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By joint decree of 13th April 1961 of the Ministers of Transport, Roads and Waterways, of Defence, of Economic Affairs, of Education, Arts and Sciences, and of Finance, the name of the "Stichting Nationaal Luchtvaartlaboratorium" (shortened N.L.L.) (= "Foundation National Aeronautical Research Institute"), was changed into "*Stichting Nationaal Lucht- en Ruimtevaartlaboratorium*" (*abbreviated by* N.L.R.) (= "Foundation National Aero- and Astronautical Research Institute").

PREFACE.

The report published in this Volume of "Verslagen en Verhandelingen" ("Reports and Transactions") of the "Nationaal Lucht- en Ruimtevaartlaboratorium" (National Aero- and Astronautical Research Institute) served the author as a thesis for the degree of doctor in the Technical Sciences at the Technical University of Delft.

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A. J. Marx
Director of the
"Nationaal Lucht- en Ruimtevaartlaboratorium"

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TECHNICAL REPORT S. 578

ON THE ANALYSIS OF SWEPT WING STRUCTURES

BY

J. P. BENTHEM

ERRATA

page 3	3rd line from below read: cross section i.s.o. cross
page 18	9th line from below read: *one system* i.s.o. **one** or both systems
page 20	11th line from below read: sin θ i.s.o. sin 0
page 32	4th line from below read: V^{*}_{A+B} i.s.o. V_{A+B}
page 93	8th line from below read: ε_x and ε_y i.s.o. ε_y
page 103	17th line read: attack i.s.o. attach

In tables 8.3–8.7 the factor 10^6 should be omitted.

Contents.

Summary.

Existing methods to analyse discontinuous aircraft wing structures (discrete-rib theories) usually start with the assumption of the only-shear carrying plate, which means that the spar- and rib webs as well as the skin panels are supposed to carry only shear stresses along their edges. If considered necessary, their normal stress-carrying capacity is added to the adjacent booms and flanges. In this way, the wing structure is made finitefold statically indeterminate and the exact solution for this simplified structure can be obtained by various methods of analysis, which, of course, all yield the same answers. This method is correct in general for rectangular or nearly rectangular panels. It appears however that this method yields erroneous results if applied to oblique panels, like those occurring in swept wings with ribs parallel to the flight direction.

Methods exist for non-swept wings where the ribs are considered to be infinitely stiff in their planes and where these ribs are assumed to be continuously distributed along the span. This scheme proves to be inadequate when applied to swept wings with ribs in flight direction.

In the present work discrete deformable ribs are assumed and the stress-strain relations of the oblique skin panels are not or hardly simplified. The structure then consists of a great number of structural elements, in which the state of stress is either one-dimensional (only normal force carrying sparbooms and rib flanges, only shear carrying spar- and rib webs) or two-dimensional (the oblique skin panels together with the continuously distributed stringers). The latter skin panels make the structure infinitely-fold statically indeterminate and only approximations for the displacements, strains and stresses can be obtained by means of the principles of variational calculus (minimum of the complementary energy and minimum of the potential energy). The type of the structure asks for a special form of application of these principles, where the use of oblique coordinates and matrix notation is of great importance.

Both the variational principles mentioned are used in order to compare their results for a five-cell swept box beam clamped at one side. Besides, for important displacements upper and lower bounds between which the exact answers must lie are determined.

With the object to establish an elementary theory for swept beams corresponding to the elementary theory for straight beams the relations between stress, curvatures and load are determined for the infinitely long beam under constant shear load. In addition, the root effect for the clamped semi-infinite swept box beam has been established.

<u>Notations.</u>

E	Elasticity modulus .
ν	Poisson ratio.
$G = \frac{E}{2(1+\nu)}$	Shear modulus.
x,y	Coordinates in an oblique plane coordinate system.
x,y,z	Coordinates in a right handed oblique coordinate system. The z-axis is perpendicular to the x and the y-axis.
θ	Angle between x and y axes .
h	Plate thickness .
σ_x, σ_y, τ	Stress components in an oblique coordinate system τ is also a shear stress (in a rectangular system) in rib- or spar webs.
$s_x = h\,\sigma_x = s_1$, $s_y = h\,\sigma_y = s_2$, $t = h\,\tau = s_3$	Stressflow components. In an oblique system, s_x and s_y are called "axial stressflows", t"tangential stressflow". A "shear stressflow" t may also occur in rib or spar webs.
$\vec{K}(K_x,K_y,K_z)$	Force with its components in the oblique system x,y,z (fig. 3.1b) .
$\vec{u}(u_x,u_y,u_z)$	Displacement with its orthogonal projections on the oblique coordinate axes x,y and z (fig. 3.1d) .
$\vec{M}(M_x,M_y,M_z)$	Righthanded moment with its projections (fig.3.1d).
$\vec{\varphi}(\varphi_x,\varphi_y,\varphi_z)$	Righthanded rotation with its components (fig.3.1b).
$\varepsilon_x = \frac{\partial u_x}{\partial x} = \varepsilon_1$, $\varepsilon_y = \frac{\partial u_y}{\partial y} = \varepsilon_2$, $\gamma = (\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) = \varepsilon_3$	Strain components in an oblique coordinate system x,y. The components ε_x and ε_y are called "axial strains" and γ "tangential strain". In spar booms ε_x is a normal strain and in rib booms ε_y is a normal strain.
$\gamma = (\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y})$	Also shear strain in a rib web .
and	
$\gamma = (\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x})$	Also shear strain in a spar web .

a_{ij} Stiffness matrix of the stress-strain relations :

$$s_x = a_{11}\varepsilon_x + a_{12}\varepsilon_y + a_{13}\gamma$$

$$s_y = a_{21}\varepsilon_x + a_{22}\varepsilon_y + a_{23}\gamma$$

$$t = a_{31}\varepsilon_x + a_{32}\varepsilon_y + a_{33}\gamma$$

$$a_{ij} = a_{ji} .$$

A_{ij} Flexibility matrix of the strain-stress relations :

$$\varepsilon_x = A_{11}s_x + A_{12}s_y + A_{13}\, t$$

$$\varepsilon_y = A_{21}s_x + A_{22}s_y + A_{23}\, t$$

$$\gamma = A_{31}s_x + A_{32}s_y + A_{33}\, t$$

$$A_{ij} = A_{ji} .$$

V^* Complementary energy, defined by (6.1) .

V Potential energy, defined by (6.27) .

For the dimensions of the swept-back box see also the notations of section 7.2 and 7.3.

1 Introduction.

The characteristic feature of the planform of an aircraft wing, called swept wing, is such that the leading edge makes a large angle with the lateral axis of the aircraft; in flight direction the wing tip is aft of the wing root. The structural consequence is that the spars are not normal or approximately normal to the plane of symmetry of the wing, which introduces a distinct obliqueness of the structure. The stiffness of the skin takes part in this obliqueness. As far as the ribs are concerned there are two possibilities; the rib planes can be either approximately normal to the spars, or they can be placed parallel to the plane of symmetry.

In the first case the structural problem differs from the problem with the straight, i.e. non swept wing, only by the fact that the analysis has to account for the structure of triangular planform between the rib at the root of the swept part of the wing and the plane of symmetry. This monograph deals with the structure where the ribs are parallel to the plane of symmetry. Then the angles between the ribs and the spars or the skin stiffeners differ considerably from the right angle and the skin panels between the ribs and the spars are distinctly oblique. Consequently the methods of analysis for straight wings are not applicable to the swept structure.

For straight wings there are two methods of stress analysis. The first method starts from the well known theories of the cylindrical beam loaded by a bending moment, by a torsional moment (De Saint Venant, ref.1) and by a shear force (De Saint Venant, ref.2). For thinwalled cylinders, where a cross section may consist of more than one cell[1], loaded in bending a specialization is superfluous. For torsional load the specialization was given by Bredt, ref.3; for shear load specializations were given by Van der Neut, ref.4, Leibenson, ref.5 and Koiter, ref.6. Ref. 4 is valid for the case where a continuum of ribs infinitely stiff in their plane is present (or if Poisson's ratio $\nu = 0$). Ref. 5 is valid for the beam without ribs. Ref. 6 applies to both cases.

The, usually so called, engineering theory considers any cross of a non-cylindrical beam to be the cross section of a cylindrical beam, carrying the same load (bending moment, torsional load or shear force, accom-

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1) Such cells are separated by walls in the longitudinal direction of the cylinder. In section 7 and the subsequent sections the name cell is used for the part of the structure between two successive ribs.

panied by a linearly varying moment) along its entire length. This assumption yields quite accurate results, except for the part of the structure close to clamped root, or more in general near a cross section where warping of the normal plane cannot occur freely due to discontinuity of torsional moment or shear load, or in the geometry of the structure. However, additional corrective computations or considerations can account with fair accuracy for the compatibility of warping to either side of the discontinuity. For a special case of non cylindrical beams, the conical one, Hadji-Argyris and Dunne have extended the theories of the thinwalled cylindrical tubes (ref.7).

Koning and Van der Neut (ref. 8, 9, 10 and 11) developed a method for tapered box beams with two parallel spars, shear resisting skin and a continuous system of infinitely stiff ribs. Koiter and Van der Neut (ref.12) extended this work so as to incorporate rib deformation in shear.

With the <u>second method</u> for straight wings discrete ribs form part of the structural scheme. To make such a structure amendable for analysis the rib webs as well as the skin panels are assumed to carry along their edges shear stresses only. The capability of these elements to carry normal stresses may be accounted for by adding their longitudinal stiffness to that of the adjacent spar booms, ribflanges, or stringers or by concentrating it linewise, so as to represent the longitudinal stiffness by a number of stringers, which are not actually present.

A finite-fold statically indeterminate structure results and the exact solution for this scheme as to displacements, strains and stresses under a given load can be established, whatever method of solution is used provided the redundancy of the structural scheme is fully taken into account.

This schematization gives in general reliable results and is able in particular to predict stresses at the root or in general the stresses affected by discontinuities, which cannot be established by the engineering theory of bending and torsion. However, the applicability of the method is subject to the condition that the skin panels are about rectangular.

Van der Neut and Plantema (ref.13 and 14) dealt with the box beam with two not necessarily parallel spars, with discrete ribs deformable in shear and with shear resisting skin, such that planes parallel to the ribs carry shear stresses only, whereas the normal strain in rib direction is considered to be zero. Also in this case a finite-fold statically indeterminate structure remains.

Both methods have been extended to <u>swept wings</u> with ribs in flight direction by several authors. Flügge, ref.15 and 16, deals with the mono-cell thin-walled cylinder, infinite in length, with a continuum of oblique ribs infinitely stiff in their planes and loaded by a moment or a shear force. Wittrick and Thomson, ref.17 to 21 give the theory of mono-cell and multi-cell thinwalled cylinders and cones, again with a continuum of oblique ribs infinitely stiff in their planes. The stiffness of the skin and of the continuized and discrete stringers may vary exponentially along the generators, as well as the bending or torsional moment. An additional method to establish perturbation stresses, which occur if an oblique section is prevented from warping, is also given. However, this additional method proved to be of little practical use.

It will be shown that the continuum of infinitely stiff ribs introduces a severe deviation from the actual behaviour of the structure.

Hemp, ref.22, continuizes the oblique ribs as well, but takes their proper bending and shear stiffness into account. However, his structure has been greatly simplified. It consists of a box beam with rectangular or nearly rectangular cross section with spar booms at the corners. The top and bottom skin are stiffened with continuized stringers. The spar webs are carrying only shear. Solutions for constant moments and constant shear force are given together with an exact as well as an approximate - and more practical - method for calculation of perturbation stresses in the region where an oblique section is prevented from warping out of the plane and from distortion in its plane. This approach has over Wittrick's and Thomson's method the advantage that it accounts for finite rib stifness, but it remains questionable what the consequences are of continuizing widely spaced ribs. Besides the method must still be considered to be little suitable for wings where successive ribs are loaded quite differently due to inhomogenity of the structure (e.g. discontinuity of angle of obliqueness) or due to discontinuity of the lateral loads (e.g. ribs introducing large shear loads in the beam).

In order to take into account discrete ribs, the same supposition could be made as for beams with ribs normal to the webs, the oblique skin panels are capable only to carry shear stresses along their edges, whereas again their stiffness with respect to normal stresses is carried over to the stiffeners by increasing their stiffness with an appropriate additional amount. Such a supposition was used by Levy, ref.23, but it fails to predict an important aspect of the root perturbation stresses. This would also be the case if it would be tried to apply the already mentioned schematization of ref.13 and 14.[1)]

In the present work the stress-strain relations of the oblique skin panels are not violated, but this introduces particular difficulties. The structure consists of a great number of structural elements, in which the state of stress is one-dimensional (only normal forces carrying spar booms, rib flanges or other stiffeners, only shear carrying spar and rib webs) or two-dimensional (the oblique skin panels). These skin panels make the structure infinitelyfold statically indeterminate. Therefore solutions can be obtained only by introducing approximations for the displacements and strains or for the stresses. The use of the principles of variational calculus is required (minimum of the potential energy and minimum of the complementary energy), but the type of the structure asks special forms of application, which are examined in detail. It has been the purpose not to be satisfied with approximate answers only, but to get upper and lower bounds for some important displacements. Determination of these bounds is only possible if the stress-strain relations of the structure are not, or hardly, simplified.

For the present oblique structure, use of oblique coordinates is very fruitful and there is a better correspondence with the methods used for straight wings.

For the development of the theory and for performing numerical calculations, matrix notation and matrix calculus are of invaluable importance. In recent years a great stimulus to the use of matrix calculus in the stress analysis of aircraft structures was given by Langefors, ref.25 and Hadji-Argyris (ref.26 and 27). The structures considered in these references are finite-fold statically indeterminate and only consist of elements in which the state of stress is one-dimensional. Then the solutions obtained are exact solutions for the schematized structures.

Matrix calculus is the more important, since the number of unknowns to be solved from linear equations will be greater than in calculations for straight wings. Using an electronic computer a great number of unknowns is no longer prohibitive. Results can get every desired accuracy, which accuracy may be estimated by determining upper and lower bounds. It is therefore believed that the significance of the present methods may be not confined to oblique wing structures. They may enable to refine the

1) The work on swept wings hitherto mentioned is summarized and discussed by the author in ref.24.

analysis of straight wings, or the methods may be applied to the triangular root structure of swept wings with ribs normal to the spar webs (ref.24).

Everywhere in the numerical work the unit of force is the lb, the unit of length the inch. The reason for the adaption of these units is that the numerical applications refer to a structure (ref.36,37), the data of which were given in these units.

2 The thin isotropic plate in orthogonal coordinates.

The two-dimensional problem of the plate loaded in its plane is governed by the following 6 conditions:

(1) The geometry of the plate.

(2) The prescribed forces of stresses along the boundary of the plate.

(3) The prescribed displacements.

(4) The equilibrium equations, which read

$$\frac{ds_x}{dx} + \frac{dt}{dy} = 0 \tag{2.1}$$

$$\frac{ds_y}{dy} + \frac{dt}{dx} = 0 \tag{2.2}$$

with

$$\begin{aligned} s_x &= h\sigma_x \\ s_y &= h\sigma_y \\ t &= h\tau \end{aligned} \tag{2.3}$$

where σ_x, σ_y, and τ are stress components and h = plate thickness. The quantities s_x, s_y and t are called stressflow components, s_x and s_y are normal stressflows, t is a shear stressflow.

(5) The strain-stress relations which read

$$\begin{aligned} \varepsilon_x &= \frac{1}{Eh} s_x - \frac{\nu}{Eh} s_y \\ \varepsilon_y &= -\frac{\nu}{Eh} s_x + \frac{1}{Eh} s_y \\ \gamma &= \frac{2(1+\nu)}{Eh} t \end{aligned} \tag{2.4}$$

or in a general form to use later

$$\varepsilon_x = A_{11}s_x + A_{12}s_y + A_{13}t$$
$$\varepsilon_y = A_{21}s_x + A_{22}s_y + A_{23}t \qquad (2.5)$$
$$\gamma = A_{31}s_x + A_{32}s_y + A_{33}t$$

where the symmetrical matrix $A_{ij} = \begin{Vmatrix} \frac{1}{Eh} & -\frac{\nu}{Eh} & 0 \\ -\frac{\nu}{Eh} & \frac{1}{Eh} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{Eh} \end{Vmatrix}$. (2.6)

A_{ij} is called the flexibility matrix.
With the notation

$$\varepsilon_x = \varepsilon_1 \qquad s_x = s_1$$
$$\varepsilon_y = \varepsilon_2 \qquad s_y = s_2 \qquad (2.7)$$
$$\gamma = \varepsilon_3 \qquad t = s_3$$

(2.5) can be written

$$\varepsilon_i = \sum_{j=1}^{3} A_{ij}s_j \qquad (2.8)$$

or with the summation convention

$$\varepsilon_i = A_{ij}s_j \quad . \qquad (2.9)$$

The inverse relations of (2.4), the stress-strain relations, are

$$s_x = \frac{Eh}{1-\nu^2}\varepsilon_x + \frac{\nu Eh}{1-\nu^2}\varepsilon_y$$
$$s_y = \frac{Eh}{1-\nu^2}\varepsilon_x + \frac{Eh}{1-\nu^2}\varepsilon_y \qquad (2.10)$$
$$t = \frac{Eh}{2(1+\nu)}\gamma$$

or, in a general form

$$s_x = a_{11}\varepsilon_x + a_{12}\varepsilon_y + a_{13}\gamma$$
$$s_y = a_{21}\varepsilon_x + a_{22}\varepsilon_y + a_{23}\gamma \qquad (2.11)$$
$$t = a_{31}\varepsilon_x + a_{32}\varepsilon_y + a_{33}\gamma$$

with

$$a_{ij} = \begin{Vmatrix} \frac{Eh}{1-\nu^2} & \frac{\nu Eh}{1-\nu^2} & 0 \\ \frac{\nu Eh}{1-\nu^2} & \frac{Eh}{1-\nu^2} & 0 \\ 0 & 0 & \frac{Eh}{2(1+\nu)} \end{Vmatrix} , \tag{2.12}$$

a_{ij} is called the stiffness matrix.
With the notation (2.7) and the summation convention the equations (2.11) take the form

$$s_i = a_{ij}\varepsilon_j \; . \tag{2.13}$$

The symmetrical matrices (2.6) and (2.12) are inverse to each other. It is noted that

$$A_{ij}^{-1} = a_{ij} \quad \text{or} \quad A_{ij} = a_{ij}^{-1} \; . \tag{2.14}$$

(6) The compatibility condition for the strains

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma}{\partial x \partial y} \tag{2.15}$$

which is obtained from the requirement that it must be possible to derive the strains from the (small) continuous displacement components u_x and u_y (respectively, in x and y direction) as follows:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u_x}{\partial x} \\ \varepsilon_y &= \frac{\partial u_y}{\partial y} \\ \gamma &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \; . \end{aligned} \tag{2.16}$$

The equilibrium equations (2.1), (2.2) are satisfied when s_x, s_y and t are derived from Airy's stress function Ψ:

$$\begin{aligned} s_x &= \frac{\partial^2 \Psi}{\partial y^2} \\ s_y &= \frac{\partial^2 \Psi}{\partial x^2} \\ t &= -\frac{\partial^2 \Psi}{\partial x \partial y} \; . \end{aligned} \tag{2.17}$$

After substitution of (2.17) in (2.5) and thereafter of (2.5) in (2.15), it follows with h = constant and A_{ij} is constant, that the compatibility condition expressed in the stress function is

$$A_{22}\frac{\partial^4\Psi}{\partial x^4}-2A_{23}\frac{\partial^4\Psi}{\partial x^3\partial y}+(2A_{12}+A_{33})\frac{\partial^4\Psi}{\partial x^2\partial y^2}-2A_{13}\frac{\partial^4\Psi}{\partial x\partial y^3}+A_{11}\frac{\partial^4\Psi}{\partial y^4}=0. \quad (2.18)$$

When A_{ij} is taken according to (2.6) the equation (2.18) yields

$$\frac{\partial^4\Psi}{\partial x^4}+2\frac{\partial^4\Psi}{\partial x^2\partial y^2}+\frac{\partial^4\Psi}{\partial y^4}=0 \quad . \quad (2.19)$$

The expression for the strain energy per unit of plate area is

$$A=\frac{1}{2}(s_x\varepsilon_x+s_y\varepsilon_y+t\gamma) \quad (2.20)$$

and after substitution of (2.5)

$$A=\frac{1}{2}(A_{11}s_x^{\,2}+2A_{12}s_xs_y+2A_{13}s_xt+A_{22}s_y^{\,2}+2A_{23}s_yt+A_{33}t^2) \quad (2.21)$$

or with the notation (2.7)

$$A=\frac{1}{2}A_{ij}s_is_j \quad . \quad (2.22)$$

Substitution of (2.11) in (2.20) yields

$$A=\frac{1}{2}(a_{11}\varepsilon_x^{\,2}+2a_{12}\varepsilon_x\varepsilon_y+2a_{13}\varepsilon_x\gamma+a_{22}\varepsilon_y^{\,2}+2a_{23}\varepsilon_y\gamma+a_{33}\gamma^2) \quad (2.23)$$

or with the notation (2.7)

$$A=\frac{1}{2}a_{ij}\varepsilon_i\varepsilon_j \quad . \quad (2.24)$$

3 The thin plate in oblique coordinates.

3.1 Forces, stressflows, displacements and strains.

An oblique system of coordinates x, y (fig.3.1) is introduced as was done by Hemp (ref.22) when investigating the swept wing structure. The distances between point x = 0, y = 0 and the points x = 0, y = 1 and x = 1, y = 0 respectively, are both the unit of length. Fig. 3.1a shows the coordinates of a point P; fig. 3.1b the components K_x and K_y of the force $\vec{K}$.

The magnitude of $\vec{K}$, expressed in its components, is

$$|\vec{K}|=\sqrt{K_x^{\,2}+K_y^{\,2}+2K_xK_y\cos\Theta} \quad . \quad (3.1)$$

A force $(K_x,0)$ is a force in the direction of the x axis of magnitude

K_x, a force $(0,K_y)$ is a force in the direction of the y axis of magnitude K_y.

Fig.3.1c represents the stressflow components s_x, s_y and t for an oblique element with the sides dx and dy. On the righthand side of this parallelogram acts a force per unit of length, whose components are s_x and t, on the lower side a force per unit of length with the components t and s_y. From equilibrium of moments it follows that the two components t are identical. However, they do not represent the total shear stressflows along the boundaries of the parallelogram, because the stressflows s_x and s_y contribute as well since they are not normal to the surfaces to which they are applied. Therefore, the stressflow t will be designated as "tangential stressflow". The stressflows s_x and s_y will now be called "axial stressflows".

The displacement vector $\vec{u}$ is not expressed by its components in the way as is done with the force $\vec{K}$, but by means of its normal projections u_x and u_y (fig.3.1d). The magnitude of $\vec{u}$ expressed in its projections is

$$|\vec{u}| = \frac{\sqrt{u_x^2 + u_y^2 - 2u_x u_y \cos\theta}}{\sin\theta} . \qquad (3.2)$$

A displacement $\vec{u}(u_x,0)$ is a displacement perpendicular to the y-axis with a magnitude $u_x/\sin\theta$. A displacement $\vec{u}(0,u_y)$ is a displacement perpendicular to the x-axis with a magnitude $u_y/\sin\theta$.

