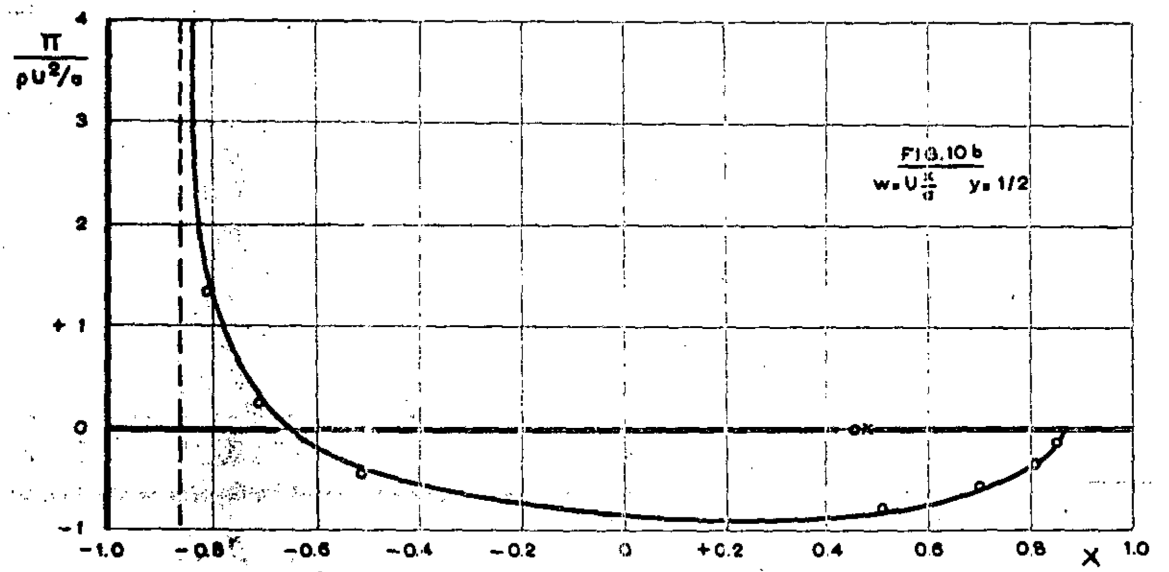


FIG. 10b
w = U x/a y = 1/2
π / ρU²/a
x

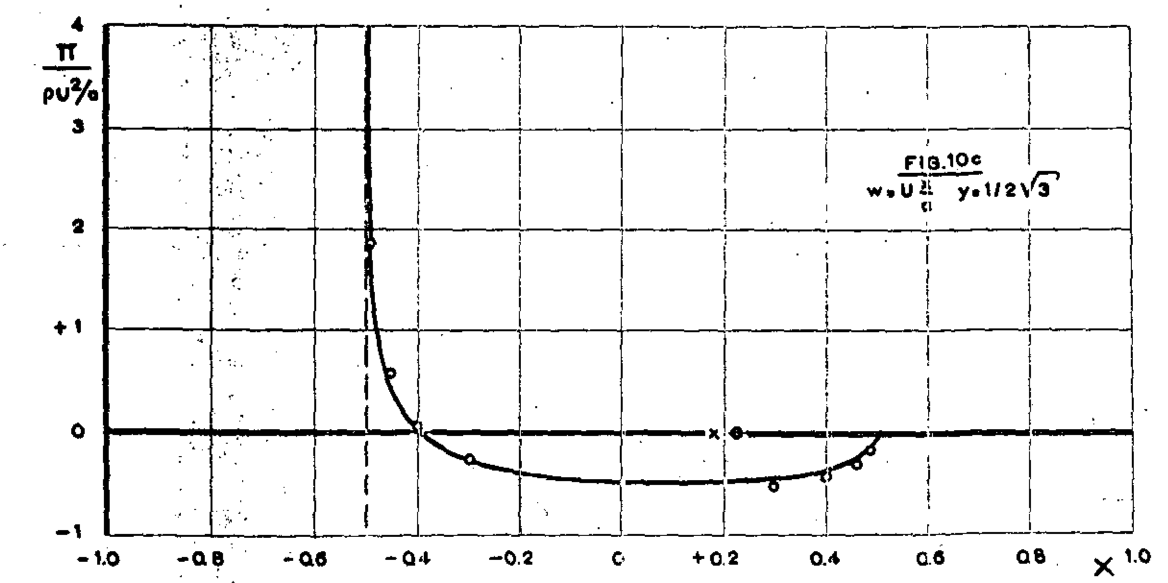


FIG. 10c
w = U x/a y = 1/2√3
π / ρU²/a
x

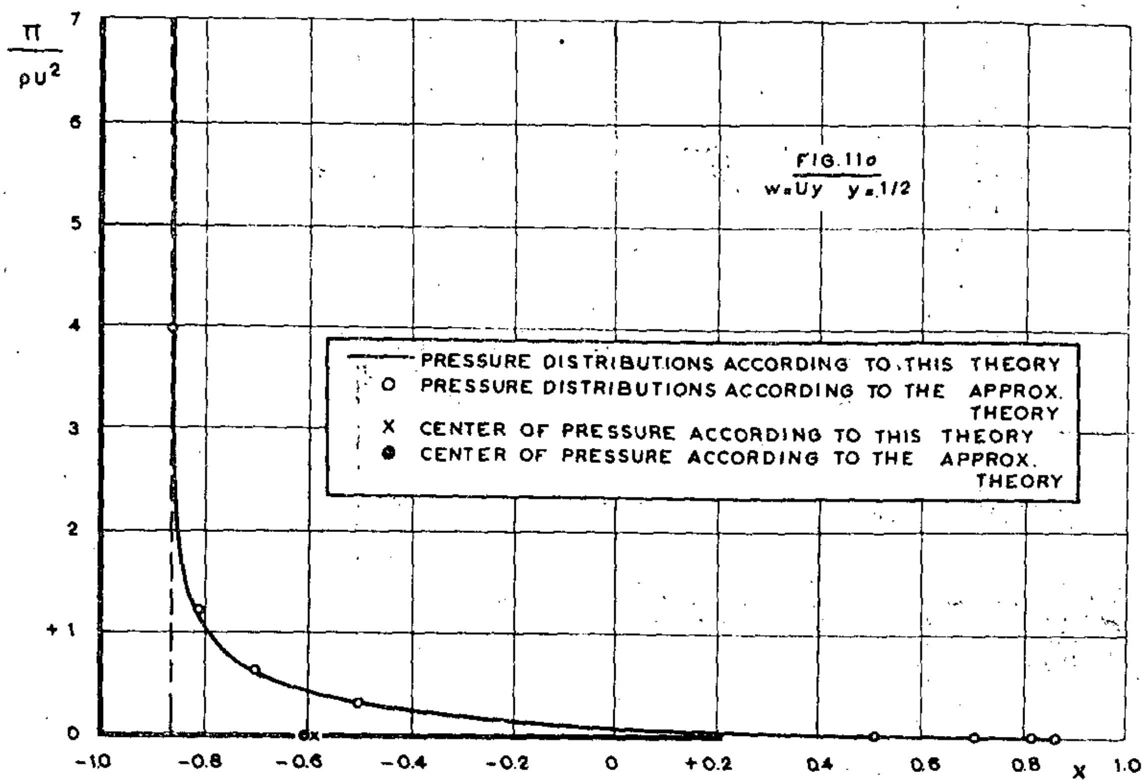


FIG. 11a
w=Uy y=1/2
PRESSURE DISTRIBUTIONS ACCORDING TO THIS THEORY
O PRESSURE DISTRIBUTIONS ACCORDING TO THE APPROX. THEORY