In agreement with (2.16), strains are again defined as

$$\varepsilon_x = \frac{du_x}{dx}$$

$$\varepsilon_y = \frac{du_y}{dy} \qquad (3.3)$$

$$\gamma = \frac{du_x}{dy} + \frac{du_y}{dx} .$$

Then the physical meaning of the strain ε_x is the specific extension $\frac{ds-ds_o}{ds_o}$ of a line element dx (side of the oblique element, fig.3.1c), where ds is the length in the strained state and ds_o the length in the unstrained state of the line element. The same is true for ε_y and a line element dy. The strains ε_x and ε_y are called axial strains. The γ component has no simple geometrical meaning. For sake of uniformity with the name tangential stressflows, the strain component γ will be called tangential strain. If γ' represents the decrease of the angle between dx and dy (the sides of the oblique element), the relation between ε_x, ε_y, γ and γ' is

$$\gamma' = \frac{\gamma}{\sin\theta} - (\varepsilon_x + \varepsilon_y)\,\mathrm{ctn}\,\theta \quad . \tag{3.4}$$

If a rectangular coordinate system $\bar{x}$, $\bar{y}$ is assumed in figure 3.1, so that the x and $\bar{x}$ coordinate axes coincide, the transformation formulas for the coordinates are

$$\begin{aligned} x &= \bar{x} - \bar{y}\,\mathrm{ctn}\,\theta & \bar{x} &= x + y\cos\theta \\ y &= \bar{y}/\sin\theta & \bar{y} &= y\sin\theta \quad . \end{aligned} \tag{3.5}$$

The force $\vec{K}$ with the components $\bar{K}_x$ and $\bar{K}_y$ in the rectangular system has in the oblique system the components K_x and K_y and the relation is analogeous to (3.5).

$$\begin{aligned} K_x &= \bar{K}_x - \bar{K}_y\,\mathrm{ctn}\,\theta & \bar{K}_x &= K_x + K_y\cos\theta \\ & \text{or} & & \\ K_y &= \bar{K}_y/\sin\theta & \bar{K}_y &= K_y\sin\theta \quad . \end{aligned} \tag{3.6}$$

A force $\vec{K}(\bar{K}_x, \bar{K}_y)$applied at the point $(\bar{x},\bar{y})$ yields a moment M with respect to the origin. This moment is a vector perpendicular to the $\bar{x},\bar{y}$ plane, hence with one component $M_z = M$, if the z-axis is normal to the x,y-plane. Its magnitude is

$$M = \bar{K}_y\bar{x} - \bar{K}_x\bar{y}$$

which becomes in oblique coordinates x,y

$$M = (K_y x - K_x y)\sin\theta \quad . \tag{3.7}$$

The relations between the stressflow components $\bar{s}_x$, $\bar{s}_y$, $\bar{t}$ in the rectangular system and the stressflow components s_x, s_y, t in the oblique system become

$$\begin{aligned} s_x &= \bar{s}_x\sin\theta + \bar{s}_y\cos\theta\,\mathrm{ctn}\,\theta - 2\bar{t}\cos\theta \\ s_y &= \bar{s}_y/\sin\theta \\ t &= -\bar{s}_y\,\mathrm{ctn}\,\theta + \bar{t} \end{aligned} \tag{3.8}$$

or

$$\left.\begin{aligned}\bar{s}_x &= s_x/\sin\theta + s_y\cos\theta\,\mathrm{ctn}\,\theta + 2t\,\mathrm{ctn}\,\theta\\ \bar{s}_y &= s_y\sin\theta\\ \bar{t} &= s_y\cos\theta + t\ .\end{aligned}\right\}\qquad(3.9)$$

The displacement vector $\vec{u}$ with components $\bar{u}_x$ and $\bar{u}_y$ in the rectangular system (they are also the projections) has, in the oblique system, the projections u_x and u_y

$$\left.\begin{aligned}u_x &= \bar{u}_x & &\qquad & \bar{u}_x &= u_x\\ & & &\text{or} & &\\ u_y &= \bar{u}_x\cos\theta + \bar{u}_y\sin\theta & & & \bar{u}_y &= -u_x\,\mathrm{ctn}\,\theta + u_y/\sin\theta\end{aligned}\right\}\qquad(3.10)$$

The strain components $\bar{\varepsilon}_x$, $\bar{\varepsilon}_y$, and $\bar{\gamma}$ are expressed in ε_x, ε_y, and γ by means of (3.10), (3.5) and (3.3)

$$\left.\begin{aligned}\bar{\varepsilon}_x &= \varepsilon_x\\ \bar{\varepsilon}_y &= \varepsilon_x\,\mathrm{ctn}^2\,\theta + \varepsilon_y/\sin^2\theta - \gamma\,\mathrm{ctn}\,\theta/\sin\theta\\ \bar{\gamma} &= -2\varepsilon_x\,\mathrm{ctn}\,\theta + \gamma/\sin\theta\ .\end{aligned}\right\}\qquad(3.11)$$

or, when inverted

$$\left.\begin{aligned}\varepsilon_x &= \bar{\varepsilon}_x\\ \varepsilon_y &= \bar{\varepsilon}_x\cos^2\theta + \bar{\varepsilon}_y\sin^2\theta + \bar{\gamma}\sin\theta\cos\theta\\ \gamma &= 2\bar{\varepsilon}_x\cos\theta + \bar{\gamma}\sin\theta\ .\end{aligned}\right\}\qquad(3.12)$$

With (small) displacements $\vec{u}(\bar{u}_x,\bar{u}_y)$, the rotation φ in the $\bar{x},\bar{y}$ plane is given by

$$\varphi = \frac{1}{2}\left(\frac{\partial\bar{u}_y}{\partial\bar{x}} - \frac{\partial\bar{u}_x}{\partial\bar{y}}\right)\ .\qquad(3.13)$$

If the small rotation φ is constant throughout the $\bar{x},\bar{y}$ plane, for example by rotation over the small angle φ about the origin, the displacements in the x,y plane are

$$\begin{aligned}\bar{u}_x &= -\varphi\bar{y}\\ \bar{u}_y &= +\varphi\bar{x}\ .\end{aligned}\qquad(3.14)$$

The corresponding relations for the oblique system x,y follow from the relations (3.13) and (3.14) by means of (3.10) and (3.5)

$$\varphi = \frac{\frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)}{\sin\theta} \tag{3.15}$$

$$\begin{aligned} u_x &= -\varphi y \sin\theta \\ u_y &= \varphi x \sin\theta \end{aligned} \qquad . \tag{3.16}$$

3.2 Equilibrium equations and compability condition.

The equations of equilibrium in the oblique system are identical with (2.1) and (2.2)

$$\begin{aligned} \frac{\partial s_x}{\partial x} + \frac{\partial t}{\partial y} &= 0 \\ \frac{\partial s_y}{\partial y} + \frac{\partial t}{\partial x} &= 0 \end{aligned} \qquad . \tag{3.17}$$

The equilibrium conditions are satisfied when s_x, s_y, t are derived from a stress function like that occurring in (2.17) :

$$\begin{aligned} s_x &= \frac{\partial^2 \Psi}{\partial y^2} \\ s_y &= \frac{\partial^2 \Psi}{\partial x^2} \\ t &= -\frac{\partial^2 \Psi}{\partial x \partial y} \end{aligned} \qquad . \tag{3.18}$$

Even the overall static equilibrium conditions remain applicable, that is the sum of the external force components in x direction, the sum of the external force components in y direction and the sum of the external moments (to be computed with (3.7)) must be zero.

Since the relations between strain and displacement components for rectangular and oblique coordinates are identical (equations (2.16) and (3.3)), the compatibility condition must be likewise identical

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma}{\partial x \partial y} \qquad . \tag{3.19}$$

3.3 Stress-strain relations for the isotropic plate.

The definitions for coordinates, forces, stressflows, displacements, and strains are chosen such that the equilibrium equations have the same form. The compatibility condition remains in the same form, and the strain components ε_x and ε_y have still a clear geometric meaning. However, the stiffness and flexibility matrices a_{ij} and A_{ij} are different for oblique and rectangular coordinates.

Maintaining the definition of these matrices in the form (2.5) and (2.11), one obtains from (3.12), (2.4) and (3.9)

$$A_{ij} = a_{ij}^{-1} = \begin{Vmatrix} \frac{1}{Eh\sin\theta} & \frac{\cos^2\theta - \nu\sin^2\theta}{Eh\sin\theta} & \frac{2}{Eh}\operatorname{ctn}\theta \\ \frac{\cos^2\theta - \nu\sin^2\theta}{Eh\sin\theta} & \frac{1}{Eh\sin\theta} & \frac{2}{Eh}\operatorname{ctn}\theta \\ \frac{2}{Eh}\operatorname{ctn}\theta & \frac{2}{Eh}\operatorname{ctn}\theta & \frac{2(1+\cos^2\theta+\nu\sin^2\theta)}{Eh\sin\theta} \end{Vmatrix} \tag{3.20}$$

and from (3.8), (2.10) and (3.11)

$$a_{ij} = \frac{Eh}{(1-\nu^2)\sin^3\theta}\begin{Vmatrix} 1 & \cos^2\theta+\nu\sin^2\theta & -\cos\theta \\ \cos^2\theta+\nu\sin^2\theta & 1 & -\cos\theta \\ -\cos\theta & -\cos\theta & \frac{1+\cos^2\theta-\nu\sin^2\theta}{2} \end{Vmatrix} . \tag{3.21}$$

These matrices A_{ij} and a_{ij} are symmetrical with respect to the main diagonal. Hemp (ref.22) computed the elements of the matrix a_{ij} by means of vector calculus, but the derivations can also be obtained by means of tensor calculus. A note on the tensor calculus derivation of (3.6) and (3.8) to (3.11) incl. is given in appendix B of ref.24. (For detailed treatment see, for example, ref. 28).

3.4 Stress-strain relations for a continuously stiffened plate.

Fig.3.2 represents a grid of stiffeners R in the direction of the y-axis and stiffeners S in the direction of the x-axis. The stiffeners are closely spaced. The distance measured in x direction of the stiffeners R is a_R, the distance measured in y direction of the stiffeners S is a_S. The area of the normal cross section of a stiffener R is A_R, and of a stiffener S is A_S. The stiffeners carry only normal forces and have no stiffness with respect to other loads. Both systems can be idealized to systems of continuously distributed stiffeners. Then the stiffeners are made equivalent to an anisotropic plate whose stress-strain relations with respect to the oblique x,y-system are

$$\left.\begin{aligned} s_x &= \frac{EA_S}{a_S}\,\varepsilon_x \\ s_y &= \frac{EA_R}{a_R}\,\varepsilon_y \\ t &= 0 \end{aligned}\right\} . \tag{3.22}$$

By (2.11) the matrix a_{ij} of this plate is

$$(a_{ij})_1 = \begin{Vmatrix} \frac{EA_S}{a_S} & 0 & 0 \\ 0 & \frac{EA_R}{a_R} & 0 \\ 0 & 0 & 0 \end{Vmatrix} . \tag{3.23}$$

The grid - hence also the anisotropic plate - is then considered to be attached to an isotropic skin in order to serve as a system of stiffeners. The matrix a_{ij} of this isotropic plate is $(a_{ij})_2$ and is given by (3.21).

In view of the meaning of the quantities s_x, s_y, t (forces per unit of length) the addition of the stressflow components in the skin and in the anisotropic plate yields the stress flow components of the stiffened skin; and the same holds for the addition of the stiffness matrices $(a_{ij})_1$ and $(a_{ij})_2$. Then the matrix a_{ij} of the composite plate (that is the combination of the isotropic skin and the anisotropic plate equivalent to the stiffeners) is

$$a_{ij} = (a_{ij})_1 + (a_{ij})_2 \quad , \tag{3.24}$$

$$a_{ij} = \frac{Eh}{(1-\nu^2)\sin^3\theta} \begin{Vmatrix} P & \cos^2\theta + \nu\sin^2\theta & -\cos\theta \\ \cos^2\theta + \nu\sin^2\theta & Q & -\cos\theta \\ -\cos\theta & -\cos\theta & \frac{1+\cos^2\theta-\nu\sin^2\theta}{2} \end{Vmatrix} \tag{3.25}$$

where

$$P = 1 + \frac{A_S(1-\nu^2)\sin^3\theta}{a_S h} \tag{3.26}$$

and

$$Q = 1 + \frac{A_R(1-\nu^2)\sin^3\theta}{a_R h} \tag{3.27}$$

From (3.25)...(3.27) follows that the parameters that govern the anisotropy of the composite structure are $A_S/a_S h$, $A_R/a_R h$ and the angle θ.

The inverted matrix $A_{ij} = a_{ij}^{-1}$ is

$$A_{ij} = \frac{1}{|a_{ij}|} \begin{Vmatrix} a_{22}a_{33}-a_{23}^2 & a_{23}a_{31}-a_{21}a_{33} & a_{21}a_{32}-a_{31}a_{22} \\ a_{13}a_{32}-a_{12}a_{33} & a_{11}a_{33}-a_{13}^2 & a_{12}a_{31}-a_{11}a_{32} \\ a_{12}a_{23}-a_{22}a_{13} & a_{13}a_{21}-a_{11}a_{23} & a_{11}a_{22}-a_{12}^2 \end{Vmatrix} \tag{3.28}$$

where $|a_{ij}|$ is the determinant of a_{ij}. The elements of A_{ij} are

$$\left.\begin{aligned} A_{11} &= \frac{1}{2R}\,(Q+Q\cos^2\theta - Q\nu\sin^2\theta - 2\cos^2\theta) \\ A_{12} = A_{21} &= \frac{1}{2R}\,(\cos^2\theta - \nu\sin^2\theta)(1-\nu)\sin^2\theta \\ A_{13} = A_{31} &= -\frac{1}{R}\,(-\cos^3\theta - \nu\sin^2\theta\cos\theta + Q\cos\theta) \\ A_{22} &= \frac{1}{2R}\,(P+P\cos^2\theta - P\nu\sin^2\theta - 2\cos^2\theta) \\ A_{23} = A_{32} &= -\frac{1}{R}\,(P\cos\theta - \cos^3\theta - \nu\sin^2\theta\cos\theta) \\ A_{33} &= \frac{1}{R}\left\{PQ-(\cos^2\theta+\nu\sin^2\theta)^2\right\} \end{aligned}\right\} \tag{3.29}$$

in which

$$\begin{aligned} R = \frac{Eh}{(1-\nu^2)\sin^3\theta} \Bigg\{ & \frac{PQ}{2}\,(1+\cos^2\theta - \nu\sin^2\theta) + \\ & + \frac{(\cos^4\theta - \nu^2\sin^4\theta)}{2}\,(1-\nu)\sin^2\theta + \\ & - \cos^2\theta(-\cos^2\theta - \nu\sin^2\theta + P+Q) \Bigg\}. \end{aligned} \tag{3.30}$$

If the system of stiffeners R is missing (Q=1) the elements A_{ij} of (3.29) become

$$A_{11} = \frac{1}{E\left(\frac{A_S}{a_S} + h\sin\theta\right)}$$

$$A_{12}=A_{21} = A_{11}(\cos^2\theta - \nu\sin^2\theta)$$

$$A_{13}=A_{31} = 2A_{11}\cos\theta$$

$$A_{22} = A_{11}\left\{1+\frac{A_S(1+\nu)\sin\theta}{a_S h}\;(1+\cos^2\theta-\nu\sin^2\theta)\right\}$$

$$A_{23}=A_{32} = 2A_{11}\left\{\frac{A_S(1+\nu)\sin\theta}{a_S h}+1\right\}\cos\theta$$

$$A_{33} = A_{11}\left\{\frac{2A_S(1+\nu)\sin\theta}{a_S h}+2(1+\cos^2\theta+\nu\sin^2\theta)\right\}. \qquad (3.31)$$

Lewis (ref.29) gives numerical values of the matrix A_{ij} for different angles θ and several stiffness ratios for system S, system R and the isotropic skin (with $\nu = 0.3$).

The matrices a_{ij} (3.25) and A_{ij} (3.29) are symmetrical and remain so when a transformation into another oblique or rectangular system of axes takes place.

The characteristic feature of the matrices (3.20), (3.21) for the isotropic plate and of the matrices (3.25), (3.29) for the composite plate in comparison with the matrices (2.6) and (2.12) is, that the axial stressflows s_x and s_y are dependent not only on the axial strains ε_x and ε_y, but as well on the tangential strain γ. Likewise, the tangential stressflow t depends on the axial strains ε_x and ε_y. This is the direct consequence of the elements a_{13}, a_{23}, A_{13} and A_{23} not being zero.

Appendix B of ref.24 deals with the transformation of the matrices A_{ij} and a_{ij} for an arbitrary anisotropic plate from one arbitrary, rectangular or oblique system of coordinates into another by means of tensor calculus. If this is done such that $a_{13}=a_{23}=A_{13}=A_{23}=0$ is obtained, the directions of the new coordinates represent the principal directions of anisotropy. For a plate with two perpendicular systems of stiffeners, the directions of the stiffeners are obviously the principal directions of anisotropy. Such a stiffened plate is called orthotropic.

The directions of the stiffeners of the continuously stiffened isotropic plate in fig.3.2 are not principal directions of anisotropy, nor will they become so if one or both systems of stiffeners is infintely rigid.

If the stressflow components s_x, s_y, t of a composite plate are known the computation of the stressflow components in the separate parts requires to calculate first the corresponding strains with the matrix A_{ij}. These strains together with the matrices a_{ij} of the skin and of the plate, which is equivalent to the stiffeners, yield the stressflows in the separate parts. If a stressflow component in the composite plate happens to be zero, it does not imply that the corresponding stressflow components in the separate parts are zero.

Not much is known about the accuracy of the calculations if the discrete stiffeners are replaced by continuously distributed stiffeners. For example, with the rectangular plate in fig.3.3 of width c and loaded at the sides $x = 0$ and $x = b$ by a stressflow s_x, it is not known how the average extension in x-direction depends on the distance a_R of infinitely rigid stiffeners in y-direction. The relation for continuously distributed stiffeners is according to (3.25) or (3.29) with $P = 1$, $Q \to \infty$ and $\theta = 90^o$

$$s_x = \frac{Eh}{1-\nu^2} \varepsilon_x \quad . \tag{3.32}$$

However, for widely spaced stiffeners s_x will be a function of x and y if the width c is finite. The stress distribution near the edges $y = 0$ and $y = c$ is complex and comprises components s_y and t as well as s_x. Only when $\frac{c}{a_R}$ goes to infinity (3.32) is again valid. Obviously the replacement by continuously distributed stiffeners is better for larger ratios between the length of the stiffeners and their distance.

3.5 Differential equations for the stress function Ψ. Strain energy.

For orthogonal coordinates, the compatibility condition (2.18) was derived from (2.17), (2.5) and (2.15). From (3.18), (2.5) and (3.19) follows the same equation as (2.18):

$$A_{22} \frac{\partial^4\Psi}{\partial x^4} - 2A_{23} \frac{\partial^4\Psi}{\partial x^3 \partial y} + (2A_{12}+A_{33}) \frac{\partial^4\Psi}{\partial x^2 \partial y^2} - 2A_{13} \frac{\partial^4\Psi}{\partial x \partial y^3} + A_{11} \frac{\partial^4\Psi}{\partial y^4} = 0 \tag{3.33}$$

which is valid for the isotropic plate and the continuously stiffened plate both.

The expressions for the strain energy (2.20) to (2.24) also remain applicable, if A is considered as strain energy per unit rhomb, that is a rhomb with sides 1 in the directions of the coordinate axes and an area $\sin\theta$. This appears when (3.9) and (3.11) are substituted into (2.20)

$$A = \frac{1}{2} s_i \varepsilon_i \tag{3.34}$$

$$A = \frac{1}{2} A_{ij} s_i s_j \tag{3.35}$$

$$A = \frac{1}{2} a_{ij} \varepsilon_i \varepsilon_j \quad . \tag{3.36}$$

The quadratic forms (3.35) and (3.36) are positive definite, that means that $A_{ij}s_is_j$ and $a_{ij}\varepsilon_i\varepsilon_j$ never become negative for whatever values of

s_x, s_y, t or ε_x, ε_y, γ are taken. This yields for the elements A_{ij} (and of a_{ij}) of an arbitrary continuously stiffened plate in an arbitrary oblique coordinate system the requirements (ref.30, page 30)

$$A_{11}+A_{22}+A_{33} > 0$$

$$\begin{vmatrix} A_{22} A_{23} \\ A_{32} A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} A_{13} \\ A_{31} A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} A_{12} \\ A_{21} A_{22} \end{vmatrix} > 0 \qquad (3.37)$$

and

$$\begin{vmatrix} A_{11} A_{12} A_{13} \\ A_{21} A_{22} A_{23} \\ A_{31} A_{32} A_{33} \end{vmatrix} > 0 \quad .$$

Dependent on these conditions are the conditions that the separate terms in the inequalities of (3.37) are greater than zero.

In some indealisations to be used some of the forms (3.37) are zero.

3.6 Affine transformation of a plate.

Suppose that the (two-dimensional) state of stress in an isotropic plate is known. The geometry of the plate, the forces and stressflows are known in orthogonal coordinates. Then visualize in an oblique system of coordinates an isotropic plate of the same thickness and elasticity constants, the geometry, forces and stresses of which are described with the oblique coordinates in exactly the same way (coordinates for boundaries, force and stressflow components have the same numerical values) as for the orthogonal coordinates. Then equivalent moments differ by a factor sin O (according to (3.7)). It can then be said that the first plate with forces and stressflows is affinely transformed into the second plate.

It appears from (3.17) that in the oblique plate the equilibrium conditions are satisfied again. The state of stress thus obtained is however, in general, not attainable because the strains are different from those of the orthogonal case due to the alteration of the matrix A_{ij}. These strains do in general not satisfy the compatibility condition (3.19), since the stress functions are identical and their differential equations (3.33) are not identical. The compatibility condition is a second order partial differential equation in the strains, hence in the stressflows. Therefore if in the untransformed and also in the transformed plate the stressflows are linear functions of the coordinates, the stressflows in the transformed plate do not violate the compatibility condition and the system of stressflows is attainable in the oblique plate. Nevertheless, corresponding displacement projections and strain components have no longer the same value in both plates.

A plate can also be affinely transformed with its displacement projections and strains. All rotations in this process are multiplied by the factor $1/\sin\theta$, (compare (3.15)). Of course the compatibility condition for this plate is then satisfied but the stresses in the transformed plate deduced from the strains by means of the matrix a_{ij} are in general not in equilibrium. Even stressflows may occur along boundaries which are stress-free in the orthogonal case.

4 Some relations for a three-dimensional oblique coordinate system.

The oblique system is right-handed, the positive x and y axes include the angle θ, while the z-axis is perpendicular to the x-y plane (fig.4.1).

Stresses and strains in this three-dimensional oblique system can be defined also in such a way that equilibrium and compatibility equations preserve their shape, but this will not be discussed here.

The coordinates of the point P are given in fig.4.1. The transformation formulae between these coordinates and those of the orthogonal system $\bar{x}$, $\bar{y}$, $\bar{z}$ are $z = \bar{z}$ together with (3.5).

A force $\vec{K}$ is expressed by its components K_x, K_y, K_z (fig.4.1).

The magnitude of the force $\vec{K}$ is (compare (3.1))

$$|\vec{K}| = \sqrt{K_x^2 + K_y^2 + K_z^2 + 2K_xK_y\cos\theta} \quad . \qquad (4.1)$$

The relation with the components $\bar{K}_x$, $\bar{K}_y$, $\bar{K}_z$ of the orthogonal system is $K_z = \bar{K}_z$ together with (3.6).

The displacement vector $\vec{u}$ is represented by its projections u_x, u_y, u_z (fig.4.1). The relation between these projections and the projections (that is components) $\bar{u}_x$, $\bar{u}_y$, $\bar{u}_z$ in the orthogonal system is $u_z = \bar{u}_z$ together with (3.10).

The magnitude of a displacement is (compare (3.2))

$$|\vec{u}| = \sqrt{\frac{u_x^2 + u_y^2 - 2u_xu_y\cos\theta}{\sin^2\theta} + u_z^2} \quad . \qquad (4.2)$$

The righthanded moment vector $\vec{M}$ is also represented by its projections M_x, M_y, M_z in just the same way as is done for $\vec{u}$. If, in the orthogonal system the projections (= components) are equal to $\overline{M}_x$, $\overline{M}_y$, $\overline{M}_z$, the transformation formulas are

$$M_z = \overline{M}_z$$

and in analogy to (3.10)

$$M_x = \overline{M}_x \tag{4.3}$$

$$M_y = \overline{M}_x \cos\theta + \overline{M}_y \sin\theta \quad \text{or} \quad \overline{M}_y = -M_x \operatorname{ctn}\theta + \frac{M_y}{\sin\theta} .$$

In the righthanded oblique system of axes, a force $\vec{K}(K_x, K_y, K_z)$ at the point x, y, z gives the righthanded moments, with respect to the origin

$$\begin{aligned} M_x &= (K_z y - K_y z) \sin\theta \\ M_y &= (K_x z - K_z x) \sin\theta \\ M_z &= (K_y x - K_x y) \sin\theta \quad . \end{aligned} \tag{4.4}$$

The magnitude of a moment $\vec{M}(M_x, M_y, M_z)$ is (compare (4.2))

$$|\vec{M}| = \sqrt{\frac{M_x^2 + M_y^2 - 2M_xM_y\cos\theta}{\sin^2\theta} + M_z^2} \tag{4.5}$$

The moment $\vec{M}(0,0,M_z)$ is perpendicular to the x,y plane, its magnitude is M_z. The moment $\vec{M}(M_x,0,0)$ is perpendicular to the y,z plane and its magnitude is $M_x/\sin\theta$, see fig.4.2. A moment $\vec{M}(0,M_y,0)$ is perpendicular to the x,z plane, its magnitude is $M_y/\sin\theta$, see fig.4.3. A rotation vector $\vec{\varphi}$ is expressed by its components φ_x, φ_y, φ_z in the same way as a force $\vec{K}$ (fig.4.1). If small displacements $\vec{u}(u_x,u_y,u_z)$ occur in the righthanded oblique system x,y,z, the rotation components are

$$\begin{aligned} \varphi_x &= \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)/\sin\theta \\ \varphi_y &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)/\sin\theta \\ \varphi_z &= \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)/\sin\theta \quad . \end{aligned} \tag{4.6}$$

If the small rotation vector $\vec{\varphi}(\varphi_x,\varphi_y,\varphi_z)$ is constant throughout, for example by a rigid body rotation about an axis through the origin, the displacements are

$$u_x = (\varphi_y z - \varphi_z y)\sin\theta$$

$$u_y = (\varphi_z x - \varphi_x z)\sin\theta \qquad (4.7)$$

$$u_z = (\varphi_x y - \varphi_y x)\sin\theta \quad .$$

If $\bar{\varphi}_x$, $\bar{\varphi}_y$, $\bar{\varphi}_z$ are the components of the rotation in the orthogonal system $\bar{x}$, $\bar{y}$, $\bar{z}$ of figure 4.1, the transformation formulae are $\varphi_z = \bar{\varphi}_z$ together with (3.6).

The work done by the force $\vec{K}(K_x,K_y,K_z)$ through the displacement $\vec{u}(u_x,u_y,u_z)$ amounts to

$$\vec{K}.\vec{u} = K_x u_x + K_y u_y + K_z u_z \qquad (4.8)$$

and the work done by the moment $\vec{M}(M_x,M_y,M_z)$ through the rotation $\vec{\varphi}(\varphi_x,\varphi_y,\varphi_z)$ amounts to

$$\vec{M}.\vec{\varphi} = M_x\varphi_x + M_y\varphi_y + M_z\varphi_z \quad . \qquad (4.9)$$

The expressions (4.8) and (4.9) are identical with those for orthogonal coordinates.