X CENTER OF PRESSURE ACCORDING TO THIS THEORY
● CENTER OF PRESSURE ACCORDING TO THE APPROX. THEORY
$\frac{\Pi}{\rho U^2}$
7
6
5
4
3
2
+1
0
-1.0
-0.8
-0.6
-0.4
-0.2
0
+0.2
0.4
0.6
0.8
x
1.0

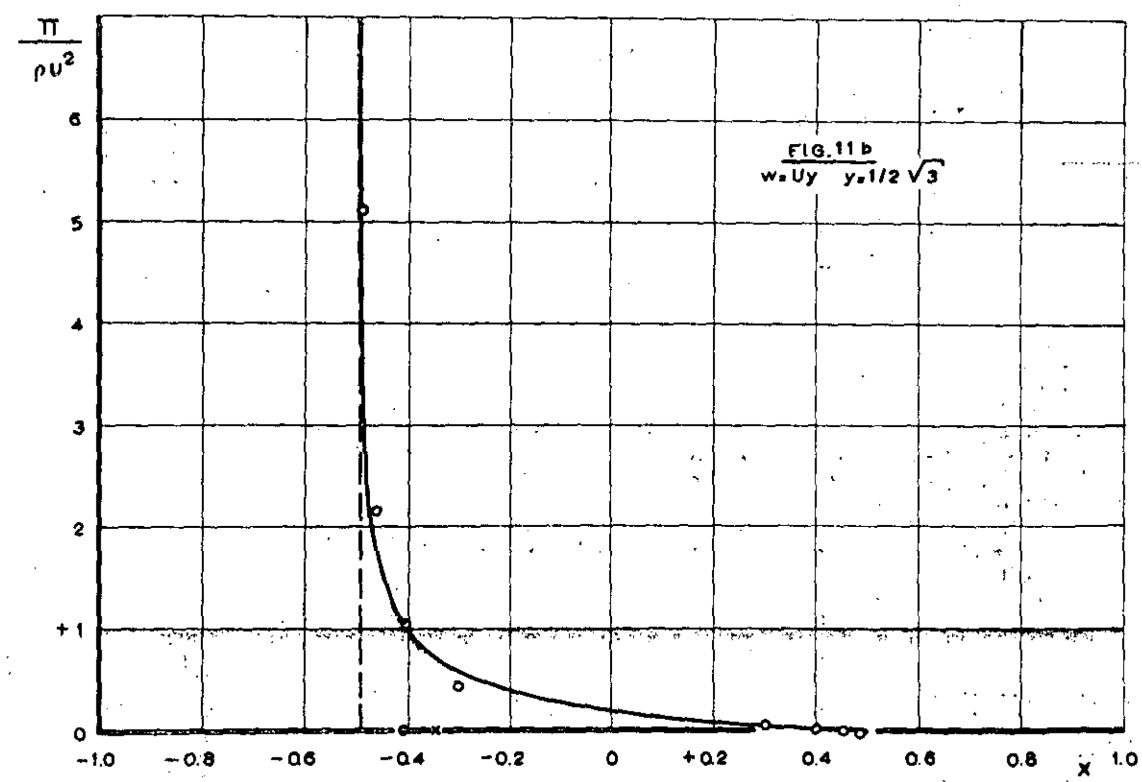


FIG. 11b
w=Uy y=1/2 $\sqrt{3}$
$\frac{\Pi}{\rho U^2}$
6
5
4
3
2
+1
0
-1.0
-0.8
-0.6
-0.4
-0.2
0
+0.2
0.4
0.6
0.8
x
1.0

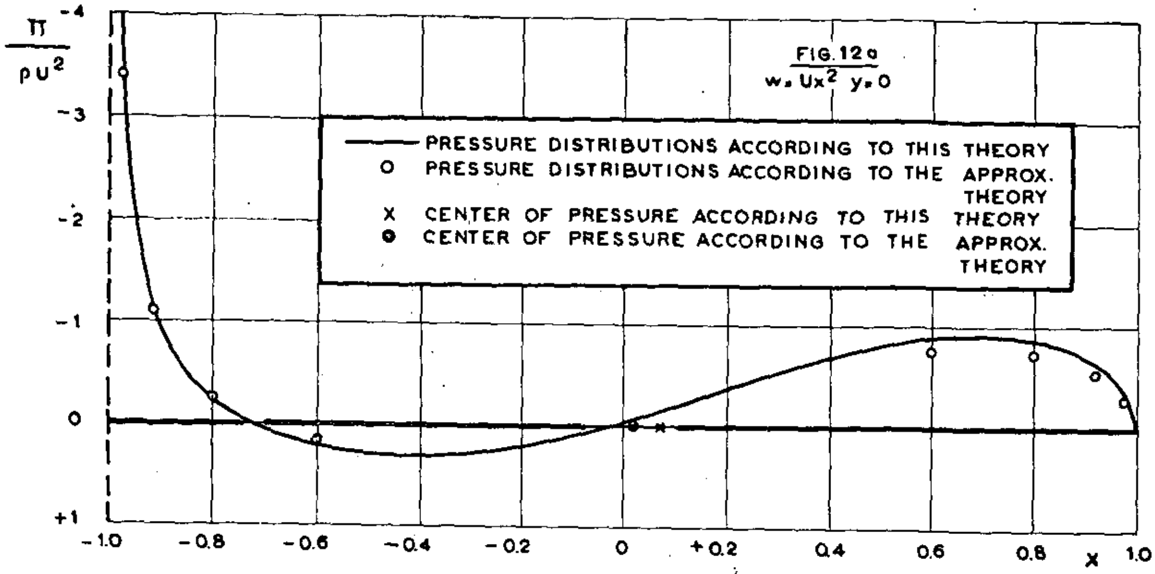


FIG. 12a
w = Ux² y = 0
PRESSURE DISTRIBUTIONS ACCORDING TO THIS THEORY
O PRESSURE DISTRIBUTIONS ACCORDING TO THE APPROX. THEORY
X CENTER OF PRESSURE ACCORDING TO THIS THEORY
● CENTER OF PRESSURE ACCORDING TO THE APPROX. THEORY
π / ρU²
-4
-3
-2
-1
0
+1
-1.0
-0.8
-0.6
-0.4
-0.2
0
+0.2
0.4
0.6
0.8
1.0
X

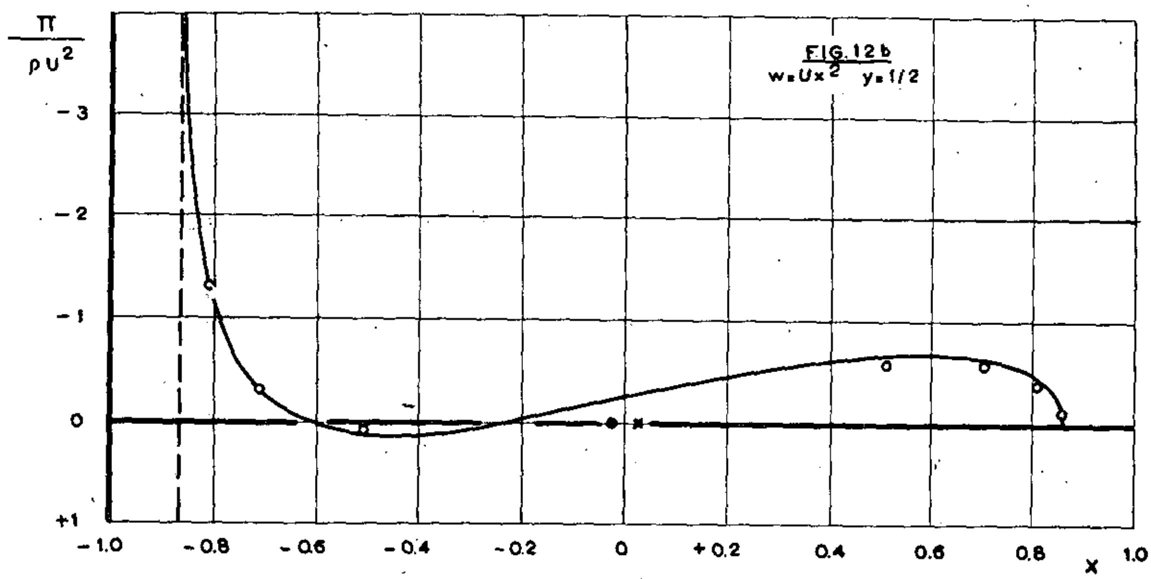


FIG. 12b
w = Ux² y = 1/2
π / ρU²
-3
-2
-1
0
+1
-1.0
-0.8
-0.6
-0.4
-0.2
0
+0.2
0.4
0.6
0.8
1.0
X