5 Discussion of the shear-field scheme for skin panels.

5.1 Rectangular panels.

Consider the rectangular isotropic skin (fig.5.1), which is provided with stiffeners S parallel to the x-axis on a pitch s_S and with a cross sectional area A_S per stiffener. The stiffeners R parallel to the y-axis are spaced at distances a_R and have a cross sectional area A_R.

The stiffeners of both groups are preliminary assumed to be continuously distributed. The stress-strain relations of the composite plate can be determined according to section 3.4.

The stiffness matrix $(a_{ij})_1$ for the plate that is equivalent to the stiffener systems S and R is given by (3.23). The stiffness matrix $(a_{ij})_2$ is given by (2.12).

Then according to (3.24) the stiffness matrix a_{ij} of the composite plate is

$$a_{ij} = \begin{Vmatrix} E'h + \frac{EA_S}{a_S} & \nu E'h & 0 \\ \nu E'h & E'h + \frac{EA_R}{a_R} & 0 \\ 0 & 0 & \frac{1-\nu}{2}E'h \end{Vmatrix} \qquad (5.1)$$

where $E' = \frac{E}{1-\nu^2}$.

The inverse matrix of a_{ij}, the flexibility matrix A_{ij} is

$$A_{ij} = \begin{Vmatrix} \dfrac{E'h + E\dfrac{A_R}{a_R}}{(E'h+\dfrac{EA_S}{a_S})(E'h+\dfrac{EA_R}{a_R})-(\nu E'h)^2} & \dfrac{-\nu E'h}{(E'h+\dfrac{EA_S}{a_S})(E'h+\dfrac{EA_R}{a_R})-(\nu E'h)^2} & 0 \\ \dfrac{-\nu E'h}{(E'h+\dfrac{EA_S}{a_S})(E'h+\dfrac{EA_R}{a_R})-(\nu E'h)^2} & \dfrac{E'h + E\dfrac{A_S}{a_S}}{(E'h+\dfrac{EA_S}{a_S})(E'h+\dfrac{EA_R}{a_R})-(\nu E'h)^2} & 0 \\ 0 & 0 & \dfrac{2}{(1-\nu)E'h} \end{Vmatrix} . \quad (5.2)$$

If the rectangular skin plate of fig.5.1 is slender in the direction of x it can be expected that, if the sides along $y = \pm c$ are free or provided with flanges which carry normal forces, the stressflows s_y, which must be zero at $y = \pm c$, will remain small. Thus the influence of the stressflows s_y on the strains ε_x may be neglected and in (5.2) and A_{12} may be put zero without serious error. The stressflows s_x have their influence on the strains ε_y by means of the element A_{21} in (5.2). These strains, however, contribute little to the displacements in y direction, since the dimensions in this direction are small. Therefore also the element A_{21} may be neglected. Besides, generally numerical values of A_{21} are small with respect to A_{11} (for the isotropic plate $A_{21}/A_{11} = -\nu$).

Then the inverse matrix of the in this way simplified matrix A_{ij} is

$$a_{ij} = \begin{Vmatrix} E'h+\dfrac{EA_S}{a_S} - \dfrac{(\nu E'h)^2}{E'h+E\dfrac{A_R}{a_R}} & 0 & 0 \\ 0 & E'h+\dfrac{EA_R}{a_R} - \dfrac{(\nu E'h)^2}{E'h+E\dfrac{A_S}{a_S}} & 0 \\ 0 & 0 & \dfrac{1-\nu}{2}E'h \end{Vmatrix} . \quad (5.3)$$

Matrices of the general pattern of (5.3)

$$a_{ij} = \begin{Vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{Vmatrix} \quad (5.4)$$

can also be obtained along other, less reasonable lines, by putting $A_{12} = A_{21} = 0$ in the matrix of the isotropic skin, or by putting $a_{12} = a_{21} = 0$ in the matrix of the isotropic skin or of the composite plate. Numerical differences with the result (5.3), however will, in general be small.

A composite plate with matrix of the pattern (5.4) can be visualized as pertaining to two systems of continuized stiffeners, the equivalent plate of which has the matrix

$$a_{ij} = \begin{Vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{Vmatrix} \qquad (5.5)$$

together with a skin that can merely take shear stress, the matrix of which is

$$a_{ij} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{Vmatrix} . \qquad (5.6)$$

Next the equivalent plate, with matrix a_{ij} of (5.5) is replaced by a finite number of "replacement" stiffeners, which not only represent the actual stiffeners but as well the stiffness (with respect to stressflows s_x and s_y) of the skin. The replacement stiffeners are preferably situated along actual stiffeners, but they may be more in number (e.g. if there are no actual stiffeners but only an isotropic plate) or less in number (if there is a great number of closely spaced actual stringers). The stiffness of the replacement stiffeners is such that, if continuized, their matrix a_{ij} is (5.5), where a_{11} and a_{22} are given the values of (5.3) or reduced values which account for post-buckling behaviour of the skin panels.

Between the replacement stringers a skin capable of carrying shear stresses only remains (shear field scheme). Its matrix a_{ij} is (5.6) where the element a_{33} has the value of (5.3) or a reduced value if buckling occurs.

The property of a plate to carry shear stresses only, $s_x=0$, $s_y=0$, $t \neq 0$ is confined to a special system of axes. If the property is valid for a certain orthogonal system of axes x,y, the matrix a_{ij} becomes

$$a_{ij} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Gh \end{Vmatrix}$$

and after transformation to the system of orthogonal axes, rotated through 45^0 with respect to the former, the matrix a_{ij} becomes

$$a_{ij} = \begin{Vmatrix} Gh & -Gh & 0 \\ -Gh & Gh & 0 \\ 0 & 0 & 0 \end{Vmatrix} .$$

This means that the plate is anisotropic.

The equilibrium equations (2.1) together with $s_x = 0$, $s_y = 0$, $t \neq 0$ yield that in such a plate the shear stressflow between two stiffeners of the system R and of the system S (fig.5.1) can only be constant[1)].

In a structure consisting of such panels and of a finite number of stiffeners, only a finite number of stress systems satisfying the conditions of equilibrium conditions are possible. Thus the structure is made finite fold statically indeterminate and all methods of analysis give the same - and for the idealized structure - exact answers.

5.2 Oblique panels.

Not it will be examined whether the shear field scheme is applicable to oblique panels. Again there may be two systems of stiffeners R and S (fig.5.2). The stiffness matrix a_{ij} of the composite panel is given by (3.25) and the inverse matrix A_{ij} is given by (3.29). These matrices a_{ij} and A_{ij} are such that not a single element is zero.

Again considering the general strain-stress relations in their general form

$$\begin{aligned} \varepsilon_x &= A_{11}s_x + A_{12}s_y + A_{13}t \\ \varepsilon_y &= A_{21}s_x + A_{22}s_y + A_{23}t \qquad (5.7) \\ \gamma &= A_{31}s_x + A_{32}s_y + A_{33}t \quad , \end{aligned}$$

1) Garvey, ref.31 establishes by means of equations of equilibrium a stress distribution within an arbitrary quadrilateral plate field, that carries only shear stressflow along its edges. Compare footnote p.27.

it is seen that a diagonal matrix

$$A_{ij} = \begin{Vmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{Vmatrix} \qquad (5.8)$$

could be obtained by putting $A_{12}=A_{21}=0$ as with the rectangular plate and in addition $A_{13}=A_{31}=A_{23}=A_{32}=0$.

The neglection of $A_{12}=A_{21}$ can be justified by the same reasoning as used in section 5.1 for the rectangular plate (s_y is small and has little influence on ε_x, s_x has influence on ε_y, but these strains are not important for overall displacements). But, generally stressflows t are not small, nor their influence on the strains ε_x. The stressflows s_x contribute to the strains γ, and these strains are important for overall displacements. These facts are the reason that it is quite inadmissible to put $A_{13}=A_{31}=0$, the more since the ratio A_{13}/A_{11} is not small. For example, the isotropic plate has with $\nu = 0.3$ and $\theta = 45^{\circ}$ the matrix

$$A_{ij} = \frac{1}{Eh} \begin{Vmatrix} 1.414 & 0.495 & 2 \\ 0.495 & 1.414 & 2 \\ 2 & 2 & 4.667 \end{Vmatrix} .$$

Since the matrix A_{ij} cannot be reduced to the form (5.8), it is not possible to split up in parts pertaining to two systems of stiffeners with a matrix like (5.5) and a shear field with a matrix like (5.6). In other words it is impossible to represent the behaviour of an oblique panel adequately by two systems of stiffeners and an oblique panel that can carry only shear stresses along its edges.

The other still less reasonable attempts that could be made are to put $A_{12}=A_{21}=A_{13}=A_{31}=A_{23}=A_{32}=0$ in the matrix of the isotropic skin[1)] or to put $a_{12}=a_{21}=a_{13}=a_{31}=a_{23}=a_{32}=0$ in the matrix of the isotropic skin or of the composite plate. Numerical results will often be quite different, and neither will give useful results.

Methods will be developed where the stress-strain relations will not, or only little, be simplified. Instead of an exact solution for a

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1) In Levy's work, ref.23, a supposition was made which is equivalent with this attempt. Garvey (ref.31) extends the supposition of Levy to the arbitrary quadrilateral plate field, but this supposition seems only useful if the platefield is close to rectangular.

simplified structure approximate solutions for a non- (or slightly) simplified structure will be obtained; and it will be possible, by means of different types of solutions, to derive upper and lower bounds for the numerical results. These approximate solutions will be obtained by means of the energy theorems of the theory of elasticity.

6 The variational principles of elasticity theory.

6.1 Minimum of the complementary energy.

6.1.1 Determination of stresses.

The theorem of the minimum of the complementary energy can be stated in the following way (ref.32, page 286).

Of all states of stress satisfying the conditions of equilibrium in the interior and on that portion of the surface where the surface forces are prescribed, the actual state of stress is such as to minimize the expression for the complementary energy:

$$V^{*} = \frac{1}{2}\int S.R\ dv - \int_{u} \vec{k}.\vec{u}\ df \quad . \tag{6.1}$$

The scalar product S.R is the sum of the scalar products of the stress components S acting on the volume element dv and the strains R of the volume element. The product $\frac{1}{2}$ S.R dv is called the strain energy of the volume element dv and the first integral of (6.1), which extends over the whole volume of the body, the strain energy of the body.

Further, $\vec{k}$ are the forces (per unit area), acting at the surface f of the body and $\vec{u}$ the displacements through which these forces act. The symbol u at the integral sign means that the integral extends only over that portion of the surface where the displacements are prescribed.[1)]

It must be kept in mind, that applying the minimum principle to (6.1) primarily only the state of stress is varied and that the strains R in (6.1) follow from the stresses, which are subject to variation, by means of the stress-strain relations. Therefore the expression (3.35) will be used for a unit rhomb (of the oblique coordinate system) of an anisotropic plate. Unless V^{*} is indeed the minimum, the strains R do not satisfy compatibility conditions.

The state of stress S, together with its external forces $\vec{k}$, is now considered to be the sum of a number of states of stress each with their

1) The refinement where at a surface element df only one or two projections of $\vec{u}$ are prescribed together with two or one force component of course does not introduce any special difficulty. Nor the introduction of prescribed mass forces, which are excluded here.

external forces

$$S = S_0 + X_i S_i$$
$$\vec{k} = \vec{k}_0 + X_i \vec{k}_i \qquad (6.2)$$

The state of stress S_0, together with its external load $\vec{k}_0$ and which is called the "basic stress system", satisfies the equilibrium conditions everywhere. Where the external load is prescribed, $\vec{k}_0$ is equal to this prescribed load. A state of stress S_i, together with its external loads $\vec{k}_i$, satisfies the equilibrium conditions everywhere. Since $\vec{k}(=\vec{k}_0+X_i\vec{k}_i)$ for sake of equilibrium must be equal to the prescribed loads in any point of the surface where the external load is prescribed, $\vec{k}_i$ (of every system S_i) vanishes in that part of the surface. However, where the displacements are prescribed $\vec{k}_i$ may be non zero. The resultant of $\vec{k}_i$ (of every system S_i) on the whole body vanishes for sake of equilibrium. The stress systems S_i, which are to be linearly independent, are called "supplementary stress systems". In an n-fold statically indeterminate structure n such systems may be constructed, and in (6.2) the unknowns X_i are the statically indeterminate quantities, which have to be determined with the aid of the minimum theorem (6.1). This "minimum theorem of the complementary energy" will be called here for the sake of brevity and so as to indicate that stresses are varied, "minimum principle for the stresses".

The quantities X_i can also be called the participation factors of the supplementary stress systems S_i.

The usual terminology with statically indeterminate structures originates from considerations on (finite fold) statically indeterminate structures, such as trusses or beam systems, the members of which can carry only one or a small number of load types (normal force, bending moment). Part of the members, called redundant members, may be removed or bereaved of the capability to take a certain load (a bending moment), thereby leaving the remaining structure capable to carry the load in a statically determinate way.

The loads that occur in these members in the actual structure are called redundant forces (moments) or redundancies.

In the present work the participation factors of the supplementary stress systems take the place of these redundancies. The notions redundant member and redundancy are not used here because the supplementary stress systems relate to plates with non-simplified stress-strain relations, which carry an infinite number of statically indeterminate stress systems because their state of stress is two-dimensional. Besides removal of such a plate may possibly make the structure kinematically undetermined. Such an element can hardly considered to be redundant in the strict sense of the word.

Coming back now to the application of the minimum principle for the stresses the equations (6.2) are substituted in (6.1)

$$V^{*}= \frac{1}{2}\int(S_0+X_iS_i).(R_0+X_jR_j)dv-\int_u(\vec{k}_0+X_i\vec{k}_i).\vec{u}\,df \tag{6.3}$$

$$V^{*}= \frac{1}{2}\lambda_{00}+X_i\lambda_{0i}+\frac{1}{2}X_iX_j\lambda_{ij}-\int_u(\vec{k}_0+X_i\vec{k}_i).\vec{u}\,df \tag{6.4}$$

where

$$\left.\begin{aligned}\lambda_{00} &= \int S_0.R_0dv\\ \lambda_{0i} &= \int S_0.R_idv = \int S_i.R_0dv\\ \lambda_{ij} &= \lambda_{ji} = \int S_i.R_jdv = \int S_j.R_idv\quad.\end{aligned}\right\} \tag{6.5}$$

Since V^{*} must be minimal the equations

$$\frac{dV^{*}}{dX_i} = \lambda_{0i}+X_j\lambda_{ij}-\int_u\vec{k}_i.\vec{u}\,df = 0 \tag{6.6}$$

must be satisfied.

The solution for the unknowns X_j is

$$X_j = (\lambda_{ij})^{-1}(-\lambda_{0i}+\int_u\vec{k}_i.\vec{u}\,df) = 0 \tag{6.7}$$

where $(\lambda_{ij})^{-1}$ is the inverse matrix of the matrix λ_{ij}.

With the solution of X_j the stress distribution is known. If the structure is infinitely fold statically indeterminate, usually only a finite number of internal systems of stress will be introduced and the solution is only an approximate one. That means that the stresses do not yield completely compatible strains, which consequently are not integrable to displacements in a completely unambiguous way. In section 6.1.3 displacements are determined without integration of strains.

In choosing the basic stress system S_0, there is of course a good reason to take it as simple as possible. Values of λ_{0i} are then simple to compute. However, there may be also a reason to choose it such that it resembles as much as possible the solution for the actual state of stress S which is to be expected. Such a system S_0 could for example be achieved by deriving it by means of engineering methods of stress analysis. Then S_0 has the significance of a first approximation and the statically undetermined analysis yields a correction to this first approximation. With such basic stress system the values of the unknowns X_i are small and may be computed with less accuracy. Besides some of the unknowns even may be put zero, which means that the number of supplementary stress systems is reduced.

The possibility to use this refined type of basic stress system depends largely on the possibility which the structure and its type of loading offer for an approximate analysis of straightforward methods.

If the structure is loaded merely as a result of its prescribed displacements, $\vec{k}_0$ vanishing throughout, the most simple basic stress system is $S_0 = 0$. There is however in this case no objection, if desired, to take S_0 such that it approaches S as much as possible. This basic stress system may be, though not necessarily, a linear combination of the adopted supplementary stress systems.

For the actual state of stress, therefore the state that yields compatible strains and uniquely determinable displacements, the theorem of elasticity (ref.32, p.90) states, that the strain energy is equal to the externally applied energy

$$\frac{1}{2}\int S.R \, dv = \frac{1}{2}\int \vec{k}.\vec{u} \, df \quad . \tag{6.8}$$

In contrast to (6.1) the integral on the right hand side of (6.8) extends over the whole surface. Substitution of the equation (6.8) into (6.1) yields another expression for the complementary energy

$$V^{*} = \frac{1}{2}\int_{k} \vec{k}.\vec{u} \, df - \frac{1}{2}\int_{u} \vec{k}.\vec{u} \, df \tag{6.9}$$

and the symbol k at the integral sign means that the integral extends only over that part of the surface, where the external load is prescribed.

The unknowns X_j in (6.4) are for future use eliminated by means of (6.6). This yields

$$V^{*} = \frac{1}{2}\lambda_{00} + \frac{1}{2} X_i \lambda_{0i} - \frac{1}{2} X_i \int_{u} \vec{k}_i.\vec{u} \, df - \int_{u} \vec{k}_0.\vec{u} \, df \quad . \tag{6.10}$$

For the sake of simplicity when forming the equations (6.7), it is in general desirable that as many supplementary stress systems S_i as possible be orthogonal, which means that

$$\lambda_{ij} = \int S_i.R_j \, dv = 0 \tag{6.11}$$

for as many $i,j (i \neq j)$ as possible. Non-overlapping supplementary stress systems are, of course, always orthogonal.

Suppose that a certain problem in rectangular coordinates has been solved by a suitably chosen set of stress systems $S_0 \ldots S_i$, then in the affinely transformed structure (see par.3.6) the stress systems $S_0 \ldots S_i$ satisfy the conditions of equilibrium as well. Only the constants X_i assume different values.

Initial orthogonality between the systems S_i is usually lost in the affinely transformed structure, because the matrix A_{ij} and consequently R_j have assumed different values. The orthogonality of systems not overlapping each other is of course maintained.

6.1.2 Complementary energy at combination of two loading cases.

Suppose the loading case A

$$S_A = S_{AO} + X_{Ai} S_i$$
$$\vec{k}_A = \vec{k}_{AO} + X_{Ai} \vec{k}_i \tag{6.12}$$

and B

$$S_B = S_{BO} + X_{Bi} S_i$$
$$\vec{k}_B = \vec{k}_{BO} + X_{Bi} \vec{k}_i \tag{6.13}$$

for the same structure and assume that for both cases the unknowns X_{Ai}, respectively X_{Bi} have been solved by the minimum principle for the stresses. In both problems the surface region where forces and where displacements are prescribed are the same and the same supplementary stress systems have been used. The complementary energies for the two cases are V_A^* and V_B^* respectively.

If both stress systems are present simultaneously (that means prescribed forces, prescribed displacements, stresses and strains are the sum of these quantities of the separate cases), the total complementary energy V_{A+B} in general does not equal $V_A^* + V_B^*$ but follows from (6.3)

$$V^*_{A+B} = \frac{1}{2}\int (S_{AO}+S_{BO}+X_{Ai}S_i+X_{Bi}S_i).(R_{AO}+R_{BO}+X_{Aj}R_j+X_{Bj}R_j)dv -$$
$$- \int_u (\vec{k}_{AO}+X_{Ai}\vec{k}_i+\vec{k}_{BO}+X_{Bi}\vec{k}_i).(\vec{u}_A+\vec{u}_B)df$$

where $\vec{u}_A$ and $\vec{u}_B$ are the prescribed displacements for the two cases respectively.

With the notation (6.5)

$$V^*_{A+B} = V^*_A+V^*_B+\int S_{AO}.R_{BO}dv + X_{Ai}(\lambda_{BOi}-\int_u \vec{k}_i.\vec{u}_B df) +$$

$$+ X_{Bi}(\lambda_{AOi}-\int_u \vec{k}_i.\vec{u}_A\, df + X_{Aj}\lambda_{ij}) +$$

$$-\int_u \vec{k}_{AO}.\vec{u}_B df - \int_u \vec{k}_{BO}.\vec{u}_A\, df \quad . \tag{6.14}$$

In (6.14) the form

$$\lambda_{AOi} - \int_u \vec{k}_i.\vec{u}_A df + X_{Aj}\lambda_{ij} = 0$$

because of (6.6), therefore

$$V^*_{A+B} = V^*_A+V^*_B+\int S_{AO}.R_{BO}dv + X_{Ai}(\lambda_{BOi}-\int_u \vec{k}_i.\vec{u}_B df)-\int_u \vec{k}_{AO}.\vec{u}_B df-\int_u \vec{k}_{BO}.\vec{u}_A df \tag{6.15}$$

or

$$V^*_{A+B}= V^*_A+V^*_B+\int S_{AO}.R_{BO}dv+X_{Bi}(\lambda_{AOi}-\int_u \vec{k}_i.\vec{u}_A df)-\int_u \vec{k}_{AO}.\vec{u}_B df-\int_u \vec{k}_{BO}.\vec{u}_A df \quad . \tag{6.16}$$

A further expression for V^*_{A+B} can be derived using (6.9)

$$V^*_{A+B}= \frac{1}{2}\int_k (\vec{k}_A+\vec{k}_B).(\vec{u}_A+\vec{u}_B)df- \frac{1}{2}\int_u (\vec{k}_A+\vec{k}_B).(\vec{u}_A+\vec{u}_B)df =$$

$$= V^*_A+V^*_B+ \frac{1}{2}\int_k (\vec{k}_A.\vec{u}_B+\vec{k}_B.\vec{u}_A)df - \frac{1}{2}\int_u(\vec{k}_A.\vec{u}_B+\vec{k}_B.\vec{u}_A)df = \tag{6.17}$$

$$= V^*_A+V^*_B+ \frac{1}{2}\int(\vec{k}_A.\vec{u}_I+\vec{k}_B.\vec{u}_A)df- \int_u(\vec{k}_A.\vec{u}_B+\vec{k}_B.\vec{u}_A)df \quad . \tag{6.18}$$

The reciprocal theorem of Betti and Rayleigh (ref.32, p.297) states that

$$\int \vec{k}_A.\vec{u}_B df = \int \vec{k}_B.\vec{u}_A\, df \quad \text{or}$$

$$\int_k \vec{k}_A.\vec{u}_B df + \int_u \vec{k}_A.\vec{u}_B df = \int_k \vec{k}_B.\vec{u}_A df + \int_u \vec{k}_B.\vec{u}_A df \tag{6.19}$$

and this simplifies (6.17) to

$$V^*_{A+B} = V^*_A+V^*_B+ \int_k \vec{k}_A.\vec{u}_B df - \int_u \vec{k}_B.\vec{u}_A df \tag{6.20}$$

or

$$V^*_{A+B} = V^*_A + V^*_B + \int_k \vec{k}_B.\vec{u}_A df - \int_u \vec{k}_A.\vec{u}_B df \quad . \qquad (6.21)$$

The equations (6.15) and (6.21) will be used for the determination of displacements.

6.1.3 Determination of displacements.

The displacement vector $\vec{u}_A$ of the surface element df, where the external load is prescribed is to be determined for a given loading case A :

$$S_A = S_{AO} + X_{Ai}S_i$$
$$\vec{k}_A = \vec{k}_{AO} + X_{Ai}\vec{k}_i \quad . \qquad (6.22)$$

This will be done by means of the formula (6.15). Take an auxiliary system B

$$S_B = S_{BO} + X_{Bi}S_i$$
$$\vec{k}_B = \vec{k}_{BO} + X_{Bi}\vec{k}_i \qquad (6.23)$$

as described in section 6.1.2. The external load of system B is taken zero everywhere, where the loads are prescribed, except in the element df where the external load is a vector $\vec{K}$ of magnitude 1. The prescribed displacements may be arbitrary.

Equating the right hand sides of (6.15) and (6.21) the result is

$$\int_k \vec{k}_B.\vec{u}_A df = \int S_{AO}.R_{BO}dv + X_{Ai}\lambda_{BOi} - \int_u \vec{k}_{BO}.\vec{u}_A df \quad . \qquad (6.24)$$

The integrand of the left hand side vanishes except for the surface element df where the displacement $\vec{u}_A$ is to be determined. The external load of the system B was taken such that

$$\vec{k}_B \, df = \vec{K}_B \qquad (6.25)$$

was a unit vector.

With (6.25) and (4.8) equation (6.24) becomes

$$\vec{K}_B.\vec{u}_A = K_{Bx}u_{Ax} + K_{By}u_{Ay} + K_{Bz}u_{Az} = \int S_{AO}.R_{BO}dv + X_{Ai}\lambda_{BOi} - \int_u \vec{k}_{BO}.\vec{u}_A df \quad ^{1)} \tag{6.26}$$

By taking successively three auxiliary systems B, with on the surface element df forces $\vec{K}_B(1,0,0)$, $\vec{K}_B(0,1,0)$ and $\vec{K}_B(0,0,1)$, the displacement projections u_{Ax}, u_{Ay} and u_{Az} are obtained separately. Along the same lines the rotation components φ_{Ax}, φ_{Ay} and φ_{Az} of the surface element df are found if three auxiliary systems B are taken with on the surface element df unit moments (compare (4.9)) $\vec{M}_B(1,0,0)$, $\vec{M}_B(0,1,0)$ and $\vec{M}_B(0,0,1)$.