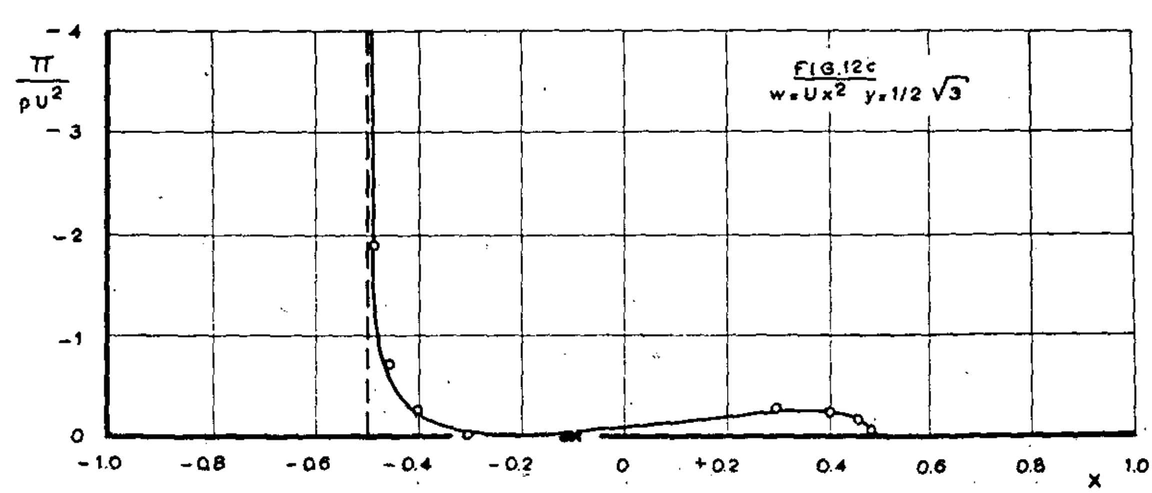


FIG. 12c
w = Ux² y = 1/2 √3
π / ρU²
-4
-3
-2
-1
0
-1.0
-0.8
-0.6
-0.4
-0.2
0
+0.2
0.4
0.6
0.8
1.0
X

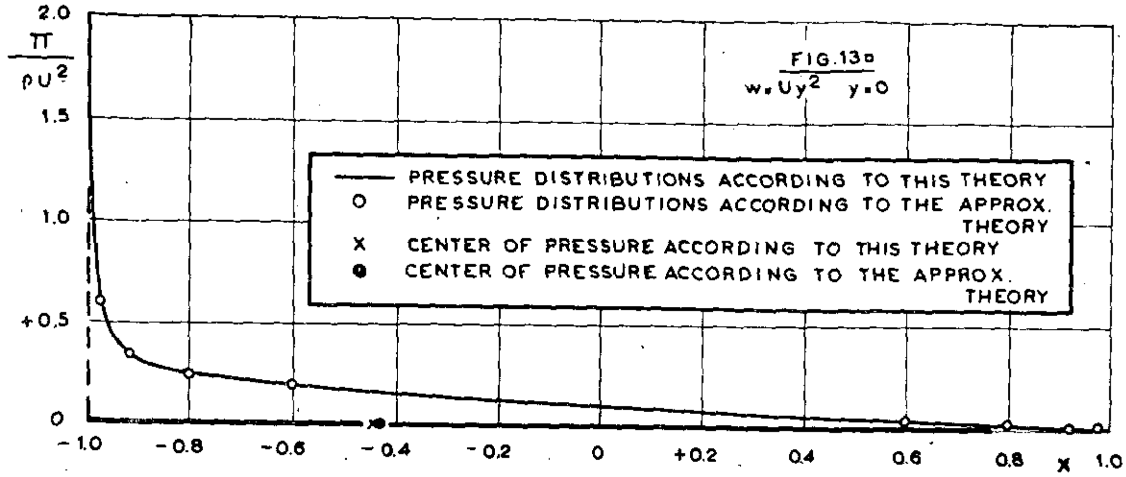


FIG. 13a
w = Uy² y = 0
PRESSURE DISTRIBUTIONS ACCORDING TO THIS THEORY
O PRESSURE DISTRIBUTIONS ACCORDING TO THE APPROX. THEORY
X CENTER OF PRESSURE ACCORDING TO THIS THEORY
● CENTER OF PRESSURE ACCORDING TO THE APPROX. THEORY
Π / ρU²
2.0
1.5
1.0
+0.5
0
−1.0
−0.8
−0.6
−0.4
−0.2
0
+0.2
0.4
0.6
0.8
X
1.0

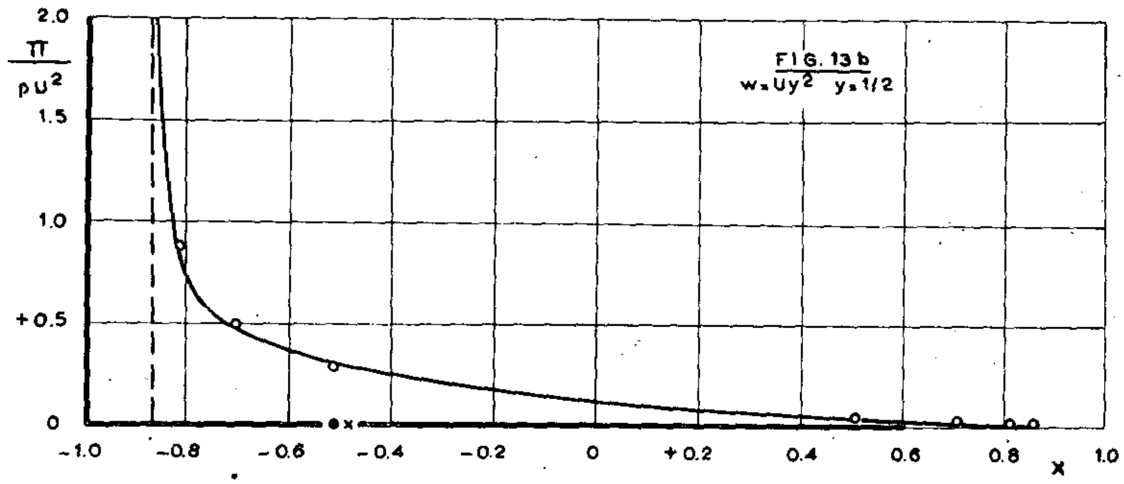


FIG. 13b
w = Uy² y = 1/2
Π / ρU²
2.0
1.5
1.0
+0.5
0
−1.0
−0.8
−0.6
−0.4
−0.2
0
+0.2
0.4
0.6
0.8
X
1.0

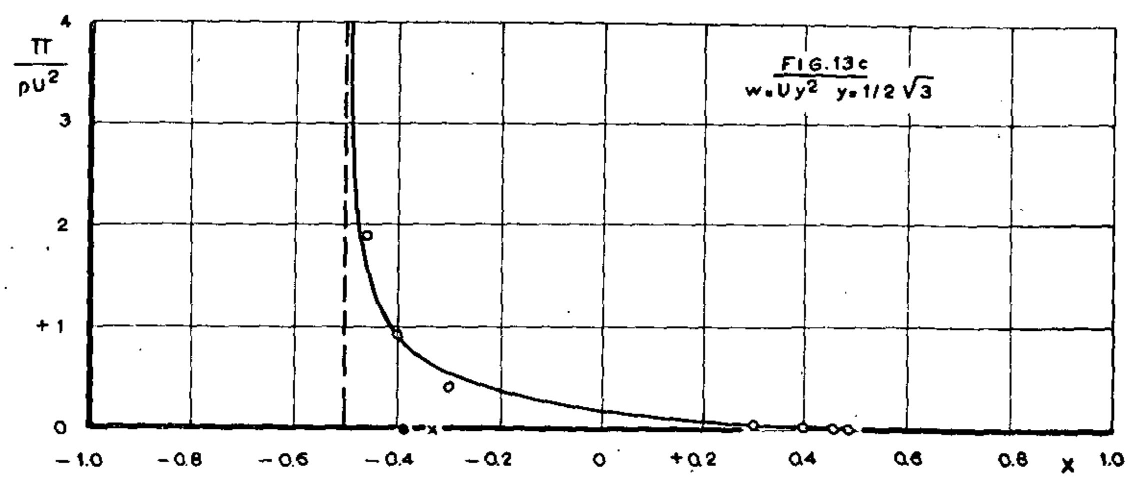


FIG. 13c
w = Uy² y = 1/2 √3
Π / ρU²
4
3
2
+1
0
−1.0
−0.8
−0.6
−0.4
−0.2
0
+0.2
0.4
0.6
0.8
X
1.0