Note that in applying (6.26) it is not required to determine the unknowns X_{Bi} [2]. Only the basic stress system S_{BO} and load $\vec{k}_{BO}$ enter the equation. With any other basic stress system, say S'_{BO}, $\vec{k}'_{BO}$ yields in (6.26) the same result, provided S'_{BO}, $\vec{k}'_{BO}$ is a linear combination of S_{BO}, $\vec{k}_{BO}$ and the supplementary stress systems S_i used in (6.22) and (6.23). If the number of possible supplementary stress systems is larger than the number of the systems used, S'_{BO}, $\vec{k}'_{BO}$ could be adopted in such a manner that they are not related to S_{BO}, $\vec{k}_{BO}$ by equations similar to (6.23). Then the systems S_{BO}, $\vec{k}_{BO}$ and S'_{BO}, $\vec{k}'_{BO}$ would yield when applied to equation (6.26) different results for $\vec{u}_A$ (however usually only slightly different). This is a consequence of the fact that the strains of the stress system A are not completely compatible.

6.2 Minimum of the potential energy.

6.2.1 Determination of the strains.

The theorem of the minimum of the potential energy can be stated in the following way (ref.32, page 281)

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1) Of course also

$$\vec{K}_B.\vec{u}_A = \int S_A.R_B dv - \int_u \vec{k}_B.\vec{u}_A\, df .$$

This formula is very easily derived by starting from the reciprocal theorem

$$\int \vec{k}_B.\vec{u}_A df = \int S_A.R_B\, dv$$

instead of the reciprocal theorem (6.19).
Further substitution of $S_A = S_{AO} + X_{Ai}S_i$ and $R_B = R_{BO} + X_{Bi}R_i$ leads also to (6.26).

2) This is also the reason why the prescribed displacements of system B may be taken arbitrary. Prescribed displacements only affect the unknowns X_{Bi}.

Of all states of strain satisfying the compatibility conditions and of which the displacements satisfy prescribed boundary conditions at the surface, the actual state of strain is such as to minimize the expression for the potential energy

$$V = \frac{1}{2}\int S.R \, dv - \int_k \vec{k}.\vec{u} \, df \quad . \tag{6.27}$$

In (6.27) the stresses S follow by means of the stress-strain relations from the strains which are subject to variation. These stresses do not satisfy equilibrium conditions unless V is indeed the minimum. The minimum theorem (6.27), usually called the "minimum theorem of the potential energy" will be called here for the sake of brevity and in order to indicate that the strains are varied, "minimum principle for the strains". Note that the starting point as well as further developments of this section show remarkable parallelism with section (6.1). In fact in all corresponding formulas the symbols k and u are interchanged and so are S and R. To distinguish further the two minimum principles, the symbol X for the unknowns in section 6.1 is changed in the symbol Y here.

The state of strain R, together with its displacements $\vec{u}$ is considered to be the sum of a number of states of strain, each with their compatible displacements

$$\left.\begin{aligned} R &= R_0 + Y_i R_i \\ \vec{u} &= \vec{u}_0 + Y_i \vec{u}_i \end{aligned}\right\} \tag{6.28}$$

The state of strain R_0, which yields the displacements $\vec{u}_0$, complies with the prescribed displacements at the boundary and it is called "basic strain system". With the other systems of strain R_i the displacements $\vec{u}_i$ are zero where displacements are prescribed. These systems are called "supplementary strain systems". The unknowns Y_i may be called the participation factors of the supplementary strain systems R_i.

The analogy of the deviation from (6.1) to (6.6), obtained by interchanging S,R and k,u yields the equation for the unknowns Y_i

$$\lambda_{0i} + Y_j \lambda_{ij} - \int_k \vec{k}.\vec{u}_i \, df = 0 \quad . \tag{6.29}$$

When the unknowns Y_j are solved the complete solution for the state of strain is established, and with the strain-stress relations also the state of stress. If due to a limited number of unknowns Y_i the solution is not exact, the state of stress does not satisfy the equilibrium conditions.

For an n-fold statically indeterminate structure of the type to be considered generally more than n linearly independent supplementary strain systems can be constructed. This number may even be infinite. For this reason the minimum principle for the strains is not suitable for structures which are finite-fold statically indeterminate and which have an infinite number of degrees of freedom for the strains . But also in structures which are infinitely-fold or many-fold statically indeterminate the use of the minimum principle for the strains has a disadvantage. It will be confirmed in the following chapters that in order to obtain an equal degree of accuracy a much larger number of supplementary strain systems than of supplementary stress systems are required. Usually the forces are prescribed over the greater part of the surface of the structure (be it that these forces are often zero-forces), whereas the displacements are prescribed usually in a limited area of the surface. This fact reduces the freedom for stresses, which are subject to variation, more than the freedom for displacements (strains), which are subject to variation.

Also when using the minimum principle for the strains there is a reason to choose the basic strain system R_0 as simple as possible. Often there are only forces and zero-displacements prescribed. Then R_0 may be such that it gives zero displacements (and strains) everywhere. However, (again in order to reduce the required number of supplementary strain systems)it may be advantageous to choose R_0 such that it approaches as much as possible the expected solution for R. (Compare the discussion of the systems S_0 given in section 6.1.1).

For the actual state of strain another expression for the potential energy is derived from (6.27) and (6.8)

$$V = \frac{1}{2} \int_u \vec{k}.\vec{u}\, df - \frac{1}{2} \int_k \vec{k}.\vec{u}\, df \quad . \qquad (6.30)$$

By comparison of (6.30) and (6.9) it appears that for the exact solutions V and V^* holds

$$-V^* = V \quad . \qquad (6.31)$$

In analogy to (6.10) another expression for V is

$$V = \frac{1}{2}\lambda_{00} + \frac{1}{2} Y_i \lambda_{0i} - \frac{1}{2} Y_i \int_k \vec{k}.\vec{u}_i df - \int_k \vec{k}.\vec{u}_0 df \quad . \qquad (6.32)$$

6.2.2 Potential energy at combination of two loading cases.

Suppose the loading cases A

$$\begin{aligned} R_A &= R_{AO} + Y_{Ai}R_i \\ \vec{u}_A &= \vec{u}_{AO} + Y_{Ai}\vec{u}_i \end{aligned} \tag{6.33}$$

and B

$$\begin{aligned} R_B &= R_{BO} + Y_{Bi}R_i \\ \vec{u}_B &= \vec{u}_{BO} + Y_{Bi}\vec{u}_i \end{aligned} \tag{6.34}$$

for the same structure and assume that for both cases the unknowns Y_{Ai}, respectively Y_{Bi} have been solved by the minimum principle for the strains. In both problems the surface regions, where forces and where displacements are prescribed, are the same and the same supplementary strain systems have been used.

Following the derivations of (6.12) to (6.16) the analogous results are obtained

$$V_{A+B} = V_A + V_B + \int R_{AO}\cdot S_{BO}dv + Y_{Ai}(\lambda_{BOi} - \int_k \vec{k}_B\cdot\vec{u}_i df) - \int_k \vec{k}_B\cdot\vec{u}_{AO}df - \int_k \vec{k}_A\cdot\vec{u}_{BO}df \tag{6.35}$$

$$V_{A+B} = V_A + V_B + \int R_{AO}\cdot S_{BO}dv + Y_{Bi}(\lambda_{AOi} - \int_k \vec{k}_A\cdot\vec{u}_i df) - \int_k \vec{k}_B\cdot\vec{u}_{AO}df - \int_k \vec{k}_A\cdot\vec{u}_{BO}df \ . \tag{6.36}$$

According to (6.30) (compare the derivation of (6.17) and (6.18))

$$V_{A+B} = V_A + V_B + \frac{1}{2}\int_u(\vec{k}_A\cdot\vec{u}_B + \vec{k}_B\cdot\vec{u}_A)df - \frac{1}{2}\int_k(\vec{k}_A\cdot\vec{u}_B + \vec{k}_B\cdot\vec{u}_A)df = \tag{6.37}$$

$$= V_A + V_B + \frac{1}{2}\int(\vec{k}_A\cdot\vec{u}_B + \vec{k}_B\cdot\vec{u}_A)df - \int_k(\vec{k}_A\cdot\vec{u}_B + \vec{k}_B\cdot\vec{u}_A)df \tag{6.38}$$

and from the reciprocal theorem of Betti and Rayleigh (6.19)

$$V_{A+B} = V_A + V_B + \int_u \vec{k}_A\cdot\vec{u}_B\, df - \int_k \vec{k}_B\cdot\vec{u}_A df \ . \tag{6.39}$$

The equations (6.35) and (6.39) will be used for the determination of external loads.

6.2.3 Determination of external loads.

The external load $\vec{k}_A$ at the surface element df, where the displacement is prescribed is to be determined for a given loading case A

$$\left.\begin{aligned} R_A &= R_{AO} + Y_{Ai}R_i \\ \vec{u}_A &= \vec{u}_{AO} + Y_{Ai}\vec{u}_i \end{aligned}\right\} \tag{6.40}$$

The solution is readily obtained from the solution for the state of strain which gives stresses also at the surface element df. If, however, the solution for the strains is not the exact one, the method presented in this section gives usually more accurate results.

Take an auxiliary system B

$$\left.\begin{aligned} R_B &= R_{BO} + Y_{Bi}R_i \\ \vec{u}_B &= \vec{u}_{BO} + Y_{Bi}u_i \end{aligned}\right\} \qquad (6.41)$$

as described in section 6.2.2. As far as the displacements are prescribed those of the system B are taken zero everywhere, except in the element df considered where the displacement is a vector $\vec{u}$ of magnitude 1. The prescribed external forces may be arbitrary.

Equating the right hand sides of (6.35) and (6.39) the result is (also if the prescribed external forces of system B are not zero)

$$\int_u \vec{k}_A.\vec{u}_B df = \int R_{AO}.S_{BO}dv + Y_{Ai}\lambda_{BOi} - \int_k \vec{k}_A.\vec{u}_{BO}df \ . \qquad (6.42)$$

The integrand of the left hand side vanishes except for the surface element df where the external load $\vec{K}_A = \vec{k}_A df$ is to be determined.

Equation (6.42) now becomes

$$\vec{K}_A.\vec{u}_B = K_{Ax}u_{Bx} + K_{Ay}u_{By} + K_{Az}u_{Bz} = \int R_{AO}.S_{BO}dv + Y_{Ai}\lambda_{BOi} - \int_k \vec{k}_A.\vec{u}_{BO}df \ . \qquad (6.43)$$

By taking successively three auxiliary systems B, with at the surface element df the prescribed displacement vectors $\vec{u}_B(1,0,0)$, $\vec{u}_B(0,1,0)$ and $\vec{u}_B(0,0,1)$ the force components K_{Ax}, K_{Ay}, K_{Az} are established separately. Along the same lines the external moment projections $\vec{M}_{Ax}$, $\vec{M}_{Ay}$, $\vec{M}_{Az}$ acting at the surface element df can be established if three auxiliary systems B are taken with at the surface element df prescribed unit rotation components $\vec{\varphi}_B(1,0,0)$, $\vec{\varphi}_B(0,1,0)$ and $\vec{\varphi}_B(0,0,1)$ respectively.

Like in section 6.1.3 it is not necessary to solve the unknowns Y_{Bi}. Only the basic strain system $R_{BO}, \vec{u}_{BO}$ is needed in the equation. Every other zero system R'_{BO}, $\vec{u}'_{BO}$ yields the same results provided R'_{BO} is a linear combination of the system R_{BO} and the supplementary strain systems R_i.

6.3 Enclosing numerical results between bounds.

6.3.1 Displacements.

In section 6.1.3 is shown how, for a loading case A, the three displacement projections u_{Ax}, u_{Ay}, u_{Az} at a surface element df, where the external load is prescribed, are determined by means of an auxiliary system B. The only prescribed external load of system B consists of the unit forces $\vec{K}_B(1,0,0)$, $\vec{K}_B(0,1,0)$ and $\vec{K}_B(0,0,1)$ at df.

Use of $\vec{K}_B(K_{Bx} = 1,0,0)$ gives the displacement projection u_{Ax}. One of the ways to get the expression for the complementary energy of the structure if indeed the loading case A and loading case B are present simultaneously is formula (6.21)

$$V^*_{A+B} = V^*_A + V^*_B + u_{Ax} \tag{6.44}$$

provided the prescribed displacements $\vec{u}_B$ are taken equal to zero. In order to calculate V^*_A and V^*_B by means of (6.10), the unknowns X_{Bi} must now be solved.

If all stresses, forces and displacements of loading case A are multiplied by α, and those of loading case B by β, the expression for the complementary energy becomes

$$V^* = \alpha^2\, V^*_A + \beta^2 V^*_B + \alpha\beta\, u_{Ax} \quad . \tag{6.45}$$

With the notation

$$V^*_A = P'_{11},\ V^*_B = P'_{22} \quad \text{and} \quad u_{Ax} = 2P'_{12} = 2P'_{21} \tag{6.46}$$

V takes the form

$$V^* = P'_{11}\, \alpha^2 + 2P'_{12}\, \alpha\beta + P'_{22}\, \beta^2 \quad . \tag{6.47}$$

If the number of redundancies of the structure is larger than the number of supplementary stress systems which are used, the value of V^* is for arbitrary values of α and β larger[1)] than the exact value V^*_e. If the exact values of P'_{11}, $P'_{12} = P'_{21}$, P'_{22} are respectively P_{11}, $P_{12} = P_{21}$ and P_{22} it is obvious that the quadratic form

1) For a special ratio α/β an exact solution may be achieved, because the participation factors of all stress systems not used are in fact zero for that special ratio α/β. Then of course $V^* = V^*_e$. Compare also section 8.2.6.3.

$V^* - V_e^* = (P'_{11} - P_{11})\alpha^2 + 2(P'_{12} - P_{12})\alpha\beta + (P'_{22} - P_{22})\beta^2$ is positive definite, and therefore (page 30, ref.30)

$$P'_{11} - P_{11} > 0 \tag{6.48}$$

$$P'_{22} - P_{22} > 0 \tag{6.49}$$

and

$$\begin{vmatrix} P'_{11} - P_{11} & P'_{12} - P_{12} \\ P'_{21} - P_{21} & P'_{22} - P_{22} \end{vmatrix} > 0$$

or

$$(P'_{11} - P_{11})(P'_{22} - P_{22}) > (P'_{12} - P_{12})^2 \quad . \tag{6.50}$$

Next the loading cases A and B will be analyzed also by means of the minimum principle for the strains. For the combination of the loading case A and the loading case B the potential energy is according to (6.39), thereby again taking the prescribed displacements $\vec{u}_B$ equal to zero and reminding that the prescribed force of loading case B is a force $\vec{K}_B(1,0,0)$

$$V = V_A + V_B - u_{Ax} \quad , \tag{6.51}$$

where V_A and V_B have to be calculated from (6.32) after solving for Y_{Ai} and Y_{Bi} .

If the loading case A is multiplied by α and the loading case B by β, expression (6.51) becomes

$$V = \alpha^2 V_A + \beta^2 V_B - \alpha\beta\, u_{Ax} \quad . \tag{6.52}$$

With the notation

$$V_A = -P''_{11} \;,\; V_B = -P''_{22} \quad \text{and} \quad u_{Ax} = 2P''_{12} = 2P''_{21} \tag{6.53}$$

V takes the form

$$-V = P''_{11}\alpha^2 + 2P''_{12}\,\alpha\beta + P''_{22}\,\beta^2 \quad . \tag{6.54}$$

In general the solutions for the loading cases will be approximations and therefore the value for V is too large, and the value for $-V$ is smaller than the exact value $-V_e$ for all values of α and β. According to (6.31), which must be hold for any set of parameters α,β, the exact solutions of P''_{ij} are the same as those of P'_{ij} in (6.47). Hence

$$P_{11} - P''_{11} > 0 \tag{6.55}$$

$$P_{22} - P''_{22} > 0 \tag{6.56}$$

and

$$\begin{vmatrix} P_{11} - P''_{11} & P_{12} - P''_{12} \\ P_{12} - P''_{12} & P_{22} - P''_{22} \end{vmatrix} > 0$$

or

$$(P_{11} - P''_{11})(P_{22} - P''_{22}) > (P_{12} - P''_{12})^2 \ . \tag{6.57}$$

From (6.48), (6.49), (6.50), (6.55) and (6.56) follows

$$(P'_{11} - P''_{11})(P'_{22} - P''_{22}) > (P'_{12} - P_{12})^2$$

or

$$-R < P'_{12} - P_{12} < R \tag{6.58}$$

where

$$R = \sqrt{(P'_{11} - P''_{11})(P'_{22} - P''_{22})} > 0 \tag{6.59}$$

and (6.58) yields for the bounds for P_{12}

$$P'_{12} - R < P_{12} < P'_{12} + R \ . \tag{6.60}$$

Likewise from (6.48), (6.49), (6.57), (6.55) and (6.56) follows

$$P''_{12} - R < P_{12} < P''_{12} + R \ . \tag{6.61}$$

If $P'_{12} < P''_{12}$ (6.60) and (6.61) yield

$$P''_{12} - R < P_{12} < P'_{12} + R \tag{6.62}$$

and if $P'_{12} > P''_{12}$

$$P'_{12} - R < P_{12} < P''_{12} + R \ . \tag{6.63}$$

The bounds for P_{12} are according to its definition also the bounds for $\frac{1}{2} u_{Ax}$.

Again the present procedure can be extended to the determination of the rotation components of the surface element df.

Likewise upper and lower bounds for the differences in displacement projections of two points may be determined by suitably choosing two auxiliary forces in the two points. If the distance between these points goes to zero these displacements get the character of strain components. So by applying the same principles it is possible to determine upper and lower bounds for strains and stresses.

The method described here is essentially the same as already used by Weber, ref.33. Cooperman, ref.34 and Synge, ref. 35 have given a general scheme for finding local bounds for the solutions, and their derivatives, of boundary value problems of mathematical physics, of which the problems of elasticity theory are special cases.

6.3.2 External loads.

In section 6.2.3 it has been shown how, for a loading case A, the three force components K_{Ax}, K_{Ay}, K_{Az} at a surface element df, where the displacement is prescribed, are determined by means of an auxiliary system B, where the prescribed displacements are confined to unit displacements $\vec{u}_B(1,0,0)$, $\vec{u}_B(0,1,0)$ and $\vec{u}_B(0,0,1)$ respectively at the element df. The displacement $\vec{u}_B(1,0,0)$ yields the force component K_{Ax}.

The expression for the complementary energy of the structure under the combined loading cases A and B is formula (6.21)

$$V^*_{A+B} = V^*_A + V^*_B - K_{Ax} \tag{6.64}$$

and the expression for the potential energy is formula (6.39)

$$V_{A+B} = V_A + V_B + K_{Ax} \tag{6.65}$$

if the prescribed external loads of system B are taken zero.

In the same way as in section 6.3.1, where (6.44) and (6.51) were used to establish the bounds for u_{Ax}, the equations (6.64) and (6.65) yield the bounds for K_{Ax}.

With the notation

$$V^*_A = P'_{11} \text{ , } V^*_B = P'_{22} \text{ and } K_{Ax} = -2\,P'_{12} = -2P'_{21} \text{ in (6.64)}$$

and

$$V_A = -P''_{11} \text{ , } V_B = -P''_{22} \text{ and } K_{Ax} = -2\,P''_{12} = -2P''_{21} \text{ in (6.65)}$$

the bounds for P_{12} are, if $P'_{12}<P''_{12}$

$$P''_{12} - R < P_{12} < P'_{12} + R$$

and if $P'_{12}>P''_{12}$

$$P'_{12} - R < P_{12} < P''_{12} + R$$

where $R = \sqrt{(P'_{11} - P''_{11})(P'_{22} - P''_{22})} > 0$.

The bounds for P_{12} are also the bounds for $-\frac{1}{2}K_{Ax}$.

7 Deformation and stress analysis of a 5 cell clamped swept back box beam with ribs in flight direction.

7.1 Introduction.

Lang and Bisplinghoff (ref.36 and 37) give results of experiments on a swept box with ribs in "flight direction". It was decided to apply the variational principles of section 6 to this swept box, in order to make a comparison with the experimental results. The swept box was clamped at the root and had 5 ribs, the rib at the free end included, thus there were 5 cells (a cell is the space between two rib planes). Also Lang and Bisplinghoff made calculations, but in these calculations they took the ribs normal to the spars instead of in flight direction, which seems to be a rather crude assumption more in particular for the region near the root.

In schematizing the structure for the present calculations, the starting point was the schematization of Lang and Bisplinghoff (of course except for the position of the ribs). However, after completion of the calculations, the schematization of Lang and Bisplinghoff proved to be not sufficiently correct. The moment of inertia of the cross section AA (fig. 7.1) with respect to the xy plane used by Lang and Bisplinghoff is 25.7 $inch^4$ (a value also following from fig.7.5 of ref. 36). The moment of inertia, calculated from the actual structural dimensions proved to be 28.7 $inch^4$, which is a difference of more than 10 °/o [1)].

Since the calculations of Lang and Bisplinghoff and equally the present ones do not refer to the actual structure on which the experiments were carried out it had to be concluded that it was not worthwhile to compare the results of the present calculations with the experiments described in ref.36 and 37. To meet this disadvantage another swept box beam was built and tested at the National Aeronautical Research Institute, Amsterdam. The dimensions of this structure were taken such that they correspond as good as possible with the schematization of the calculations. Another reason to replace Lang and Bisplinghoff's tests was that the root restraint in these tests was not completely rigid. The N.A.R.I. test specimen was a double swept box loaded symmetrically so as to provide rigid root restraint[2)]. This experimental investigation and its

1) This difference stems from the fact that in the calculation which led to the results of 25.7 $inch^4$ all upper- and lowerside material is considered to be at a distance of exactly 3 inch from the neutral plane. However, only the innersides of the top and bottom plates are 3 inch from the neutral plane, and all the material of top and bottom plates, stringers and especially of the spar booms has greater distances.

2) Lang and Bisplinghoff incorporated measured displacements at the root in their calculations. Nevertheless these displacements obscure to a certain degree the results. The special state of stress near the root becomes less pronounced.

interpretation were carried out by Van Grol, Hakkeling and Schuerman (ref.38). Also Brühl (ref.39) made experiments on a swept box beam and compared measured stresses with stresses computed according to the theory of section 7.4.2, which was made available to him. The results of these comparisons proved to be quite satisfactory.

The schematized structure will be analysed first by applying the minimum principle for the stresses, taking 10, 15 and 20 supplementary stress systems respectively. Subsequently the structure is analysed by applying the minimum principle for the strains taking into account 50 supplementary strain systems.

7.2 Dimensions.

The planform and cross section of the swept-back box are illustrated by fig. 7.1 and 7.2. The angle of sweep is $(90^{\circ} - \theta) = 45^{\circ}$. There are two parallel spars of equal and constant cross section along the span. The thickness of the spar web plates h_s is 0.051 inch and the cross sectional area of each spar boom is 0.3819 inch^2 (section normal to the boom). Top and bottom structures are identical and consist of a skin plate of thickness $h = 0.032$ inch and 3 stringers of cross sectional area $A_S = 0.059$ inch^2 each. Their distance a_S measured along a rib is $3\sqrt{2}$ inch. The webs of all five ribs have a thickness of 0.051 inch, and the upper and lower rib flanges 0.0255 inch^2 have cross sectional area each (section normal to the rib). The modulus of elasticity E is 10.5×10^6 lbs/inch^2 and Poisson's ratio $\nu = 0.3$. (Shear modulus $G = 4.0385 \times 10^6$ lbs/inch^2). A right handed oblique coordinate system x,y,z as indicated in fig. 7.1 is used. The swept box is clamped at the root at section $x = 0$.

The cross sectional area of the upper spar booms is 1.36× the cross sectional area of upper skin and stringers. It is to be expected that the particular stress distribution near the clamped root is more pronounced the lighter the spar booms are. It may then be necessary to use in the root region still more supplementary stress or strain systems than are used in the present calculations. Some calculations for a box beam with lighter spar booms were carried out by the author in ref.40. Experiments with this box beam are under progress.

7.3 The strain-stress relations.

The structure discussed in section 7.2 is schematized in the following way.

The stringers are continuously distributed to form an anisotropic plate only capable of carrying normal stresses in stringer direction. To this purpose the cross sectional area of each spar boom was diminished with an amount equal to 1/2 of the cross sectional area of a stringer and this area was added to the anisotropic plate consisting of the skin and the continuized stringers.

The spar and rib webs are assumed to carry shear stresses only. Their normal stress carrying capacity is added to spar booms and rib flanges respectively.

The applied loads give in all spars and ribs stresses which are antisymmetrical with respect to the x,y plane. If it is supposed that the normal stresses in a cross section are proportional to the distance to the x,y plane (the bending stress distribution of engineering theory), the normal stress carrying capacity of the webs is properly taken into account by adding 1/6 of the sectional area of the webs to each spar boom or rib flange (by doing so, the moment of inertia of the spar remains unchanged).

The cross sectional area of the idealized spar boom becomes now

$$A = 0.3819 - 1/2 \times 0.059 + \frac{1}{6} \times 6 \times 0.051 = 0.4034 \text{ inch}^2,$$

that of the idealized rib flange

$$0.0255 + \frac{1}{6} \times 6 \times 0.051 = 0.0765 \text{ inch}^2 .$$

The strain-force relation of a spar boom is

$$\varepsilon = \frac{P}{EA} \tag{7.1}$$

where $EA = 4.2357 \times 10^6$ lbs

and of a rib flange

$$\varepsilon = \frac{P}{EA} \tag{7.2}$$

where $EA = 0.80325 \times 10^6$ lbs.

The spar and rib webs carry only shear stressflows and their strain-stress relation is

$$\gamma = \frac{t}{Gh} \tag{7.3}$$

where $Gh = 0.205963 \times 10^6$ lbs/inch.

With the numerical values $\theta = 45^o$, $h = 0.032''$, $A_S = 0.059$ inch^2 and $a_S = 3\sqrt{2}$ inch, the stiffness matrix (3.25) of the composite plate takes the value

$$a_{ij} = 10^6 \begin{Vmatrix} 1.1903 & 0.6788 & -0.7385 \\ 0.6788 & 1.0443 & -0.7385 \\ -0.7385 & -0.7385 & 0.7049 \end{Vmatrix} \text{ lbs/inch .} \qquad (7.4)$$

It be remembered that the rib flanges are not continuously distributed. Therefore in (3.25) Q = 1.

The inverse of matrix (7.4) is computed from (3.31)

$$A_{ij} = 10^{-6} \begin{Vmatrix} 2.6068 & 0.9124 & 3.6866 \\ 0.9124 & 4.0127 & 5.1594 \\ 3.6866 & 5.1594 & 10.6853 \end{Vmatrix} \text{ inch/lbs .} \qquad (7.5)$$

7.4 Analysis with the minimum principle for the stresses.

7.4.1 10 Supplementary stress systems.

A load in the direction of the positive z axis of 1 lb will be placed successively in the stations 1,2...10 (fig.7.1). These states of loading, will be called "loading case 1", "loading case 2", etc. For each loading case the stress distribution and the vertical deflections of the stations 1...10 will be established. The calculations follow the lines of section 6.1.

From symmetry considerations it follows that, in the lower skin, the projections of the displacement vectors, the strain components and the stress components have always the opposite sign as those in the upper skin.

The figures 7.3 and 7.4 show the two types of supplementary stress systems, admitted in this analysis. Each system only extends along two consecutive cells. In addition the cell next to the root carries stress systems, which consist of the right part of the two types of systems. Fig.7.5 gives the position of each system within the structure. (The systems number 11...20 are introduced in section 7.4.2 and 7.4.3). If the region of a system is not bounded by the root, the structure beyond the bounds of the region is free from stresses and forces. If the root is a part of the bound of a system, the structure may exert forces on the root the resultant of which is zero; this applies to the stress systems 1,6,11 and 16.

The systems of fig.7.3 (type 1) are the only supplementary stress systems when the structure is made 5-fold statically indeterminate by supposing that the skin bays can carry only shear stresses along their edges. These systems account for the capability of the ribs to carry shear stresses.

The addition of the systems according to fig.7.4 (type 2) presents the first step towards refinement of the behaviour of the skin. It allows the composite plate to carry longitudinal stress s_x. Strictly, each of these systems consists of two independent systems, one in the upper plate and one in the lower plate. Because it is known that stresses in upper and lower plate will be reversed in sign, they are taken together.

The limitations to which the state of stress in the composite plate is subjected are:

1^o The axial stress flow s_x is constant over the width of the skin (in y-direction), it varies linearly between two successive ribs (in x-direction).

2^o The axial stress flow s_y is zero throughout.

From the equilibrium conditions then follows that the shear stress flow t is a linear function of y.

With the supplementary stress systems to be used later, fig.7.7 and 7.8, the axial stress flow s_y still remains zero and the axial stress flow s_x still varies linearly between 2 successive ribs, but it is being admitted that s_x is a quadratic function of the chordwise coordinate.

The coefficients λ_{ij} defined by (6.5) have been calculated with the strain-stress relations of section 7.3 and are tabulated in table 7.1. Values not given are obtained by observing that $\lambda_{ij} = \lambda_{ji}$. In calculating the integrals a table given by Van Beek, ref.41, was used.

In structures were in all elements the state of stress is one dimensional (trusses, structures with only shear carrying plates) the computation of the elements λ_{ij} can be very well systematized to a matrix multiplication of three matrices (ref.26). The first and third matrix represent the supplementary stress systems, the second one the flexibilities of the elements. The presence of elements in which the state of stress is two-dimensional gives complications, but the procedure can be adapted. For the present structure, which consists of equal cells it was considered not necessary to use the matrix procedure in question.

It may cause surprise that for example λ_{ij} for j = 1 and i = 6 and 7 does not vanish; j = 1 pertaining to the type 1 systems which are "anti-symmetrical" to the axis of x, whereas i = 6 and 7 pertaining to the type 2 systems are "symmetrical" with respect to the axis of x. The reason for the fact that they nevertheless couple in the products $\int S_i . R_j dv$, $\int S_j . R_i dv$ is that the element A_{13} of the flexibility matrix of the composite plate is not zero (i.e. the expression for the strain energy contains a term $2A_{13}s_x t$).

For each loading case, the basic stress system was chosen as simple as possible. They occurred only in the spar connecting the loaded station with the root and are given in fig.7.6.

The terms λ_{0i} of the equations (6.6), for the 10 loading cases m = 1...10 indicated by λ_{m0i}, are given in table 7.2, forming a non-symmetrical matrix, which is only square by chance (because the number of supplementary stress systems happens to be the same as the numbers of stations in which a vertical unit load is placed). No non-zero displacements are prescribed. So the term $\int_u \vec{k}_i.\vec{u}_i df$ does not play a role.

The solutions of the unknowns X_j, j = 1...10 for the loading cases m(m=1...10), indicated by X_{mj}, are given in table 7.3.

Consider now the case, where the load is placed in station m, and let be asked the vertical deflection in station n.

According to (6.26), the displacement indicated by the symbol C_{mn} - which is an influence coefficient since it is the displacement for a unit load -, is

$$C_{mn} = \int S_{m0}.R_{n0}dv + X_{mi}\lambda_{n0i} \quad . \qquad (7.6)$$

S_{m0} indicates the basic stress system for loading case m, R_{n0} the state of strain corresponding with the basic stress system S_{n0} for loading case n.

The matrices $\int S_{m0}.R_{n0}dv$ and C_{mn} are given in tables 7.4, and 7.5 respectively.

It is easy to compute stresses or strains in the structural members, once the unknowns X_i have been solved.

Table 7.6 gives stresses for the 10 loading cases in the stations 1...12 of the spar booms, in the spar webs and in the rib webs.

In section 7.6.2 figures will be produced where stresses, also in the skin, are plotted for some loading cases.

7.4.2 15 Supplementary stress systems.

A third type of supplementary stress systems was assumed and is shown in fig.7.7. Again, each system corresponding to fig.7.7 consists of two independent parts for upper and lower skin, but as stresses and displacements in upper and lower skin are reversed in sign, the two parts are taken together. Fig. 7.5 gives the position of the added systems nr. 11...15.

With respect to the type 2, the present type offers the refinement that the stressflows s_x may vary linearly in y-direction.

In view of the equilibrium conditions (2.1), (2.2) the stressflows t are now quadratic functions of y.

With the three types of supplementary stress systems the general formulas for the stressflows in a composite plate are

$$
\begin{aligned}
s_x &= c_1 xy + c_2 x + c_3 y + c_4 \\
s_y &= 0 \\
t &= -\frac{1}{2} c_1 y^2 - c_2 y + c_5 \quad .
\end{aligned}
\qquad (7.7)
$$

Only apparently these formulas offer 5 degrees of freedom per cell. However, the stressflows s_x to either side of a rib must be in equilibrium. Since s_x along a rib is a linear function of y two coefficients for the n-th cell depend on the coefficients for the n-1th. In addition s_x must vanish at the free end of the beam. Then only 3 degrees of freedom per cell remain.

If no spar booms would be present in the structure the supplementary stress systems fig. 7.3, 7.4 and 7.7 could not exist because normal forces are required along the boundaries y = constant of the skin panels. However systems fig. 7.3 and fig. 7.7, taken together in the appropriate ratio, form a system of stress without forces in the spar booms.

The computation follows further quite the same lines as in section 7.4.1. The range i and j, however, is 1...15, instead of 1...10.

Table 7.7 gives the additional values of the coefficients λ_{ij} of the 15 unknowns. The other values were already given in table 7.1.

Table 7.8 gives the additional values of the terms λ_{0mi} of the equations. The other values were already given in table 7.2.

Table 7.9 gives the solutions X_{mj} of the equations for the m loading cases (Compare table 7.3).

Table 7.4 still holds for the calculation with 15 unknowns.

Table 7.10 gives the matrix of influence coefficients (compare table 7.5).

Table 7.11 gives stresses in the structure (compare table 7.6). For further discussion of stresses see also section 7.6.2.

7.4.3 20 Supplementary stress systems.

Fig. 7.8 shows the fourth type of supplementary stress systems. Note that no forces in the spar booms occur in the stress systems of this type. So these systems can be used if no spar booms are present (together with the appropriate combination of type 1 and 3).

This type offers the refinement that the axial stressflows s_x may vary quadratically in y-direction, but they still vary linearly in x-direction. The stressflows s_y remain zero. Consequently, the stressflows t are now cubic functions of y.

With the four types of supplementary stress systems the general formulas for the stressflows in a composite plate are

$$\begin{aligned} s_x &= c_1\, xy^2 + c_2 y^2 + c_3\, xy + c_4 x + c_5 y + c_6 \\ s_y &= 0 \\ t &= -\frac{1}{3} c_1\, y^3 - \frac{1}{2} c_3\, y^2 - c_4 y + c_7 \end{aligned} \qquad (7.8)$$

Apparently these formulas now offer 7 degrees of freedom per cell. But to either side of a rib, the stressflows s_x, which are there quadratic functions of y, must be the same. This means that three coefficients for the n-th cell depend on the coefficients for the 1-n th. In addition s_x must vanish at the free end. This leaves only 4 degrees of freedom per cell.

The computation follows quite the same lines as in sections 7.4.1 and 7.4.2. Matrices are given in tables 7.12...7.15, out of which table 7.14 gives the matrix of influence coefficients and 7.15 the stresses. For further discussion of stresses see also section 7.6.2.

Inspection of tables 7.6, 7.11 and 7.15 already learns that introduction of the possibility for the stressflows s_x to vary quadratically in y-direction does not offer very much improvement. A first impression is that the use of only the first three types of supplementary stress systems gives a solution sufficiently accurate for practical purposes.

7.5 Analysis with the minimum principle for the strains.

The analysis applies again to a swept back box of section 7.2.

It is the aim to assume such a combination of types of supplementary strain systems, that an accuracy can be expected of the same order as achieved with the minimum principle for the stresses and 15 supplementary stress systems. It was discussed that the general formulae for the stressflows in the skin in that case were (7.7), in which 5 constants occur though the actual number of degrees of freedom for the stresses were only 3 per cell.

Of course the formulas (7.7) were not the result of the use of the 3 types of supplementary stress systems, but in reverse were the starting point in constructing these systems. Without any change in the final result these formulas could have been given a somewhat more general shape, viz.

$$s_x = c_1xy + c_2x + c_3y + c_4$$

$$s_y = c_6xy + c_7x + c_8y + c_9 \qquad (7.9)$$

$$t = -\frac{1}{2}c_1y^2 - c_2y - \frac{1}{2}c_6x^2 - c_8x + c_5 .$$

Such an approach could be made by supposing that two stressflow components are arbitrary functions linear in x and linear in y. These two stressflows can only be s_x and s_y. Thereupon the shape of the formula for the stressflow t follows from the equilibrium condition (3.17).

Since the stressflows s_y must be zero at the spar booms, it follows immediately that $c_6 = c_7 = c_8 = c_9 = 0$ and again (7.7) remains.

In constructing supplementary strain systems, one can likewise start with the assumption that two strain components are arbitrary functions linear in x and linear in y. Next the shape of the formula for the third strain component follows from the compatibility condition (3.19).

A set of formulas to start with may be for example the formulas

$$\varepsilon_x = c_1xy + c_2x + c_3y + c_4$$

$$\varepsilon_y = c_5xy + c_6x + c_7y + c_8 \qquad (7.10)$$

$$\gamma = c_9x^2 + c_{10}x + c_{11}y^2 + c_{12}y + c_{13} .$$

Another set is

$$\varepsilon_x = c_1xy + c_2x + c_3y + c_4 \qquad (7.11)$$

$$\gamma = c_5xy + c_6x + c_7y + c_8 \qquad (7.12)$$

$$\varepsilon_y = \frac{1}{2}c_5x^2 + c_9x + c_{10}y^2 + c_{11}y + c_{12} . \qquad (7.13)$$

The set of formulas (7.11)...(7.13) seems to have a minor advantage over the set (7.10). In the set (7.10) the strains ε_y vary only linearly in x and y-direction. This means that if an infinite box beam with normal or oblique ribs is loaded by a moment, the strain ε_y must be the same linear function of y in all cells. Then the discrete character of the ribs is lost.

The set of formulas (7.11)...(7.13) does not show this defect and was chosen as starting point.

Before constructing the supplementary strain systems the number of such systems required to guarantee the freedom expressed by (7.11)... (7.13) will be established.

Again use is made of the fact that strains and displacements of the lower skin are reversed in sign as compared to strains and displacements of the upper skin.

Suppose the strains in a skin panel are those of (7.11)...(7.13) and those of the consecutive skin panel by (both panels have same x- and y axes)

$$\varepsilon'_x = c'_1\ xy + c'_2\ x + c'_3\ y + c'_4 \tag{7.14}$$

$$\gamma' = c'_5\ xy + c'_6\ x + c'_7\ y + c'_8 \tag{7.15}$$

$$\varepsilon'_y = \frac{1}{2}\ c'_5\ x^2 + c'_9\ x + c'_{10}\ y + c'_{11}\ y + c'_{12} \tag{7.16}$$

where of course the coefficients $c'_1 \ldots c'_{12}$ have, in general, other values than the values $c_1 \ldots c_{12}$ of (7.11)...(7.13).

Along the intermediate rib flange, with x coordinate x_r, the displacement projections u_x and u_y of both panels must be the same. If for both panels is put

$$\frac{d^2u_x}{dy^2} = \frac{d\gamma}{dy} - \frac{d\varepsilon_y}{dx} \tag{7.17}$$

$$\frac{du_y}{dy} = \varepsilon_y \quad , \tag{7.18}$$

the compatibility conditions at the rib flange are

$$\frac{d\gamma}{dy} - \frac{d\varepsilon_y}{dx} = \frac{d\gamma'}{dy} - \frac{d\varepsilon'_y}{dx'} \tag{7.19}$$

$$\varepsilon_y = \varepsilon'_y \tag{7.20}$$

or

$$c_7 - c_9 = c'_7 - c'_9 \tag{7.21}$$

$$\frac{1}{2}\ c_5 x_r^2 + c_9 x_r + c_{10} y^2 + c_{11} y + c_{12} = \frac{1}{2}\ c'_5 x_r^2 + c'_9 x_r + c'_{10} y^2 + c'_{11} y + c'_{12}\ . \tag{7.22}$$

Since (7.22) must hold for any value of y

$$\frac{1}{2}\ c_5 x_r^{\ 2} + c_9 x_r + c_{12} = \frac{1}{2}\ c'_5 x_r^2 + c'_9 x_r + c'_{12} \tag{7.23}$$

$$c_{10} = c'_{10} \tag{7.24}$$

$$c_{11} = c'_{11} \quad . \tag{7.25}$$

Though the strains (7.11)...(7.13) for a panel have, by means of their 12 values $c_1 \ldots c_{12}$, 12 degrees of freedom, the actual number of degrees of freedom for each panel in the structure is only 8, because of the four relations (7.21), (7.23), (7.24), (7.25).

The spar and rib webs are still considered to be of a material that is only able to carry shear-stresses and thus the strain energy of such elements is a function of the shear strain only and not of the normal strains which may be present. It is possible to assume a system of a non-constant shear strain (of zero mean value) to one spar or rib web. As such a system, superimposed on a constant shear strain, always augments the potential energy, it needs not to be taken into account. So in the solution, the shear strains and shear stresses will be constant within a bay, and so it is sure that the equilibrium conditions within an only shear carrying bay will not be violated in the approximate solution.

Though there are three webs per cell (two spar webs and one rib web), there are not 3 degrees of freedom for the constant shear strains of these webs. If the skin and spar booms are kept unstrained, the shear strain of a rib web cannot be varied without imposing shear strains in the spar webs of the two adjoining cells. Reversely, again leaving the skin and booms unstrained, one cannot apply unequal shear strains to the spar webs, without affecting the shear of an adjoining rib and of the spar webs of the cell at the other side of this rib. Equal shear of the two spar webs is, however, compatible with absence of shear in the ribs and of spar web shear in other cells. Therefore the degrees of freedom for shear of the three webs comprises one system in which the rib is affected and one with equal shear in the two spars. So the webs add per cell two degrees of freedom to those of the skin and a number of 8 + 2 = 10 supplementary strain systems per cell must be taken. The total number of unknowns, 50, is remarkably large, compared with the 15 unknowns, which were required if for the stresses a similar freedom of distribution is supposed.

From (7.24) and (7.25) follow that the coefficients c_{10}, as well as the coefficients c_{11}, for all cells are equal to each other.

Since at the clamped root $\varepsilon_y = 0$ and $c_{10} = c_{11} = 0$ for the first skin panel and thus all c_{10} and c_{11} are zero and need not further to be taken into account in constructing the supplementary strain systems.

The 10 supplementary strain systems are developed along the following systematic lines.

First of all systems are established which occur only in one cell and leave the two rib planes which bound the cell undeformed and undisplaced. Such systems are type 1, 2, 3, 4. The basic formulas for the strains in the upper skin are:

$$\begin{aligned} &\text{for type 1, fig. 7.9, } 10^3\varepsilon_x = x - 6 \\ &\text{type 2, fig. 7.10, } 10^3\gamma = x - 6 \\ &\text{type 3, fig. 7.11, } 10^3\varepsilon_x = y(x-6)/8.4852 \\ &\text{type 4, fig. 7.12, } 10^3\gamma = y(x-6)/4.2426 \ . \end{aligned} \qquad (7.26)$$

The compatibility equation (2.15), the requirement that the sparwebs at upper and lower side must follow the displacements of the skin and the fact that the rib webs at the bounds of the cell may not displace require for some of the types strains in the spar webs and some further strains in the skin, which obey like those of (7.26) the general shape of (7.11)... (7.13).

Subsequently supplementary strain systems are established which also occur only in one cell, but where the rib plane at the right hand side translates or rotates with respect to the other rib plane which is not displaced (both rib planes remaining undeformed). The outer part of the box beam then also translates and rotates as a rigid body. Such types are type 5,6 and 7. These rib displacements or rotations were:

for type 5, fig. 7.13, translation in z-direction
type 6, fig. 7.14, rotation about y-axis
type 7, fig. 7.15, rotation about x-axis

and the strains belonging to these types obey the shape of (7.11)... (7.13).

The possibilities to construct strains in one cell only, which obey the formulas (7.11)...(7.13) are now exhausted. The types 8, 9 and 10 originate from three kinds of rib deformation which must, of course, distort two cells.

These rib deformations are:

for type 8, fig. 7.16, warping out of the plane
type 9, fig. 7.17, uniform shear
type 10, fig. 7.18, uniform bending ,

and the strains belonging to these types obey the shape of (7.11)... (7.13).

From the shape of the formulas (7.26), and from the way in which the systems type 5...type 10 have been constructed, it is clear that the linear undependency of the 10 types is fully guaranteed.

Each of the types occurs 5 times in the structure. Fig. 7.19 locates the position of each system.

Of course, any other set, mutually independent, combinations of the systems used, may serve the purpose. It is, however, desirable that in the first place the strains of each system affect only a small part of the structure. Less important, but still desirable is that also the displacements, belonging to each system are non-zero only in a small part of the structure. Of course the displacements are non-zero where the strains are so, but they may be non-zero, in regions where the strains are zero. The set of 10 systems used, fits the afore mentioned requirements reasonably well, but of course there may be more or less different combinations which equally or still somewhat better do.

The vertical displacements in the stations 1, 2...10 (fig.7.1) caused by the supplementary strain systems nr. 1, 2...50 are tabulated in table 7.16.

The matrix of the "coefficients of the unknowns" λ_{ij} from (6.29) is given in table 7.17.

Ten loading cases are considered: Vertical load of 1 lb in station 1, in station 2, etc. No basic strain systems need to be used since the prescribed displacements at the root are zero; so in (6.29)$\lambda_{0i} = 0$ and only m(m=1,..10) sets of $\int_k \vec{k}_m \cdot \vec{u}_i df$ have to be computed. Since $k_m = 1$ in one station only and vanishes in the other stations the values given in table 7.16 represent the integrals $\int_k \vec{k}_m \vec{u}_i df$ for the 10 loading cases m=1,...10. In order to solve the 50 equations use was made of the fact that the coefficients of the unknowns λ_{pq} as far as p = 1...35, q = 1....35 are concerned, form a matrix which can be divided into 7 x 7 unit sub-matrices, each multiplied with a scalar number, or with zero. The inversion of this 35 x 35 matrix can easily be done by inverting the 7 x 7 matrix of the scalar multipliers of the afore mentioned unit sub-matrices. Moreover, the inversion of this 7 x 7 matrix simplifies into the inversion of a 3 x 3 matrix and a 4 x 4 matrix. By some further operations, the number of equations was reduced to 15. The solutions Y_i (i = 1..50) for the m loading cases (m=1...10) are given in table 7.18. From these solutions the matrix C_{mn} of influence coefficients for the displacements in table 7.19 is easily computed by means of

$$C_{mn} = X_{mi} M_{in}$$

where M is the matrix of table 7.16, if the index m is changed to n.

Table 7.20 gives stresses as calculated from the superposition of the

50 strain systems. In this case, the values of the stresses in the spar booms do not refer to the stations 3...12, but to points just outside these stations.

Table 7.21 gives in addition the calculated stresses just inside the stations 1...10. The fact that the calculated stresses to either side of the stations 1...10 are not equal is not astonishing; the solution is not exact and the stresses which follow from the minimum principle for the strains do in general not satisfy equilibrium conditions completely.

7.6 Discussion of results.

7.6.1 Displacements. Determination of upper and lower boundaries.

Table 7.22 repeats in 4 columns the main diagonal elements of the matrices of influence coefficients from table 7.5, 7.10, 7.14 (obtained with the minimum principle for the stresses with respectively 10, 15 and 20 unknowns) and 7.19 (obtained with the minimum principle for the strains with 50 unknowns).

If at the stations 1...10 (fig.7.1) the vertical forces $K_1 \dots K_{10}$ are applied, it follows from (6.9) since $u_m = C_{mn}K_n$ that the complementary energy of the structure is

$$V^* = \frac{1}{2} C_{mn}K_mK_n \quad (n,m = 1\dots10) \qquad (7.27)$$

and the potential energy follows from (6.30)

$$V = -\frac{1}{2} C_{mn}K_mK_n \quad (n,m = 1\dots10) \quad . \qquad (7.28)$$

If $\overline{C_{mn}}$ is the (unknown) exact matrix of influence coefficients, then $(C_{mn}-\overline{C_{mn}})K_mK_n$ is a positive definite quadratic form if C_{mn} is one of the matrices established with the minimum principle for the stresses since $V > \overline{V}$ and $(C_{mn}-\overline{C}_{mn})K_mK_n$ is a negative definite quadratic form if C_{mn} is the matrix determined from the minimum principle for the strains, since $V<\overline{V}$. Also $(C'_{mn}-C''_{mn})K_mK_n$ is a positive definite quadratic form if C'_{mn} and C''_{mn} are both determined with the minimum principle for the stresses, C'_{mn} with a lower number of unknowns than C''_{mn}, provided the stress systems used for C'_{mn} are available among those used for C''_{mn}.

In accordance with these statements the rows of table 7.22 consist of numbers C_{mm}, decreasing in value and the (unknown) exact values $\overline{C}_{mm}$ lie between those of the third and fourth column. Also for the other elements of the exact matrix of influence coefficients upper and lower bounds can easily be computed. As to obtain these bounds for an element C_{ij}, the 2×2 submatrix

$$\begin{Vmatrix} C_{ii} & C_{ij} \\ C_{ji} & C_{jj} \end{Vmatrix}$$

is extracted from the tables 7.14 and 7.19 and the procedure on the matrices P_{ij} of section 6.3.1 is applied.

In using 10, 15 and 20 supplementary stress systems respectively, there seems to be a fairly rapid convergence of the values given in the tables 7.5, 7.10 and 7.14. This convergence is not necessarily directed towards the exact solution, because the three types of supplementary stress systems (fig.7.3, 7.4 and 7.7) have one defect in common: that the axial stressflow s_y in the composite plate is zero. In section 8.3 supplementary stress systems with a non-zero stressflow s_y will be introduced.

7.6.2 Stresses and strains.

For loading case m = 3 (that means a vertical load of 1 lb in station 3), fig.7.20 gives the normal stresses in the upper spar booms. Fig. 7.21 gives the shear stresses in the spar and rib webs. Fig.7.22 the stressflows s_x and t for the composite plate (combination of skin and equivalent stringer plate).

Figures 7.24, 7.25 and 7.26 are similar to figs. 7.20, 7.21 and 7.22 respectively, but for loading case m = 4 (load of 1 lb in station 4).

As to the curves obtained with the minimum principle for the stresses with 10, 15 and 20 unknowns respectively it is seen from the figures 7.20 and 7.22 and 7.24 - 7.26 that there is again a fair convergence. Therefore, there seems to be little need to use the last 5 supplementary stress systems. It could be expected that the addition of parabolic stressflows s_x would mainly affect the stresses in the skin (figs.7.22 and 7.26) and indeed a noticeable improvement has been obtained. However, it is unlikely that further improvement would be obtained by taking a more refined distribution-function for the stressflows s_x in y-direction, thereby maintaining $s_y = 0$.