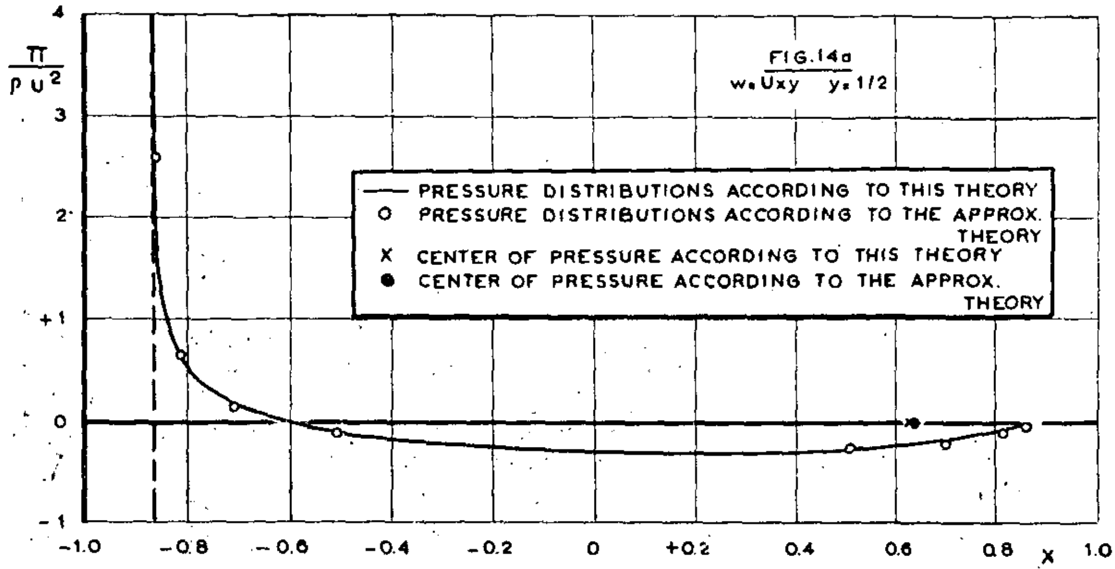


FIG. 14a
w = Uxy y = 1/2
PRESSURE DISTRIBUTIONS ACCORDING TO THIS THEORY
PRESSURE DISTRIBUTIONS ACCORDING TO THE APPROX. THEORY
CENTER OF PRESSURE ACCORDING TO THIS THEORY
CENTER OF PRESSURE ACCORDING TO THE APPROX. THEORY
$\frac{\pi}{\rho U^2}$
X

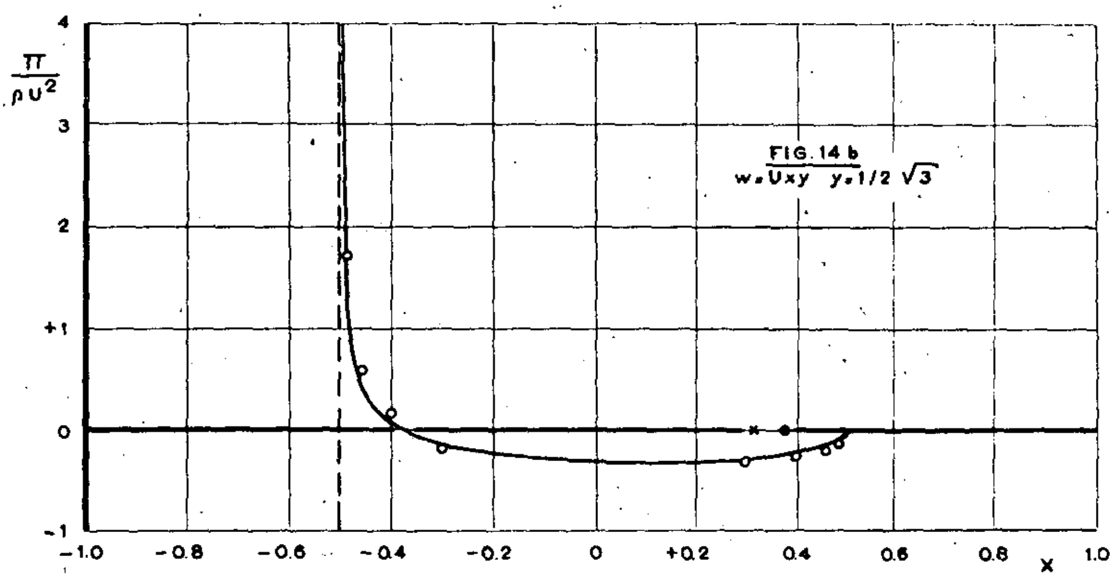
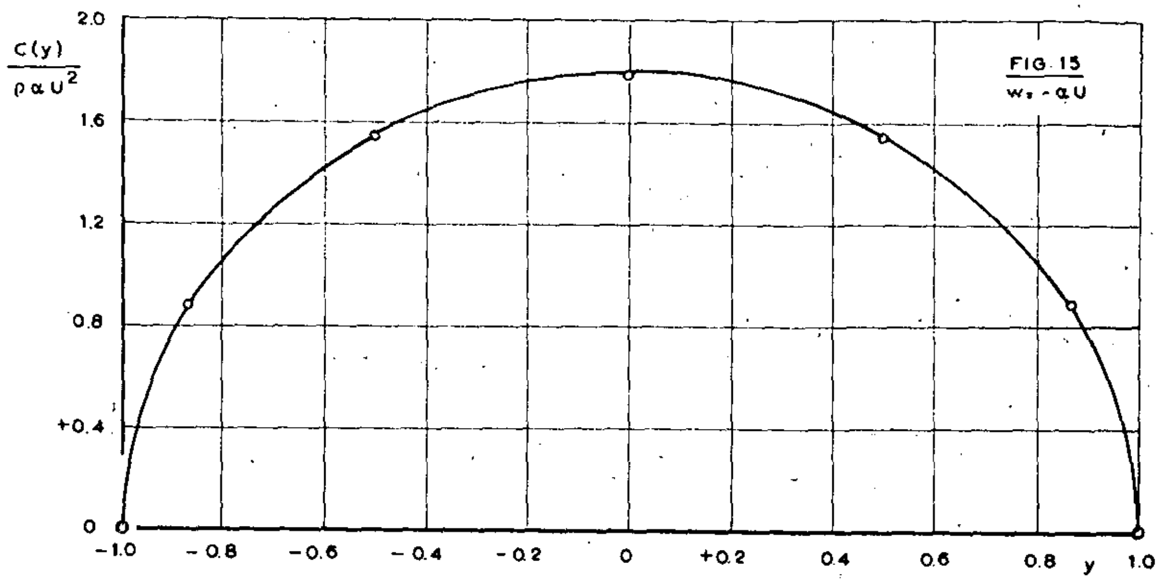
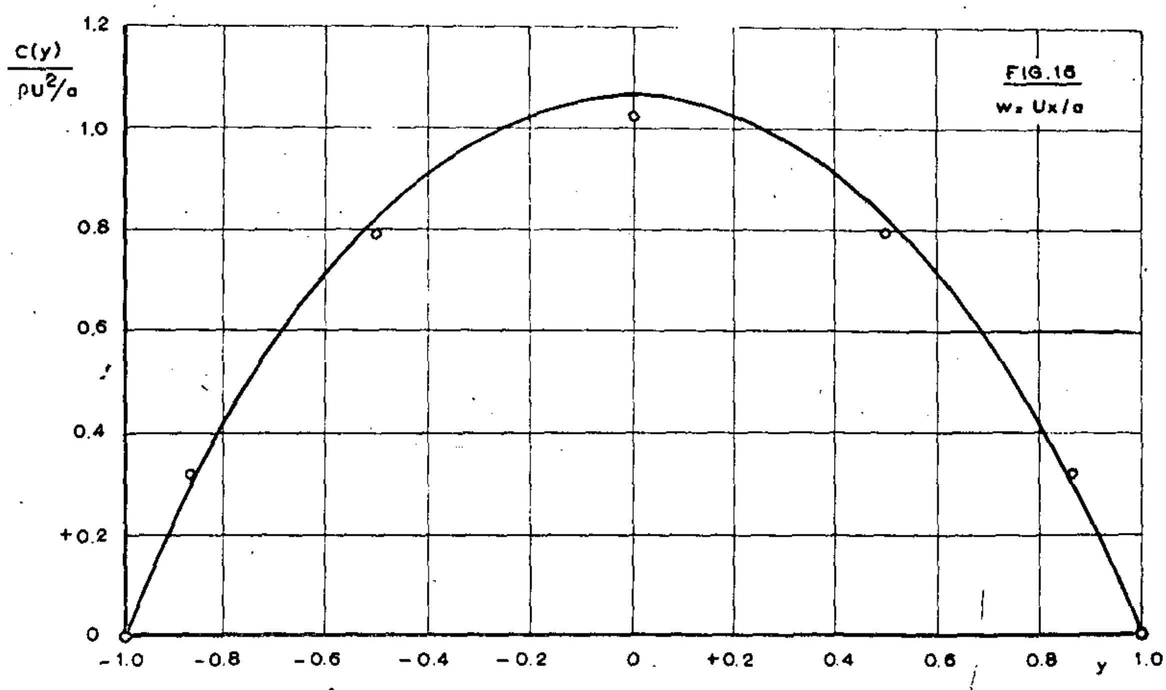