If it had been supposed that the oblique skin fields could carry only shear stresses along their edges, the results for the loading cases m = 3 and m = 4 would have been such that in figures 7.20 and 7.21 the plots for the front spar were identical to the plots for the rear spar in the figures 7.24 and 7.25 respectively. It is seen that this supposition would have led to highly misleading results.

The tables 7.23 and 7.24 give again results of stresses like tables 7.6, 7.11, 7.15, 7.20 and 7.21. Each of these two tables only refers to one loading case, but gives results for all the ways by which the computations were made.

In the case of table 7.23, vertical downwards directed loads of 1 lb at all stations are present. The agreement between the different columns is satisfactory. In the case of table 7.24 vertical downwards directed loads of 1 lb are placed at the stations of the rear spar, and upwards directed loads at the front spar. The absolute differences between the respective columns are generally not greater than those of table 7.23. The relative errors are somewhat greater.

Figures 7.23 and 7.27 give for the 2 loading cases the strain ε_x in the spar booms and in the skin immediately adjacent to the spar booms, which must be equal for the exact solutions. The calculations were performed with the minimum principle for the stresses with 20 unknowns. The solution is not exact and some incompatibilities have to be expected. The strain in the skin ($s_y = 0$) follows from the stressflows by

$$\varepsilon_x = A_{11} s_x + A_{13} t \tag{7.29}$$

or with (3.31)

$$E\,\varepsilon_x = \frac{s_x/\sin\theta + 2t\,\mathrm{ctn}\,\theta}{h + \dfrac{A_S}{a_S \sin\theta}} \tag{7.30}$$

(for the meaning and the numerical values of θ, h, A_S and a_S see section 7.2).

It is seen that there are large discontinuities in both figures at the station where the load is applied and in the rear spar at station 9.

These discontinuities correspond to the discontinuities of the term

$$\frac{2t\,\mathrm{ctn}\,\theta}{h + \dfrac{A_S}{a_S \sin\theta}} \tag{7.31}$$

in (7.30), since only t, and not s_x, is discontinuous at the intersections of the ribs and the skin. So at a rib the discontinuity is

$$E\Delta\varepsilon_x = \frac{2(\Delta t)\,\mathrm{ctn}\,\theta}{h + \dfrac{A_S}{a_S \sin\theta}} \tag{7.32}$$

This discontinuity of t, for which already a very good convergence is achieved after applying respectively 10, 15 and 20 unknowns (figs. 7.22 and 7.26) and which would not be present in the exact solution is caused by the fact that the stress systems used do not allow t to vary in x-direction within a skin field because $s_y = 0$. The discontinuity can be diminished by allowing s_y to have non-zero values by which t may vary in x-direction (whereas of course s_y remains zero at the spar boom, which has no bending stiffness). However, a complete removal of the discontinuity of t along the spar boom cannot be obtained by such means. The reason is that the exact solution for the present schematized structure will have in general stress singularities in the corners (intersections of rib flanges with spar boom). See also the discussion in section 10.

In section 8.2.6 discontinuities in strains along rib flanges, which will prove to be still more severe, will be discussed.

7.6.3 Comparison with Morley's work.

Morley, ref. 42, analyses a similar swept box, but his method is quite different. This method allows for a completely arbitrary distribution of the axial stressflows s_x in the composite plate along the ribplanes and supposes that the stressflow s_y is zero. Simultaneous differential equations for the distributions of s_x along the rib planes are formed. The rib flanges, however, are considered to be infinitely stiff with respect to normal forces and later, in the numerical work, the rib webs are taken infinitely stiff with respect to shear stresses.[1)]

For two sets of concentrated loads at the tip (which may be replaced by any other statically equivalent load in the tip rib plane, because the rib web in its plane is infinitely stiff), Morley performed calculations. The figures 7.28...7.31 are derived from figures of Morley. The figures 7.23 and 7.27 are to be compared with the figures 7.28 and 7.30. They show quite the same type of discrepancy between the strains ε_x of the spar booms and the adjacent skin. The discontinuities are again large in the

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1) It is also possible to obtain along the lines of the present method such a set of simultaneous linear differential equations for the distributions of s_x along the rib-planes ($s_y=0$), even if the rib flanges and rib webs are allowed to have a finite stiffness. Supplementary stress systems of the type 7.3 and of a type similar to fig. 7.4, 7.7 or 7.8 are to be used. However, in the 5 systems of the latter type for the distribution of s_x along the rib planes 5 different unknown arbitrary functions $f_1(y)...f_5(y)$ are to be taken. These 5 arbitrary functions replace the 5 quadratic functions, which in fact are used when the 20 supplementary stress systems were applied. The minimum principle for the stresses and the application of the principles of variational calculus then lead to the coupled differential equations for $f_1(y)...f_5(y)$.

neighbourhood of the point of the application of the loads and at the intersections of the rib next to the root and the rear spar.

Two sets of plots for the stressflows t between the ribs are reproduced in the figs. 7.29 and 7.31. They correspond to the lower parts of the figs. 7.22 and 7.26. Most of the curves in the figures 7.29 and 7.31 could be fairly well approximated by 3rd degree curves, which in fact the curves for t in figs. 7.22 and 7.26 actually are.

Morley derives his solution with much more trouble than attendant on the present method. Even, if the ribs had been supposed to be infinitely stiff in their plane, it is questionable whether Morley's solution would be much better than that according to the present method. The inaccuracy resulting from the assumption that the ribs are infinitely stiff in bending and shear presumably overrides the slight improvement resulting from Morley's more accurate treatment of the stresses in the skin.

8 Application to the infinitely long swept box.

8.1 Introduction.

It is interesting to study the results which the methods of section 7 yield when applied to a box beam of infinite length, consisting of equal cells with oblique ribs. The methods are not particularly suitable for a box beam of infinite length (i.e. a box of m cells, where $m \to \infty$), since they would lead to 2 m, 3 m, 4 m and 10 m equations with as many unknowns if the supplementary stress systems of section 7.4.1, 7.4.2 and 7.4.3 or the supplementary strain systems of section 7.5 were used respectively.

However, if the cells are identical, the solution for the case of loading by a constant moment or a constant shear force[1)] is practicable. In the following numerical application the dimensions of the cell of the swept box are equal to those of the 5 cell swept box of section 7. The moment vector is parallel to the xy-plane and the shear force is parallel to the axis of z. For these loads, like in section 7, the stresses in top

1) With loading by a constant moment (or constant shear force) is meant a loading case where in every cross section the moment (or shear force) has the same value. The external load is thought to be applied at both ends which are supposed to be infinitely far from each other.

and bottom composite plate are of equal magnitude and opposite sign.

The main purpose of the present investigation is to obtain the equivalent of the elementary beam theory (the so called engineering theory), for the oblique beam, which takes the form of simple differential equations for the relations between displacements, flexibilities and loads. Like the engineering theory for straight beams this theory for oblique beams might be used for the practical analysis of finite non-prismatic swept box beams where moment and shear load are functions of x, and the results so obtained would be slightly in error only, if the dimensions and the loads would vary sufficiently slowly along the longitudinal axis.

Results of this section (upper and lower bounds for the flexibilities are established) will be compared with the results obtained by other authors from equivalents of the elementary beam theory for the oblique beam.

Section 8.2 gives the calculations for constant moment and constant shear force according to the minimum principle for the stresses. Three types of supplementary stress systems are used (see fig.7.3, 7.4 and 7.7).

It proved to be possible to find general formulae, which are applicable to any combination of angle of sweep and structural dimensions, provided of course that the cells are equal.

In section 8.3 the calculations are repeated (only for a constant moment) after the introduction of two more types of supplementary stress systems: that of fig.7.8 and a new type. It will appear that the addition of the type of fig.7.8 only, yields the same results as section 8.2 (their coefficients X_i becoming zero).

In section 8.4 the calculations for constant moment and constant shear force are performed according to the minimum principle for the strains. Comparisons are made with the numerical results of section 8.2 and 8.3. The sections 8.1, 8.2, 8.3 and 8.4 have in common the exact solution for a constant moment $\vec{M}(M_x,\ M_y = \xi M_x)$, ξ having a particular value.

In section 8.5 the obtained flexibilities are compared with those, obtained from other methods.

8.2 Calculations according to the minimum principle for the stresses with 3 types of supplementary stress systems.

8.2.1 The supplementary stress systems.

The systems of supplementary stress systems used (figs.8.2, 8.3, 8.4) are of the same type as those given in figs. 7.3, 7.4 and 7.7, however they are generalized to arbitrary dimensions of the cell. The 3 types will be indicated by the suffices 1, 2 or 3. The stress system applying to the cells n and n+1 obtains as a second suffix n.

The symbol Z_n relating to cells n and n+1 (see fig.8.1) denotes the column matrix of the multiplication factors of the supplementary stress systems occurring in these two cells

$$Z_n = \begin{Vmatrix} X_{1.n} \\ X_{2.n} \\ X_{3.n} \end{Vmatrix} . \qquad (8.1)$$

So the unknowns of the problem are X_p,

$$p = \ldots 3.(n-2),\ 1.(n-1),\ 2.(n-1),\ 3.(n-1),\ 1.n,\ 2.n\ ,\ 3.n,\ 1.(n+1),\ 2.(n+1),\ 3.(n+1),\ 1.(n+2),\ \ldots\ldots\ldots\ . \qquad (8.2)$$

If no non-zero displacements are prescribed, the equations (6.6) take the form

$$\lambda_{pq} X_p = -\lambda_{0q} \qquad (8.3)$$

where p and q have the values of (8.2).

Furthermore, section 6.1.1 gives

$$\lambda_{pq} = \int S_p . R_q dv \qquad (8.4)$$

i.e. the work done by the stresses of the supplementary stress system number p through the (incompatible) strains of the supplementary stress system number q, and

$$\lambda_{0q} = \int S_0 . R_q dv \qquad (8.5)$$

i.e. the work done by the basic stress system (a system satisfying the external load and internal equilibrium conditions but not the compatibility conditions) through the strains of the supplementary stress system number q.

It is obvious from fig.8.1 that for the infinite box beam with equal cells only a limited number of values λ_{pq} have to be calculated, because the supplementary stress systems in the cells pertaining to Z_n do not interfere with the supplementary stress systems pertaining to Z_{n-3} and Z_{n+3}. The numerical values of λ_{pq} are given in table 8.1 and taken together in square matrices D, C, B, C' and D'. (In general matrix M' is the transposed of matrix M). They refer to an infinite box beam with cell dimensions corresponding to those in figs.7.1 and 7.2. Therefore they could be selected from the tables 7.1, 7.7 and 7.12.

8.2.2 Constant moment $(M_x,0,0)$.

8.2.2.1 Solution of the unknowns.

The moment has projections M_x, $M_y=0$, $M_z=0$ and thus has its vector parallel to the x,y plane and normal to the y-axis (see fig.4.2). This vector is the load upon the cross section, which forms the right side of a part of the box beam.

The basic stress system occurring in each cell is given in fig.8.5. The values λ_{oq} from (8.5) are equal for all cells. They are combined in a column matrix E

$$\begin{Vmatrix} \lambda_{0;\ 1.n} \\ \lambda_{0;\ 2.n} \\ \lambda_{0;\ 3.n} \end{Vmatrix} = \begin{Vmatrix} 0 \\ aA_{13}M_x/b \sin\theta \\ 0 \end{Vmatrix} = E \quad . \tag{8.6}$$

The infinite set of simultaneous equations (8.3) now reduces to

$$DZ_{n-2} + CZ_{n-1} + BZ_n + C'Z_{n+1} + D'Z_{n+2} = -E \quad . \tag{8.7}$$

Since any cell of the infinitely long beam must have the same state of stress $\ldots Z_n = Z_{n+1} = \ldots$ by which (8.7) takes the form

$$FZ_n = -E \tag{8.8}$$

with

$$F = D + C + B + C' + D' \quad . \tag{8.9}$$

The numerical values of F are (cell dimensions shown in figs.7.1 and 7.2)

$$F = 10^{-6} \begin{Vmatrix} 6526 & 0 & -3846 \\ 0 & 7509 & 0 \\ -3846 & 0 & 11112 \end{Vmatrix} \tag{8.10}$$

Recalculation of the elements of F for arbitrary cell dimensions and stiffnesses with the figures 8.2, 8.3 and 8.4 (however with equal front and rear spars) learns that the elements of F are in terms of the cell dimensions and stiffnesses

$$F = \begin{Vmatrix} 16\,\frac{a^3}{EA} & 0 & -\frac{40}{3}\,\frac{a^2c}{EA} \\ 0 & 16\,ac(A_{11} + \frac{c}{EA}) & \\ -\frac{40}{3}\,\frac{a^2c}{EA} & 0 & \frac{100}{3}\,ac\,(\frac{1}{3}\,\frac{c}{EA} + A_{11}) \end{Vmatrix} \tag{8.11}$$

with EA and A_{11} as given in section 7.3. F is not affected by the stiffnesses of the ribflanges and rib webs, though these stiffnesses are represented in the elements of D, C, B, C' en D'.

The solution for Z_n from (8.8) now takes the form

$$Z_n = \begin{Vmatrix} X_{1.n} \\ X_{2.n} \\ X_{3.n} \end{Vmatrix} = \begin{Vmatrix} 0 \\ \dfrac{-M_x A_{13}}{16\ bc\ \sin\theta(A_{11}+c/EA)} \\ 0 \end{Vmatrix} . \qquad (8.12)$$

8.2.2.2 Determination of stresses.

To the stresses of the basic stress system, fig.8.5, only stresses proportional to those of the supplementary stress systems type 2, fig. 8.3, are to be added, because $X_{1.n} = X_{3.n} = 0$.

The stresses of fig.8.3 are to be multiplied with $X_{2.n}$ according to (8.12). The results are then as follows:

Stressflow in the top skin

$$s_x = -\frac{M_x A_{13}}{8\ bc\ \sin\theta(A_{11}+\frac{c}{EA})} ,$$

$$s_y = 0 \qquad (8.13)$$

$$t = \frac{M_x}{8\ bc\ \sin\theta} .$$

Shear stressflow in front spar web (y= -c)

$$t = -\frac{M_x}{8\ bc\ \sin\theta} .$$

Shear stressflow in rear spar web (y=c) (8.14)

$$t = \frac{M_x}{8\ bc\ \sin\theta} .$$

Normal force in upper spar booms

$$N = \frac{M_x A_{13}}{8b\ \sin\theta\ (A_{11}+\frac{c}{EA})} . \qquad (8.15)$$

Rib flanges and rib webs remain unstressed.

It can easily be verified that the resultant of these stresses is, indeed, a moment $\vec{M}(M_x,0,0)$. The axial stressflows s_x in the skins are in equilibrium with the normal forces of the spar booms. If the latter would have been absent (A=0) s_x would vanish and the stresses would be tangential stresses t only. This suggests that the loading case (M_x,0,0) may be characterized as "torsional load".

For a further discussion of the results reference is made to section 8.2.6 .

8.2.3 Constant moment (0,M_y,0).

8.2.3.1 Solution of the unknowns.

The moment has projections $M_x = 0$, M_y, $M_z = 0$ and thus has its vector parallel to the x,y plane and normal to the x axis (see fig.4.3). This vector is the load upon the cross section which forms the right side of a part of the box beam. The type of loading could be characterized as a "bending load".

The basic stress system occurring in each cell is given in fig.8.6. The calculations follow the lines of section 8.2.2. The matrix E becomes

$$\left\| \begin{matrix} \lambda_{0;1.n} \\ \lambda_{0;2.n} \\ \lambda_{0;3.n} \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ \dfrac{2\ ac\ M_y}{EA\ b\ \sin\theta} \\ 0 \end{matrix} \right\| = E . \qquad (8.16)$$

Of course the matrix F remains the same and the solutions become

$$Z_n = \left\| \begin{matrix} X_{1.n} \\ X_{2.n} \\ X_{3.n} \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ \dfrac{-M_y}{8b(EAA_{11}+c)\sin\theta} \\ 0 \end{matrix} \right\| . \qquad (8.17)$$

8.2.3.2 Determination of stresses.

To the stresses of the basic stress system fig.8.6 only the stresses of the supplementary stress systems, type 2, fig. 8.3, multiplied by $X_{2.n}$ according to (8.17) are to be added.

The result is for the upper skin

$$\begin{aligned} s_x &= \frac{-M_y}{4b(EAA_{11}+c)\sin\theta} \\ s_y &= 0 \\ t &= 0 \end{aligned} \qquad (8.18)$$

The normal forces in upper booms are

$$N = \frac{-M_y}{4b \sin \theta} \left(\frac{EAA_{11}}{EAA_{11}+c} \right) \quad . \tag{8.19}$$

Spar webs, rib flanges and rib webs are unstressed.

8.2.4 Mean complementary energy per unit of length of box beam under constant moment $(M_x, M_y, 0)$ and equations for displacements.

This energy can be expressed in the form

$$V^* = \frac{1}{2} \left(Q_{11} M_x^2 + 2\, Q_{12} M_x M_y + Q_{22} M_y^{\,2} \right) \quad . \tag{8.20}$$

If the loading case $(M_x, 0, 0)$ is called case A and loading case $(0, M_y, 0)$ case B, the expression (6.15) gives the complementary energy for simultaneous application of the two loading cases:

$$V_{A+B}^{\;*} = V_A^* + V_B^* + \int S_{AO} . R_{BO} dv + X_{Ai} \lambda_{BOi} = V_A^* + V_B^* + \int S_{AO} . R_{BO} dv + + X_{Bi} \lambda_{AOi} \quad , \tag{8.21}$$

where V_A^* and V_B^* are the complementary energies for the two loading cases separately. Thus accordingly (6.10)

$$\begin{aligned} V_A^{\,*} &= \frac{1}{2} \lambda_{AOO} + \frac{1}{2} X_{Ai} \lambda_{AOi} \\ V_B^{\,*} &= \frac{1}{2} \lambda_{BOO} + \frac{1}{2} X_{Bi} \lambda_{BOi} \quad . \end{aligned} \tag{8.22}$$

If the expressions (8.21) and (8.22) are applied to the energy of one cell only, and subsequently to the mean value of the energy per unit of length (along the x axis), it follows that in (8.20) the coefficients $Q_{11}, Q_{12} = Q_{21}, Q_{22}$ have the values from

$$Q_{11} M_x^{\;2} = \frac{1}{a} \left[\lambda_{AOO} + E_A^{!}\; Z_{An} \right] \tag{8.23}$$

where $\lambda_{AOO} = \int S_{AO} .\, R_{AO} dv$ for one cell, and E_A and Z_{An} are E and Z from (8.6) and (8.12) respectively,

$$Q_{22} M_y^{\;2} = \frac{1}{a} \left[\lambda_{BOO} + E_B^{!}\; Z_{Bn} \right] \tag{8.24}$$

where $\lambda_{BOO} = \int S_{BO} . R_{BO} dv$ for one cell and E_B and Z_{Bn} from (8.16) and (8.17) respectively,

$$Q_{12} M_x M_y = \frac{1}{a} \left[\int S_{AO} . R_{BO} dv + E_B^{!}\; Z_{An} \right] = \frac{1}{a} \left[\int S_{AO} . R_{BO} dv + E_A^{!}\; Z_{Bn} \right] \tag{8.25}$$

where the integral proves to be zero.

The result for the coefficients Q_{ij} is

$$Q_{11} = \frac{1}{16\ bc\ \sin^2\theta}\left\{\frac{1}{cG\ h_s} - \frac{A_{13}^2}{b(A_{11}+\frac{c}{EA})} + \frac{A_{33}}{b}\right\} \tag{8.26}$$

$$Q_{12} = \frac{-A_{13}}{8b^2\sin^2\theta\ (EAA_{11}+c)} \tag{8.27}$$

$$Q_{22} = \frac{A_{11}}{4b^2\sin^2\theta\ (EAA_{11}+c)} \tag{8.28}$$

where h_s is the thickness of the spar web, A the cross sectional area of the spar boom and the coefficients A_{ij} are defined by (3.31). With the numerical values of (7.5), which are based on cell dimensions given in figs. 7.1 and 7.2, the numerical values for the coefficients Q_{ij} become

$$\begin{aligned} Q_{11} &= 0.015474 \times 10^{-6} \\ Q_{12} &= -0.005244 \times 10^{-6} \\ Q_{22} &= 0.007417 \times 10^{-6} \quad . \end{aligned} \tag{8.29}$$

Though, by lack of compatibility, displacements cannot be computed in a unique way from the approximate solution a reliable way was discussed in section 6.1.3 which is based on the expression for the complementary energy of two loading cases.

One of the ribs will be supposed to be not rotated nor translated. The rotation and translation of another rib plane n cells to the right hand side of this fixed rib will be computed. Though both ribplanes actually may show distortion in and warping from their plane, the notion "rotation and translation of one rib plane with respect to another rib plane" does not raise any difficulty if the two rib planes are situated at large distance from each other. The rotation and translation in question then become large and rib distortion and warping of rib planes have negligible effect on the result.

Suppose the box beam is loaded by a moment $\vec{M}(M_x,M_y,0)$. This is now the load system A of section 6.1.3 for which the unknowns (the participation factors of the supplementary stress systems) have already been computed. In order to compute the rotation component φ_x of a "right"-rib-plane situated n cells from the non rotated "left" rib-plane, an auxiliary load system B consisting of a constant moment $\vec{M}(1,0,0)$ is taken

Then φ_x is equal to the right hand side of (6.26), where the stresses pertaining to system B have to satisfy only the equilibrium conditions and as such the system of fig.8.5 with $M_x = 1$ can be taken.

It is now easily found that the rotation becomes

$$\varphi_x = na(Q_{11}M_x + Q_{12}M_y) \quad . \tag{8.30}$$

If it is imagined that this rotation φ_x occurs due to a constant curvature $d\varphi_x/dx$ this curvature is

$$\frac{d\varphi_x}{dx} = Q_{11}M_x + Q_{12}M_y \quad . \tag{8.31}$$

In order to obtain an expression for the rotation component φ_y an auxiliary system B corresponding to fig.8.6 (M_y=1) yields

$$\frac{d\varphi_y}{dx} = Q_{12}M_x + Q_{22}M_y \quad . \tag{8.32}$$

The relations (8.31) and (8.32) are confirmed by noting that the mean complementary energy per unit of length is

$$V^* = \frac{1}{2}\left(M_x \frac{d\varphi_x}{dx} + M_y \frac{d\varphi_y}{dx}\right) \tag{8.33}$$

and again (8.20) is obtained.

In reverse (8.33) could have been used to define "specific rotations" φ_x and φ_y. Then the relations (8.31) and (8.32) are obtained by equating (8.33) and (8.20).

Also the vertical deflection w of a rib plane at the distance x=na from a non rotated and non translated rib plane will be determined. The auxiliary system B is a load system where the rib n is loaded by two downward directed forces 1/2 at its ends, whereas this load is carried at the rib 0 by two upwards directed forces 1/2 and an appropriate moment.

The result is

$$w = -\frac{1}{2}n^2a^2(Q_{12}M_x + Q_{22}M_y)\sin\theta \quad .$$

If it is imagined that the deflection w follows from a constant curvature of the elastic line, which in undeflected state, is situated along the x axis, this curvature is

$$\frac{d^2w}{dx^2} = -(Q_{12}M_x + Q_{22}M_y)\sin\theta \tag{8.34}$$

and from (8.32)

$$\frac{d^2w}{dx^2} = -\frac{d\varphi_y}{dx}\sin\theta \quad . \tag{8.35}$$

Integration yields

$$\frac{dw}{dx} = -\varphi_y \sin\theta \tag{8.36}$$

since the integration constant is zero because the mean shear angle of the two spar webs is zero.

8.2.5 Constant shear force $(0,0,K_z)$.

8.2.5.1 Solution of the unknowns.

Suppose the box beam (fig.8.1) carries a constant shear force $\vec{K}$ with components $K_x = 0$, $K_y = 0$, K_z acting upon the cross section which forms the right side of a part of the box beam. The shear force K_z is accompanied by a linearly varying moment $(0,M_y,0)$

$$M_y = x\,K_z \sin\theta + \text{constant} \; .$$

This moment is taken zero in the point $y = 0$ of the rib 0 (fig.8.1). The basic stress system in the environment of rib n is given in fig.8.7.

The infinite set of equations (8.3) reduces to (compare (8.7))

$$DZ_{n-2} + CZ_{n-1} + BZ_n + C'Z_{n+1} + D'Z_{n+2} = -nR \tag{8.37}$$

where the meaning of D,C,B,C',D' and Z_n is that of section 8.2.2.1. The column matrix R is

$$R = \left\| \begin{matrix} 0 \\ 2\,\dfrac{a^2c}{bEA} \quad K_z \\ 0 \end{matrix} \right\| \quad . \tag{8.38}$$

Suppose Z_n is of the form

$$Z_n = n\,U + L \tag{8.39}$$

where U and Y are unknown column matrices.