FIG. 14 b
w = Uxy y = 1/2 $\sqrt{3}$
$\frac{\pi}{\rho U^2}$
X



—— SPANWISE LIFT DISTRIBUTION ACCORDING TO THIS THEORY

o SPANWISE LIFT DISTRIBUTION ACCORDING TO APPROX. THEORY



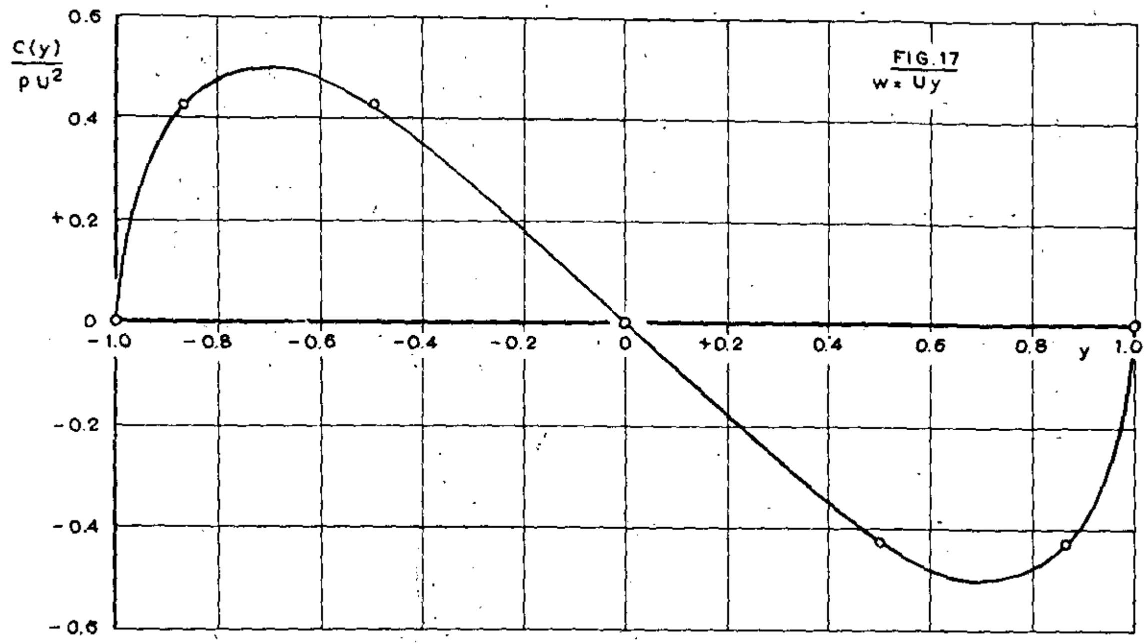


FIG. 17
$w = Uy$

———	SPANWISE LIFT DISTRIBUTION ACCORDING TO THIS THEORY
o	SPANWISE LIFT DISTRIBUTION ACCORDING TO APPROX. THEORY

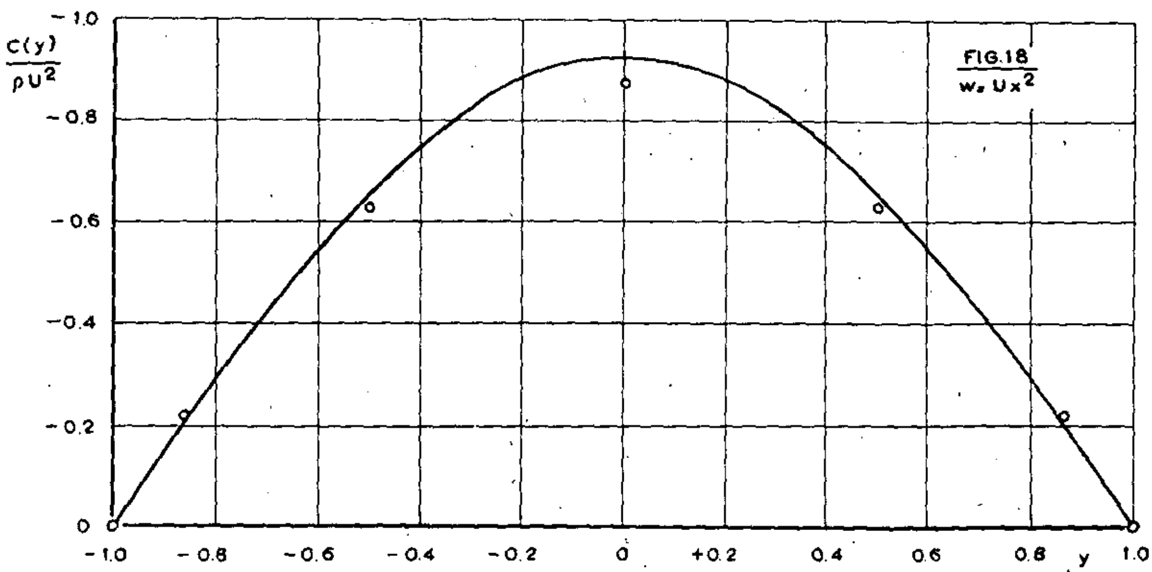
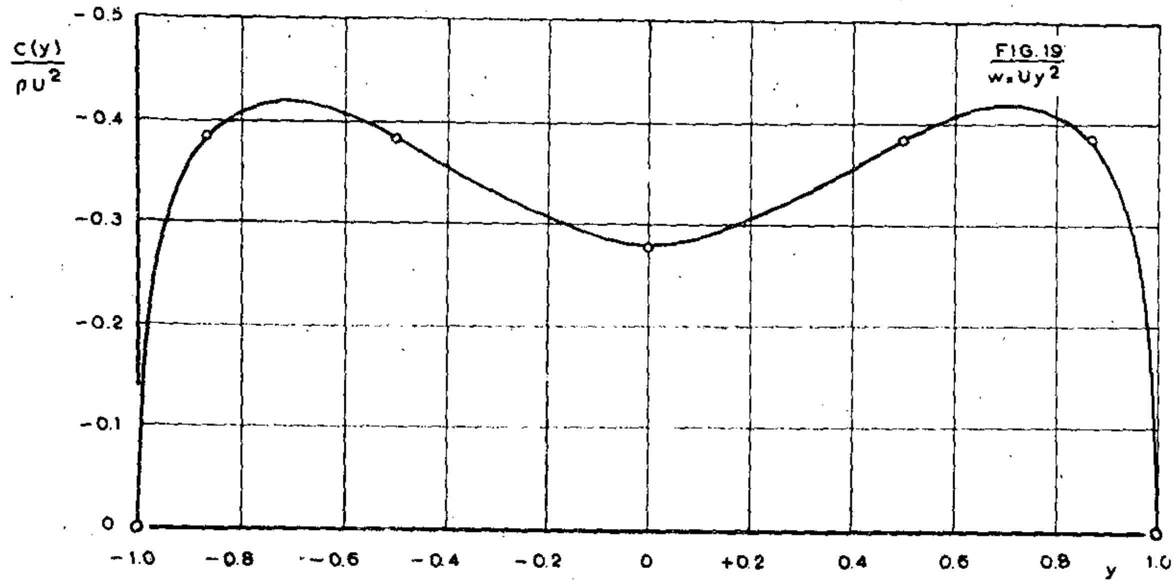


FIG. 18
$w = Ux^2$



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o	SPANWISE LIFT DISTRIBUTION ACCORDING TO APPROX. THEORY