Equation (8.39) substituted into (8.37) yields

$$\begin{aligned} &D\left\{(n-2)U + L\right\} + C\left\{(n-1)U + L\right\} + B\left\{nU+L\right\} + \\ &+ C'\left\{(n+1)U + L\right\} + D'\left\{(n+2)U + L\right\} + nR = 0 \end{aligned} \tag{8.40}$$

or

$$nFU - GU + FL + nR = 0$$

where F is as defined in (8.11) and

$$G = \left\{ 2\ D + C - C' - 2D' \right\}. \tag{8.41}$$

Numerical values of matrix G are computed from table 8.1

$$G = 2\times10^{-6} \begin{Vmatrix} 0 & 1502 & 0 \\ -1502 & 0 & -1770 \\ 0 & 1770 & 0 \end{Vmatrix} \tag{8.42}$$

For arbitrary cell dimensions and stiffnesses

$$G = \begin{Vmatrix} 0 & 8\ ac\ A_{31} & 0 \\ -8\ ac\ A_{31} & 0 & -\frac{40}{3}\ c^2 A_{31} \\ 0 & \frac{40}{3}\ c^2 A_{31} & 0 \end{Vmatrix} \tag{8.43}$$

The assumption (8.39) gives the solution if it applies to any value of n, which requires that

$$\begin{aligned} FU + R &= 0 \\ GU - FL &= 0 \end{aligned} \tag{8.44}$$

From the two matrix equations with the unknown column matrices U and L the solutions for U and L are obtained.

They are

$$U = \begin{Vmatrix} 0 \\ \dfrac{-aK_z}{8b(EAA_{11}+c)} \\ 0 \end{Vmatrix} \tag{8.45}$$

$$L = \begin{Vmatrix} \dfrac{-cA_{31}K_z}{16\ ab\ A_{11}} \\ 0 \\ \dfrac{-3c\ A_{31}K_z}{40b\ A_{11}(EAA_{11}+c)} \end{Vmatrix} \tag{8.46}$$

8.2.5.2 Determination of stresses.

Herewith the participation factors of the supplementary stress systems are known and stresses can easily be computed. If the plane x=0 coincides with rib 0, the stresses in the upper reinforced skin are

$$\left.\begin{aligned} s_x &= \frac{-x\,K_z}{4b(EAA_{11}+c)} - \frac{3A_{31}y\,K_z}{8b\,A_{11}(EAA_{11}+c)} \\ s_y &= 0 \\ t &= \frac{K_z y}{4b(EAA_{11}+c)} \quad . \end{aligned}\right\} \qquad (8.47)$$

The shear stressflows in the spar webs are:

$$t = \frac{K_z}{4b} \quad . \qquad (8.48)$$

The normal force in the upper front spar at $y = -c$ is

$$N = \frac{-x}{4b} K_z + \frac{c\,A_{31}K_z}{8b\,A_{11}} + \frac{c\,x\,K_z}{4b(EAA_{11}+c)} - \frac{c^2A_{31}K_z}{8b\,A_{11}(EAA_{11}+c)} \qquad (8.49)$$

and at the upper rear spar at $y = +c$

$$N = \frac{-x}{4b} K_z - \frac{c\,A_{31}K_z}{8b\,A_{11}} + \frac{cxK_z}{4b(EAA_{11}+c)} + \frac{c^2A_{31}K_z}{8b\,A_{11}(EAA_{11}+c)} \quad . \qquad (8.50)$$

8.2.5.3 Equations for displacements.

As in section 8.2.4 and along the lines of section 6.1.3 the following relations are established (by means of the same auxiliary stress systems)

$$\frac{d\varphi_x}{dx} = Q_{12}M_y \qquad (8.51)$$

$$\frac{d\varphi_y}{dx} = Q_{22}M_y \qquad (8.52)$$

where

$$M_y = K_z\,x\,\sin\theta \quad . \qquad (8.53)$$

The differential equation for the line of vertical deflections of the centres of the rib planes $w(x)$ becomes (compare (8.35))

$$\frac{d^2w}{dx^2} + \frac{d\varphi_y}{dx}\sin\theta = 0 \quad . \qquad (8.54)$$

Integration of (8.54) yields

$$\frac{dw}{dx} + \varphi_y \sin\theta = \frac{K_z}{4bGh_s} \qquad (8.55)$$

where the integration constant is the angle of shear of the spar webs ($\frac{dw}{dx} = \frac{K_z}{4bGh_s}$ at $x = 0$ when $\varphi_y = 0$).

8.2.6 Discussion.

8.2.6.1 Degree of compatibility.

It is worth mentioning that in all foregoing solutions the stiffnesses of rib flanges and rib webs drop out and that rib flanges and rib webs are unstressed.

The solutions will therefore also hold for vanishing rigidity of the ribs, i.e. for a hollow box beam. In accordance with this fact, all stresses must be continuous at the rib webs. For infinitely long hollow beams exact solutions for the loading cases "constant moment" or "constant shear force" are available, and indeed, as will be shown in section 8.2.6.2, the present approximate solutions for the box beam with oblique ribs are identical to the exact solutions for the hollow box beam.

It would have been useless to try more supplementary stress systems which allow s_x to be a more general function of y, as long as s_y is assumed to be zero. The final solutions will not depend on the stiffnesses of rib flanges and rib webs and these elements remain unstressed. However, the solution which made the complementary energy in the remainder of the structure a minimum was already obtained, because it was the exact solution for the hollow box beam. Therefore the additional systems could not offer any improvement and their participation factors would become zero.

It may be proved that the stress system in the composite skin, the spar booms and the spar webs form a compatible system. Only the unstressed ribs with their webs and flanges do not fit to the deformed system of spars and skin.

Two examples of the compatibility of the system of spars and skin will be given.

For the load$(M_x,0,0)$the stresses in the composite upper skin are according to (8.13)

$$s_x = \frac{-M_x A_{13}}{8\ bc\ \sin\theta(A_{11} + \frac{c}{EA})}$$

$$t = \frac{M_x}{8\ bc\ \sin\theta}\ .$$

The strain ε_x of the skin along a spar boom is

$$\varepsilon_x = A_{11}s_x + A_{13}t \tag{8.56}$$

or

$$\varepsilon_x = \frac{-M_x A_{11} A_{13}}{8\ bc \sin\theta(A_{11} + \frac{c}{EA})} + \frac{M_x A_{13}}{8\ bc \sin\theta} \ . \tag{8.57}$$

From (8.15) follows the strain in the upper spar boom

$$\varepsilon_x = \frac{M_x A_{13}}{8b \sin\theta(EAA_{11}+c)} \tag{8.58}$$

and (8.57) and (8.58) prove to be identical.

For the load $(0,0,K_z)$ the stresses in the upper reinforced skin at the front spar $y = -c$ are from (8.47)

$$s_x = \frac{-2A_{11}x\ K_z + 3c\ A_{31}K_z}{8b\ A_{11}(EAA_{11}+c)}$$

$$t = \frac{-2K_z c\ A_{11}}{8b\ A_{11}(EAA_{11}+c)} \ .$$

The strain ε_x is with (8.56)

$$\varepsilon_x = \frac{-2A_{11}x + c\ A_{31}}{8b(EAA_{11}+c)} K_z \ . \tag{8.59}$$

The strain ε_x in the upper spar boom at $y = -c$ is

$$\varepsilon_x = \frac{N}{EA} \ .$$

Substitution of (8.49) yields that this result is identical to (8.59).

In all cases the strains in the skin are only linear functions of the coordinates and thus compatible.

8.2.6.2 Solution for oblique coordinates as derived from the exact solutions for the hollow box beam.

The fact that the best solution, obtainable with $s_y=0$, equals the exact solution for the hollow box beam of course discloses other more simple ways to obtain this solution in terms of normal or oblique coordinates. The method of the preceeding sections has been used however, not only to study the results to which the methods of section 7 lead when applied to an infinite box beam, but also because it will be the basis of further refinements to be considered in section 8.3 and of the investigations of section 9.

Some of the obtained formulas applying to loading by a constant moment will now be verified by the elementary bending theory and elementary torsion theory (the Bredt theory).

The righthanded orthogonal coordinate system be $\bar{x}$, $\bar{y}$, $\bar{z}$, the $\bar{x}_a$ and $\bar{z}$ axes being along the axes x and z of the oblique system, and the $\bar{y}$ axes perpendicular to the $\bar{x}$ and $\bar{z}$ axes (fig.8.1). A moment with components (= projections) $\bar{M}_x$, $\bar{M}_y$,0 is applied to the right end.

The shear flow in the spars and the skin follows immediately from the Bredt-formula

$$t = \frac{\bar{M}_x}{8bc \sin\theta} .$$

If the box is loaded by moments $(M_x,0,0)$ and $(0,M_y,0)$ then $\bar{M}_x = M_x$ and according to (8.13), (8.14) and (8.18) the same value for t as above is found.

The normal force in the upper spar booms is

$$N = - \frac{\bar{M}_y A}{2b(2A+2c\ h \sin\theta + 2c\ A_S/a_S)} .$$

By substitution of $\bar{M}_y = M_y/\sin\theta - M_x \operatorname{ctn}\theta$ it can easily be shown that this value is equal to the sum of (8.15) and (8.19) if in the latter expressions (3.31) is substituted.

The stresses in the upper skin,

$$\bar{s}_x = \frac{-\bar{M}_y h}{2b(2A+2c\ h \sin\theta + 2c\ A_S/a_S)} ,\quad \bar{t} = \frac{\bar{M}_x}{8bc \sin\theta}$$

must be transformed to oblique coordinates to find s_x and t as given by the sum of (8.13) and (8.18). A more convenient way, however, is to consider a cross section parallel to the yz-plane.(fig.8.1). Once N and t in the spars are known, the constant stressflows s_x and t follow from equilibrium considerations.

With regard to the stiffnesses, the Bredt-torsion theory and the elementary bending theory yield

$$\frac{d\bar{\varphi}_x}{d\bar{x}} = \frac{\bar{M}_x}{S_t} = \bar{Q}_{11}\bar{M}_x + \bar{Q}_{12}\bar{M}_y \qquad (8.60)$$

$$\frac{d\bar{\varphi}_y}{d\bar{x}} = \frac{\bar{M}_y}{S_b} = \bar{Q}_{21}\bar{M}_x + \bar{Q}_{22}\bar{M}_y \qquad (8.61)$$

where $$\overline{Q}_{12} = \overline{Q}_{21} = 0 \qquad (8.62)$$

$$\frac{1}{S_t} = \overline{Q}_{11} = \frac{\frac{1}{hb} + \frac{1}{h_s c \sin\theta}}{16\ Gbc \sin\theta} \qquad (8.63)$$

$$\frac{1}{S_b} = \frac{1}{EI} = \overline{Q}_{22} = \frac{1}{4Eb^2c\left(\frac{A_S}{a_S} + h \sin\theta + \frac{A}{c}\right)} \quad . \qquad (8.64)$$

The transformation formulae for moment projections (4.3) and for rotation components (the same as (3.6)) yield that the transformation formulae for the coefficients Q_{ij} for oblique coordinates, which must lead to (8.31), (8.32) are

$$\left.\begin{aligned} Q_{11} &= \overline{Q}_{11} + \overline{Q}_{22} \operatorname{ctn}^2\theta - 2\overline{Q}_{12} \operatorname{ctn}\theta \\ Q_{12} &= Q_{21} = -\overline{Q}_{22} \frac{\cos\theta}{\sin^2\theta} + \frac{\overline{Q}_{12}}{\sin\theta} \\ Q_{22} &= \frac{\overline{Q}_{22}}{\sin^2\theta} \quad . \end{aligned}\right\} \qquad (8.65)$$

Substitution of (8.62)...(8.64) yields

$$\left.\begin{aligned} Q_{11} &= \frac{\frac{1}{hb} + \frac{1}{h_s c \sin\theta}}{16\ Ghc \sin\theta} + \frac{\operatorname{ctn}^2\theta}{4Eb^2c\left(\frac{A_S}{a_S} + h\sin\theta + \frac{A}{c}\right)} \\ Q_{12} &= Q_{21} = \frac{-\cos\theta}{4Eb^2c\sin^2\theta\left(\frac{A_S}{a_S} + h\sin\theta + \frac{A}{c}\right)} \quad . \\ Q_{22} &= \frac{\csc^2\theta}{4Eb^2c\left(\frac{A_S}{a_S} + h\sin\theta + \frac{A}{c}\right)} \quad . \end{aligned}\right\} \qquad (8.66)$$

Since according to (3.31)

$$A_{11} = \frac{1}{E(\frac{A_S}{a_s} + h \sin \theta)}$$

$$A_{13} = 2 A_{11} \cos \theta$$

$$A_{33} = A_{11} \left\{ \frac{2 A_S(1+\nu)\sin \theta}{a_S h} + 2(1+\cos^2\theta+\nu \sin^2\theta) \right\}$$

it follows that the equations (8.26)...(8.28) and the equations (8.66) are identical.

It is also interesting to note the relation between the formulae derived by Hemp, ref. 22 and the present formulae. Hemp assumes continuously distributed rib webs, which remain unstressed in the present cases, and continuised rib flanges which contribute to the coefficients A_{ij} of the reinforced skin. This method is of course applicable to the hollow box beam. Hemp also obtains the formulae (8.31), (8.32) and (8.51)... ..(8.53) but, as mentioned, the elements A_{ij} have a somewhat different meaning.[1)]

8.2.6.3 An exact solution.

It will be clear from section 8.2.6.1, that for loading by a moment $M(M_x,M_y,0)$ the solution is exact if along the rib flanges the strain ε_y in the skin is zero. In that case the rib planes fit to the other structural elements. It will be interesting to investigate afterwards whether the minimum principle for the strains yields the same exact solution.

The strain ε_y along the flanges then must be

$$\varepsilon_y = A_{21} s_x + A_{23} t = 0 \quad . \tag{8.67}$$

Suppose $M_x = \xi M_y$.

Summation of the stresses s_x and t of (8.13) and (8.18) thereby

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1) Hemp defines his moments in a somewhat different way. Therefore some formulae will differ a factor $\sin \theta$. In some cases there is also a reversal in sign, because Hemp directs the z-axis upwards, and gives stresses and strains in the upper skin. Hemp's coordinate system is, as in this paper, right handed.

introducing (8.68) and substitution of these stresses into (8.67) yields

$$\frac{-\xi M_y A_{13} A_{21}}{8bc \sin\theta (A_{11} + \frac{c}{EA})} - \frac{M_y A_{21}}{4b(EAA_{11}+c)\sin\theta} + \frac{\xi M_y A_{23}}{8bc\sin\theta} = 0 \quad (8.69)$$

or

$$\xi = \frac{2A_{12}c}{A_{23}(EAA_{11}+c)-EAA_{12}A_{13}} \quad . \quad (8.70)$$

For the cell dimensions of fig.7.1, 7.2 the value of ξ becomes

$$\xi = 0.1792 \quad . \quad (8.71)$$

8.3 Calculations according to the minimum principle for the stresses with 5 types of supplementary stress systems.

8.3.1 Introduction.

It was seen in section 7.6.3 that when the stressflow s_x is allowed more degrees of freedom, very little improvement is obtained as long as the stressflow s_y is assumed to be zero.

In the calculations of section 8.2 it was already mentioned that the supplementary stress systems of type 4 (see fig.7.8) would have vanished when introduced without the addition of other new types. The same conclusion holds for all other types with the stressflow $s_y=0$. Therefore, introduction of new types, such as type 5 fig.8.8 and type 6 fig.8.9, is necessary if further improvement is desired. In type 5 s_y is symmetrical in y, in type 6 s_y is antisymmetrical in y. However, in the loading cases of constant moment $(M_x,0,0)$ or $(0,M_y,0)$, type 6 will vanish again because of symmetry conditions, but type 5 fig.8.8 and now also type 4 fig.7.8 will participate. Of course, further extensions with types similar to figs. 8.8 and 8.9 are possible. Much extension along this line does not seem practical because there will remain stress singularities in the corners of the parallelogram shaped plate fields. See also the discussion at section 10.

In this part of the investigation it is no longer possible to perform the analysis partly in terms of arbitrary cell dimensions, but everywhere numerical dimensions (figs.7.1 and 7.2) have to be taken. Of course the calculations follow precisely the methods of section 8.2. The internal load systems are type 1 (fig.7.3), 2 (fig.7.4), 3 (fig. 7.7), 4 (fig.7.8), type 5 (fig.8.8).

8.3.2 Constant moment ($M_x = 1,0,0$).

Numerical values of the coefficients λ_{pq} are given in table 8.2 (compare table 8.1). The matrices E, F and Z_n of (8.6), (8.10) and (8.12) now become

$$E = \begin{Vmatrix} \lambda_{0;1.n} \\ \lambda_{0;2.n} \\ \lambda_{0;3.n} \\ \lambda_{0;4.n} \\ \lambda_{0;5.n} \end{Vmatrix} = \begin{Vmatrix} 0 \\ 20.8546 \\ 0 \\ 0 \\ -26.4088 \end{Vmatrix} \times 10^{-6} \qquad (8.72)$$

$$F = 10^{-6} \times \begin{Vmatrix} +6526 & 0 & -3846 & 0 & 0 \\ 0 & +7509 & 0 & 0 & -3494 \\ -3846 & 0 & +11112 & 0 & 0 \\ 0 & 0 & 0 & +7645 & -594 \\ 0 & -3494 & 0 & -594 & +21998 \end{Vmatrix} \qquad (8.73)$$

$$Z_n = \begin{Vmatrix} X_{1.n} \\ X_{2.n} \\ X_{3.n} \\ X_{4.n} \\ X_{5.n} \end{Vmatrix} = \begin{Vmatrix} 0 \\ -2394.75 \\ 0 \\ +63.8986 \\ +821.84 \end{Vmatrix} \times 10^{-6} \; . \qquad (8.74)$$

It is easy to determine the stresses and strains in the cells from the participation factors (8.74). From these stresses were computed strains ε_x in a spar boom and in the skin immediately adjacent to the spar boom. Both strains (multiplied with E) are plotted in fig.8.10. It is seen that, though the mean values of these strains are about the same, there is an incompatibility. Such incompatibility was not present in the solution which was obtained by using only the supplementary stress systems, types 1 to 3 incl. (fig.7.3, 7.4, 7.7). But then the incompatibility along the rib flanges was very severe.

With the use of only the first three types and the load $M_x = 1$, $M_y = 0$ the strain ε_y in the skin along a rib flange is

$$E\,\varepsilon_y = 0.3230$$

whilst the rib flange itself remains unstrained.

Incorporation of the 4th and 5th type offers already a great reduction of this incompatibility, which is shown in fig.8.11; it has almost completely vanished near the middle of the rib flange.

8.3.3 Constant moment $(0, M_y = 1, 0)$.

The matrices (8.16) and (8.17) become now

$$E = \begin{Vmatrix} \lambda_{0;1.n} \\ \lambda_{0;2.n} \\ \lambda_{0;3.n} \\ \lambda_{0;4.n} \\ \lambda_{0;5.n} \end{Vmatrix} = \begin{Vmatrix} 0 \\ +22.6645 \\ 0 \\ 0 \\ -7.55482 \end{Vmatrix} \times 10^{-6} \qquad (8.75)$$

$$Z_n = \begin{Vmatrix} X_{1.n} \\ X_{2.n} \\ X_{3.n} \\ X_{4.n} \\ X_{5.n} \end{Vmatrix} = \begin{Vmatrix} 0 \\ -3086.90 \\ 0 \\ -11.4476 \\ -147.236 \end{Vmatrix} \times 10^{-6} \,. \qquad (8.76)$$

Fig.8.12 gives the strain in a spar boom and in the skin immediately adjacent to the spar boom. For this more important moment (0,1,0) the discrepancy is much smaller than in fig.8.10 for the moment (1,0,0). This discrepancy is caused by the supplementary stress system type 5. Indeed its participation factor in (8.76) is much smaller than in (8.74). If the ordinates of fig.8.11 are multiplied by −0.1792 (i.e. the value $-\xi$ of section 8.2.6.3.) it holds for the moment $(0, M_y=1, 0)$.

8.3.4 Mean complementary energy per unit of length.

The numerical values for the coefficients Q_{ij} which occur in (8.20) are

$$Q_{11} = 0.014329\times10^{-6}$$

$$Q_{12} = -0.005040\times10^{-6} \quad (8.77)$$

$$Q_{22} = 0.007379\times10^{-6} \quad .$$

8.4 Calculations according to the theorem of minimal principle for the strains with 10 types of supplementary strain systems.

8.4.1 The supplementary strain systems.

The environment of the cell n of the box will be considered (fig. 8.13). The systems of strain are the types 1 to 10 given in figs.7.9-7.18. The strain systems 1.n...7.n are confined to cell n, the systems 8.n, 9.n, 10.n apply to the 2 cells n and n+1.

The symbol Z_n (see fig.8.13) denotes the column matrix of the multiplication factors of the supplementary strain systems with index n

$$Z_n = \left\| \begin{matrix} Y_{1.n} \\ Y_{2.n} \\ Y_{3.n} \\ Y_{4.n} \\ Y_{5.n} \\ Y_{6.n} \\ Y_{7.n} \\ Y_{8.n} \\ Y_{9.n} \\ Y_{10.n} \end{matrix} \right\| \quad (8.78)$$

So the unknowns in the problem are Y_p,

$$p = \ldots 10.(n-1),\ 1.n\ ,\ 2.n \ldots 10.n,\ 1.(n+1)\ ,\ 2.(n+1)\ldots \tag{8.79}$$

The basic strain system is taken zero everywhere.

If in the problem no non-zero displacements ($\vec{u}_o=0$) are prescribed, the equation (6.29) becomes

$$\lambda_{pq} Y_p = \int_k \vec{k}.\vec{u}_q df \quad . \tag{8.80}$$

For the infinite box beam with equal cells again only a limited number of values λ_{pq} have to be computed. Since the systems with indices n-1 and n+1 do not overlap λ_{pq} vanishes for $|p-q|>1$. This reduces the number of square matrices of λ_{pq} to 3, these matrices will be denoted by T, V and T'. The left hand side of the equations (8.80) for p = 1.n, 2.n,...10n can then be written in matrix form

$$T\ Z_{n-1} + V\ Z_n + T'Z_{n+1} \quad .$$

The numerical values of λ_{pq} are given in tables 8.3, 8.4 and 8.5; they are the elements of the matrices T, V and T'. The numerical values of λ_{pq} needed no computation since they could be selected from table 7.17.

8.4.2 Constant moment (M_x,0,0).

8.4.2.1 Solution of the unknowns.

Like in section 8.2.2.1a constant moment $\vec{M}(M_x,0,0)$ is applied to the structure.

The moment $\vec{M}(M_x,0,0)$ to the right end is supposed to be applied according to fig.8.14. None of the supplementary strain systems gives any displacement at the left end of the swept-back box.

The integral $\int_k \vec{k}.\vec{u}_q df$ of (8.80) applied to the right end of the box, takes for q = 1.n, 2.n...10.n the values

$$\int_k \vec{k}.\vec{u}_{i.n} df = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.02\ M_x \\ 0 \\ 0 \\ 0 \end{matrix} \right\| = K \quad . \qquad i = 1\ldots 10 \tag{8.81}$$

The equations (8.80) for p = 1.n, 2.n...10.1 are written in matrix notation

$$T\,Z_{n-1} + V\,Z_n + T'Z_{n+1} = K \quad . \tag{8.82}$$

These equations are satisfied for all values of n by

$$\ldots\ldots = Z_{n-1} = Z_n = Z_{n+1} = \ldots\ldots \quad .$$

Then Z_n is the solution of $H\,Z_n = K$, where

$$H = T + V + T' \tag{8.83}$$

which is given in table 8.6.

The solution is

$$Z_n = \left\| \begin{matrix} Y_{1.n} \\ Y_{2.n} \\ Y_{3.n} \\ Y_{4.n} \\ Y_{5.n} \\ Y_{6.n} \\ Y_{7.n} \\ Y_{8.n} \\ Y_{9.n} \\ Y_{10.n} \end{matrix} \right\| = 10^{-6} M_x \left\| \begin{matrix} 0 \\ 0 \\ -0.251438 \\ -0.457587 \\ 0 \\ +3.50339 \\ +8.30541 \\ +6.04116 \\ 0 \\ +5.86748 \end{matrix} \right\| \quad . \tag{8.84}$$

It can be proved that any other statically equivalent way of applying the end forces of fig.8.14, does not alter the solution for which $\ldots = Z_{n-1} = Z_n = Z_{n+1} = \ldots$. For instance when instead of the forces P forces are applied in the direction of y along the upper and lower skin the column matrix K is the same as given by (8.81).

If the box beam would be finite Z_n for cells near the end would not be constant and the manner in which M_x,0,0 would be applied to the end section would affect the participation factors for the cells near the ends.

In practice, however, the solution for the parts at some distance from the ends, remain those of (8.84) (Principle of De St.Venant). The same consideration holds for sections 8.4.3 and 8.4.4.

8.4.2.2 Stresses, strains and displacements.

The normal force in the upper front spar boom at $y = -c$ in a point immediately to the right of a rib plane is (see fig.7.9...7.18)

$$N = 10^{-3}EA(-6Y_{1.n} + 6Y_{3.n} + 3Y_{6.n}) = 0.038128\ M_x \quad . \tag{8.85}$$

The normal force in the upper front spar boom at $y = -c$ in a point immediately to the left of a rib plane is

$$N = 10^{-3}EA(+6Y_{1.n} - 6Y_{3.n} + 3Y_{6.n}) = 0.050908\ M_x \quad . \tag{8.86}$$

There is a fairly large difference between these two results

$$\Delta N = -0.012780\ M_x \tag{8.87}$$

on a mean value

$$N = 0.044518\ M_x \quad . \tag{8.88}$$

The mean value is in fairly good agreement with the numerical value of (8.15) (according to the minimum principle for the stresses), which is

$$N = 0.047122\ M_x \quad . \tag{8.89}$$

The shear stressflow in the front spar web at $y = -c$ is

$$t = 10^{-3}Gh(4Y_{1.n} - 4Y_{3.n} + 10Y_{5.n} - 10Y_{7.n} + 8Y_{8.n}) \tag{8.90}$$

or

$$t = -0.006944\ M_x = -\frac{M_x}{144} \tag{8.91}$$

which complies with (8.14). This value can also be found from an elementary reasoning, which proves the correctness of the calculations.

The strain of a rib flange is

$$\varepsilon_y = 4 \times 10^{-3}\ Y_{10.n} = 0.023470 \times 10^{-6} M_x \quad . \tag{8.92}$$

The derivatives to x of the mean rotation components are

$$\frac{d\varphi_x}{dx} = 1.66667 \times 10^{-3}\ Y_{7.n} = 0.013843 \times 10^{-6}\ M_x \tag{8.93}$$

$$\frac{d\varphi_y}{dx} = -1.41421 \times 10^{-3}\ Y_{6.n} = -0.004955 \times 10^{-6}\ M_x \quad . \tag{8.94}$$

The average of the second derivative to x of the vertical deflection in y = 0 is

$$\frac{d^2w}{dx^2} = \frac{\partial^2 u_z(y=0)}{dx^2} = 10^{-3} Y_{6.n} = 0.003503 \times 10^{-6} M_x \ . \qquad (8.95)$$

8.4.3 Constant moment $(0, M_y, 0)$.

8.4.3.1 Solution of the unknowns.

A constant moment $\vec{M}(0, M_y, 0)$ is applied to the structure. The moment to the right end is supposed to be applied according to fig.8.15.

The matrix K of (8.77) now becomes

$$K = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.0169705\ M_y \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\| \ . \qquad (8.96)$$

The solution for Z_n is

$$Z_n = \left\| \begin{matrix} Y_{1.n} \\ Y_{2.n} \\ Y_{3.n} \\ Y_{4.n} \\ Y_{5.n} \\ Y_{6.n} \\ Y_{7.n} \\ Y_{8.n} \\ Y_{9.n} \\ Y_{10.n} \end{matrix} \right\| = 10^{-6} M_y \left\| \begin{matrix} 0 \\ 0 \\ +0.045083 \\ +0.082046 \\ 0 \\ -5.20861 \\ -2.97271 \\ -3.69337 \\ 0 \\ -1.05156 \end{matrix} \right\| \ . \qquad (8.97)$$

8.4.3.2 Stresses, strains and displacements.

Since (8.97) and (8.84) have zero elements at the corresponding places of the matrix the formulas of section 8.4.2.2 are valid as well in this case. The normal force in the upper front spar boom at y= -c in a point immediately to the right of a rib plane is

$$N = -0.065040\ M_y \quad . \tag{8.98}$$

The normal force in the upper front spar boom at y = -c in a point immediately to the left of a rib plane is

$$N = -0.067332\ M_y \quad . \tag{8.99}$$

The difference is only

$$\Delta N = 0.002292\ M_y \tag{8.100}$$

and the mean value is

$$N = -0.066186\ M_y \tag{8.101}$$

which corresponds very well with the numerical value of (8.19)

$$N = -0.066640\ M_y \ . \tag{8.102}$$

The shear stress in the spar webs is zero.

The strain of a rib flange is

$$\varepsilon_y = -0.0042063 \times 10^{-6}\ M_y \ . \tag{8.103}$$

The derivatives to x of the mean rotation components are

$$\frac{d\varphi_x}{dx} = -0.004955 \times 10^{-6} M_y \tag{8.104}$$

$$\frac{d\varphi_y}{dx} = 0.007366 \times 10^{-6} M_y \tag{8.105}$$

and the second derivative to x of the vertical deflection in y = 0

$$\frac{d^2 w}{dx^2} = -0.005209 \times 10^{-6} M_y \ . \tag{8.106}$$

The required reciprocity between (8.94) and (8.104) checks completely.

8.4.4 Constant moment $(M_x, M_y, 0)$

8.4.4.1 Flexibilities.

The differential equations of the swept box are for constant moment

$$\frac{d\varphi_x}{dx} = Q_{11}M_x + Q_{12}M_y \tag{8.107}$$

$$\frac{d\varphi_y}{dx} = Q_{21}M_x + Q_{22}M_y \tag{8.108}$$

$$\frac{d^2w}{dx^2} = -(Q_{21}M_x + Q_{22}M_y)\sin\theta \tag{8.109}$$

and from (8.93), (8.94) and (8.104), (8.105)

$$\begin{aligned} Q_{11} &= 0.013843\times 10^{-6} \\ Q_{12} = Q_{21} &= -0.004955\times 10^{-6} \\ Q_{22} &= 0.007366\times 10^{-6} \end{aligned} \tag{8.110}$$

The values of (8.110) are indicated by Q_{ij}". The values Q_{ij} of (8.77) as derived by using 5 types of supplementary stress systems are indicated by Q_{ij}'. The principles of section 6.3.1 are now applied to derive bounds for Q_{ij} (read Q_{ij} in stead of P_{ij}).

From (6.48) and (6.55) $0.013843\times 10^{-6} < Q_{11} < 0.014329\times 10^{-6}$,

from (6.62) and (6.59) $-0.005034\times 10^{-6} < Q_{12} < -0.004961\times 10^{-6}$, (8.111)

from (6.49) and (6.56) $0.007366\times 10^{-6} < Q_{22} < 0.007379\times 10^{-6}$.

These limits are fairly close to each other.

8.4.4.2 An exact solution.

It appears from (8.92) and (8.103) as could be expected with the systems of strain used that the strain of the rib flange is constant. This means that the normal force in the rib flange is constant, which is inconsistent with the fact that no external load is applied to the ends of the ribs. If, however, ε_y of the flange would be zero, this violation of the conditions of equilibrium vanishes. The ribs not being affected in this case can be removed and the solution then obtained applies to the hollow box beam. The systems of strain which have been used contain enough degrees of freedom to describe the exact solution for the hollow box beam; therefore they are sufficient to describe the exact solution for the box with oblique ribs in the case that $\varepsilon_y = 0$.

If $\varepsilon_y=0$ it follows from (8.92) and (8.103) that

$$0.023470\times10^{-6}\,\xi\,M_y - 0.0042063\times10^{-6}\,M_y = 0$$

$$\xi = 0.1792\,. \qquad (8.112)$$

This result corresponds to that of section 8.2.6.3 for the given dimensions which was of course to be expected.

Some other numerical results confirm that the solution for the load $\xi\,M_y, M_y, 0$ is exact.

The discontinuities ΔN of (8.87) and (8.100) disappear, because

$$-0.012780\,M_y\times0.1792 + 0.002292\,M_y = 0\,.$$

The strain components are constant, so the supplementary strain systems of types 1, 2, 3 and 4 must have vanished. This has already been obtained with respect to type 1 and 2 for arbitrary moment M_x, M_y, 0. Further, indeed

$$0.1792\,(Y_{3.n}\ \text{ of (8.84)}) + Y_{3.n}\ \text{ of (8.97)} = 0$$

$$0.1792\,(Y_{4.n}\ \text{ of (8.84)}) + Y_{4.n}\ \text{ of (8.97)} = 0\,.$$

The formulae (8.107) and (8.108), with $\xi = 0.1792$ and Q_{ij} from (8.110) or from (8.77) yield the same results; from (8.110)

$$\frac{10^6}{M_y}\,\frac{d\varphi_x}{dx} = 0.013843\times0.1792 - 0.004955 = -0.002474$$

$$\frac{10^6}{M_y}\,\frac{d\varphi_y}{dx} = -0.004955\times0.1792 + 0.007366 = 0.006478\,,$$

and from (8.77)

$$\frac{10^6}{M_y}\,\frac{d\varphi_x}{dx} = 0.014329\times0.1792 - 0.005040 = -0.002471$$

$$\frac{10^6}{M_y}\,\frac{d\varphi_y}{dx} = -0.005040\times0.1792 + 0.007379 = 0.006477\,.$$

8.4.5 Constant shear force $(0,0,K_z)$

8.4.5.1 Solution of the unknowns.

Like in section 8.2.5.1 the box beam carries a constant shear force K with components $K_x = 0$, $K_y = 0$, K_z, acting upon the cross section which forms the right side of a part of the box beam. The shear force K_z is accompanied by a linearly varying moment $(0,M_y,0)$

$$M_y = x\, K_z \sin\theta + \text{constant} \quad . \qquad (8.113)$$

This moment is taken zero in the point y = 0 of the rib 0 which is now considered as the left end of the beam. The shear force and the moment are supposed to be applied at the right end as given in fig. (8.16) and fig. (8.15) respectively.

Then in the equations (8.82)

$$K = nR + S \qquad (8.114)$$

where

$$R = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.144\ K_z \\ 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix} \quad \text{and } S = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.12\ K_z \\ 0.072\ K_z \\ 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix} \quad . \qquad (8.115)$$

Suppose the general solution for Z_n to be

$$Z_n = nU + L \quad . \qquad (8.116)$$

The equation (8.82) becomes after substitution of (8.114) and (8.116)

$$(n-1)TU + nVU + (n+1)T'U + (T+V+T')L = nR + S$$

or with (8.83)

$$nHU - (T-T')U + HL = nR + S \quad . \qquad (8.117)$$

If (8.117) shall hold for any value of n, the two unknown matrices U and Y must satisfy

$$HU = R \qquad (8.118)$$

$$-(T-T')U+HL = S \quad ,$$

R and S are given by (8.115), the matrix H is given in table 8.6 and the matrix (T-T') in table 8.7.

The solution for U from HU = R is

$$-U = 10^{-6}K_z \begin{Vmatrix} 0 \\ 0 \\ -0.382 \\ -0.696 \\ 0 \\ +44.197 \\ +25.224 \\ +31.339 \\ 0 \\ +8.923 \end{Vmatrix} . \qquad (8.119)$$

The matrix (T-T')U is

$$(T-T')U = K_z \begin{Vmatrix} +0.0098678 \\ -0.0107351 \\ +0.0252750 \\ -0.0162352 \\ 0 \\ +0.0251910 \\ +0.0111682 \\ -0.050550 \\ 0 \\ +0.0210138 \end{Vmatrix} \qquad (8.120)$$

and the solution for L from HL = S +(T-T')U becomes

$$L = 10^{-6}K_z \begin{Vmatrix} -11.032 \\ -14.673 \\ -0.191 \\ -0.348 \\ +44.873 \\ +22.098 \\ +12.512 \\ 0 \\ 0 \\ 0 \end{Vmatrix} . \qquad (8.121)$$

8.4.5.2 Stresses and displacements.

It follows already from elementary consideration that the shear stressflow in both spar webs is $K_z/4b = K_z/12$, which was also found in (8.48).

The present calculations must also lead to this result. The shear stressflow in the front spar web of the cell n, in which the supplementary stress systems with participation factors $Y_{1.n}\ldots Y_{7.n}$ occur (fig.8.13), is (see fig. 7.9...7.18)

$$t = 10^{-3}Gh(4Y_{1.n}-4Y_{3.n}+10Y_{5.n}-10Y_{7.n}+4Y_{8.n}+4Y_{8.(n-1)}) \tag{8.122}$$

where $Y_{1.n}\ldots Y_{8.n}$ follow from (8.116), (8.119) and (8.121), and $Y_{8.(n-1)} = -31.339(n-1)10^{-6}\,K_z$.

In the result n falls out and

$$t = 10^{-3}Gh\;404.602\times 10^{-6}K_z \quad . \tag{8.123}$$

With $Gh = 0.205963 \times 10^{-6}$

$$t = 0.08333\;K_z = \frac{K_z}{12} \tag{8.124}$$

which again confirms (see also (8.48)) the correctness of the calculations.

For the differential quotients $\frac{d\varphi_x}{dx}$ and $\frac{d\varphi_y}{dx}$, again use is made of (8.93) and (8.94)

$$\left.\begin{aligned} \frac{d\varphi_x}{dx} &= 1.66667\times 10^{-3}Y_{7.n} \\ \frac{d\varphi_y}{dx} &= -1.41421\times 10^{-3}Y_{6.n} \end{aligned}\right\} . \tag{8.125}$$

The second derivative of w becomes with (8.95)

$$\frac{d^2w}{dx^2} = 10^{-3}Y_{6.n} \tag{8.126}$$

After substitution of $Y_{7.n}$ and $Y_{6.n}$, together with

$$a(n-\tfrac{1}{2}) = x$$

$$K_z x \sin\theta = M_y$$

the equations (8.125) yield

$$\frac{d\varphi_x}{dx} = Q_{12}M_y \tag{8.127}$$

$$\frac{d\varphi_y}{dx} = Q_{22}M_y \tag{8.128}$$

$$\frac{d^2w}{dx^2} = -Q_{22}M_y\sin\theta \tag{8.129}$$

where Q_{12} and Q_{22} have again the values of (8.110).

The form of (8.127)...(8.129) is the same as that of (8.51), (8.52) and (8.54). By integration of (8.129) is obtained

$$\frac{dw}{dx} = -Q_{22}K_z \frac{x^2}{2} \sin^2\theta + \text{constant} \ . \qquad (8.130)$$

The constant is (if the left end at a rib of the box beam is not rotated)

$$10^{-3}(4Y_{1.n} + 10Y_{5.n}) = 0.404602 \times 10^{-6} K_z \qquad (8.131)$$

which is equal to the numerical value of the integration constant of (8.55).

8.5 Elastic energy ellipses for constant moments.

8.5.1 Box beam with oblique ribs.

The elastic energy, stored in one cell, in the case of a constant moment $M_x, M_y, 0$ is

$$E = \frac{1}{2} a(M_x \frac{d\varphi_x}{dx} + M_y \frac{d\varphi_y}{dx}) \qquad (8.132)$$

or

$$E = \frac{1}{2} a(Q_{11}M_x^2 + 2Q_{12}M_xM_y + Q_{22}M_y^2) \ . \qquad (8.133)$$

The ellipses in fig.8.17 give the end points of all moment vectors starting in 0 which cause a strain energy of 12 lbs inch in each cell.

The projections on the oblique axes M_x and M_y of a vector from 0 to a point on an ellipse therefore satisfy the equation

$$\frac{2\times 12}{a} = 2 = Q_{11}M_x^2 + 2Q_{12}M_xM_y + Q_{22}M_y^2 \ . \qquad (8.134)$$

The ellipse 1 is the result of the minimum principle for the stresses with 3 types of supplementary stress systems as derived from the numerical values given in (8.29). Since the use of the minimum principle for the stresses estimates the complementary energy too large, and in this case also the strain energy, the length of a vector from 0 to a point of a ellips is too small. The ellipse 1 is also the exact solution for the hollow box beam. The principal axes of this ellipse are respectively perpendicular and parallel to the direction of the beam (x-axis).

Ellipse 2 follows by means of the minimum principle for the stresses and 5 types of supplementary stress systems in which case the numerical values are given by (8.77). The over-estimation of the strain energy is less, and ellipse 2 lies outside ellipse 1. The unknown ellipse for the exact solution must of course lie outside ellipse 2.

Ellipse 3 follows from Q_{11}, Q_{12}, Q_{22} according to (8.110) obtained with the minimum principle for the strains. This theorem estimates the potential energy toolarge and in this case the strain energy too small.

Thus the length of a vector from 0 to a point of this ellipse is too large. It is certain that the unknown exact ellipse lies between ellipses 2 and 3, *therefore a fair enclosure is reached.*

The three ellipses touch each other in the points where $M_x = 0.1792\, M_y$, which is the load for which the solutions are exact (see section 8.2.6.3., 8.4.4.2).

It is interesting to construct the ellipses following from other methods. It was already mentioned that Hemp continuises rib flanges and rib webs (these webs do not play a role for the infinite beam loaded by a constant moment). Exact values of Q_{11}, Q_{12}, Q_{22} of (8.26)...(8.28) will be the result of this schematization. However, the coefficients A_{ij} refer now to an anisotropic plate field which comprises the continuized rib flanges. Of such a skin the matrices a_{ij} and A_{ij} become

$$a_{ij} = \begin{Vmatrix} 1.19036 & 0.67882 & -0.73846 \\ 0.67882 & 1.11128 & -0.73846 \\ -0.73846 & -0.73846 & 0.70493 \end{Vmatrix} \times 10^{6} \qquad (8.135)$$

$$A_{ij} = \begin{Vmatrix} 2.56291 & 0.71921 & 3.43823 \\ 0.71921 & 3.16308 & 4.06695 \\ 3.43823 & 4.06695 & 9.28074 \end{Vmatrix} \times 10^{-6} \qquad (8.136)$$

and the coefficients Q_{11}, Q_{12}, Q_{22} are

$$\begin{aligned} Q_{11} &= 0.013763 \times 10^{-6} \\ Q_{12} = Q_{21} &= -0.004938 \times 10^{-6} \\ Q_{22} &= 0.007362 \times 10^{-6} \end{aligned} \qquad (8.137)$$

The values differ only slightly from those of (8.110).

If in the calculations based on the minimum principle for the strains, the systems of type 3 and 4 (fig.7.11 and 7.12) would have been omitted, the strains ε_y would have been constant in the skin, and therefore also in the rib flanges. If the cross section of the rib flanges would have been for example only half the actual value, but the rib distance also half the actual value, the solution according to the minimum principle of the strains would have been quite the same. It is thus clear that at omission of systems 3 and 4 only a mean stiffness of the rib flanges per cell is taken into account and the solution that would be obtained is *the exact solution if the rib flanges are actually continuized.*

It can thus be concluded that the ellipse for Hemp's solution (ellipse 4) is the ellipse that would have been obtained if systems 3 and 4 would have been omitted when applying the minimum principle for the strains; so the ellipse for Hemp's solution must lie outside ellipse 3.

Wittrick, Thómpson and Flügge (ref.15..21) assume continuously distributed ribs, infinitely stiff in their plane. In this case the matrix a_{ij} of the skin becomes (compare (8.135) and (8.136))

$$a_{ij} = \left\| \begin{matrix} 1.19036 & 0.67882 & 0.73846 \\ 0.67882 & \Gamma & -0.73846 \\ -0.73846 & -0.73846 & 0.70493 \end{matrix} \right\| \times 10^6 \qquad (8.138)$$

with $\Gamma \rightarrow \infty$

The inverse matrix is

$$A_{ij} = \left\| \begin{matrix} 2.39938 & 0 & 2.51350 \\ 0 & 0 & 0 \\ 2.51350 & 0 & 4.05164 \end{matrix} \right\| \times 10^{-6} \qquad (8.139)$$

and with (8.25)...(8.27) follows

$$\begin{aligned} Q_{11} &= 0.007093 \times 10^{-6} \\ Q_{12} &= Q_{21} = -0.003744 \times 10^{-6} \\ Q_{22} &= 0.007148 \times 10^{-6} \end{aligned} \qquad (8.140)$$

The corresponding ellipse 5 in fig.8.17 lies considerably outside the ellipses 2 and 3, which form bounds for the region, in which the exact ellipse must lie.

From the analytical geometry of ellipses follows that in a point of the ellipse

$$Q_{11}M_x^2 + 2Q_{12}M_xM_y + Q_{22}M_y^2 = C$$

the x and y components of a vector normal to the ellipse have the proportion dC/dM_x to dC/dM_y. From (8.133), (8.31) and (8.32) follows

$$\frac{d\varphi_x}{dx} = \frac{1}{2}\,\frac{dC}{dM_x}\ , \quad \frac{d\varphi_y}{dx} = \frac{1}{2}\,\frac{dC}{dM_y}\ .$$

The vector $\frac{d\vec{\varphi}}{dx}$ belonging to a certain moment $\vec{M}$ is perpendicular to the tangent of the ellipse at the end point of the moment vector.

From (8.132) follows

$$\vec{M} \cdot \frac{d\vec{\varphi}}{dx} = 2 \text{ lbs} \quad . \tag{8.141}$$

8.5.2 Box beam with normal ribs.

In fig.8.18 similar ellipses 1, 4 and 5 are drawn as in fig.8.17 for the case that the ribs are perpendicular to the spars. The rib spacing was assumed to be 12 inches and the cross-sectional area of the rib flange $A_R = 0.0765$ inch^2 like in the swept case. Again the stiffness of the rib webs, which are only able to carry shear does not effect the result.

Ellipse 1 is the solution for the hollow box beam where the presence of rib flanges is ignored; therefore this ellipse is identical to the ellipse 1 of fig.8.17. Ellipse 4 is the solution according to Hemp where ribs are continuously distributed.

Both ellipses lie very close together and the difference cannot be shown in the figure.

Ellipse 5 is the solution for continuously distributed ribs which are infinitely stiff in their plane. Such ribs prevent lateral contraction of the skin.

Calculations are not given, they follow from the elementary bending and torsion theory. Calculations for ellipses similar to 2 and 3 of fig. 8.17 were not made at all, but they must again lie between the ellipses 1 and 4. It appears that in fig.8.18 all ellipses lie much more closely together than in fig.8.17. By comparison of the two figures it appears that the assumption of continuized (infinitely) stiff ribs is much more unreliable for the oblique box than for the straight box.

8.6 Attempt to an elementary theory for establishing the deflections of the oblique box beam.

The elementary theory for straight beams is used in practice also when the assumptions upon which its validity is based are not satisfied, e.g. variability of the load along the beam, non-prismatic structure, restraint against warping of the end sections. Under such conditions the elementary beam theory usually yields fair approximations for the deformation of the beam. The relations between deflections and loads of this elementary theory are (in orthogonal coordinates)

$$\frac{d\bar{\varphi}_x}{dx} = \bar{Q}_{11}\bar{M}_x$$

$$\frac{d\bar{\varphi}_y}{dy} = \bar{Q}_{22}\bar{M}_y \qquad (8.142)$$

$$\frac{d\bar{w}}{dx} + \bar{\varphi}_y = \bar{K}_z/S_d$$

where $1/\bar{Q}_{11}$ is the torsional stiffness, $1/\bar{Q}_{22}$ the bending stiffness and S_d the shear stiffness. The sections $\bar{x}$ = constant have rotations $\vec{\varphi}$ with components $\bar{\varphi}_x$ and $\bar{\varphi}_y$.

The object of this paragraph is to establish for swept box beams a method analogeous to the elementary theory for straight beams, which likewise can be applied in order to assess approximately the deformation of swept box beams. The relations (8.142) for the straight beam are to be replaced by those established in the paragraphs 8.3 and 8.4 for the oblique coordinate system x,y

$$\frac{d\varphi_x}{dx} = Q_{11}M_x + Q_{12}M_y \qquad (8.143)$$

$$\frac{d\varphi_y}{dx} = Q_{21}M_x + Q_{22}M_y \qquad (8.144)$$

$$\frac{dw}{dx} + \varphi_y \sin\theta = K_z/S_d \quad . \qquad (8.145)$$

In order to check the reliability of this approach the deflections following from these relations will be determined for the clamped box beam of finite length which was analysed in section 7. At a clamped section the boundary conditions are, as with the straight box beam

$$\varphi_x = 0 \qquad \varphi_y = 0 \; .$$

A vertical unit load is placed at $x = p$, $y = q$. The vertical displacement in a point x,y is to be determined.

In the sections x = constant for $x < p$

$$M_x = q \sin\theta$$
$$M_y = -(p-x)\sin\theta \quad . \qquad (8.146)$$

Formula (8.143) gives, after substitution of (8.146) and integration ($\varphi_x(0) = 0$)

$$\varphi_x = \frac{1}{2} x^2 Q_{12} \sin\theta + x(-Q_{12}p + Q_{11}q) \sin\theta \quad . \tag{8.147}$$

Formula (8.144) gives after substitution of (8.146) and integration ($\varphi_y(0) = 0$)

$$\varphi_y = \frac{1}{2} x^2 Q_{22} \sin\theta + x(-Q_{22}p + Q_{21}q)\sin\theta \quad . \tag{8.148}$$

Formula (8.145) gives

$$\frac{dw}{dx} = -\frac{1}{2} x^2 Q_{22}\sin^2\theta - x(-Q_{22}p + Q_{21}q)\sin^2\theta + \frac{1}{4b\ Gh_s} \quad . \tag{8.149}$$

Since in $x = 0$, $w = 0$ integration of (8.149) yields

$$w = \sin^2\theta \left[-\frac{x^3}{6} Q_{22} + \frac{x^2}{2} (Q_{22}p - Q_{21}q) \right] + \frac{x}{4b\ Gh_s} \quad , \tag{8.150}$$

which is the vertical deflection of the line $y = 0$. For points $y \neq 0$ the vertical deflection (8.150) has to be augmented by the amount

$$\varphi_x y \sin\theta \quad .$$

So the total vertical deflection w of a point $x,y (x<p)$ is

$$w = \sin^2\theta \left\{ -\frac{x^3}{6} Q_{22} + \frac{x^2}{2} (Q_{22}\, p - Q_{21} q) \right\} + \frac{x}{4b\ Gh_s}$$
$$+ y \sin^2\theta \left\{ \frac{1}{2} x^2 Q_{12} + x(-Q_{12}p + Q_{11}q) \right\} \quad . \tag{8.151}$$

For $x>p$, $M_x = 0$ and $M_y = 0$ and the part of the box beam for $x>p$ is considered to be unstressed.

The rotation components of the rib plane at $x = p$ and of the box beam outside of this plane are obtained by substituting $x = p$ into (8.147) and (8.148) :

$$(\varphi_x)_p = (pq\ Q_{11} - \frac{1}{2} p^2\ Q_{12}) \sin\theta \tag{8.152}$$

$$(\varphi_y)_p = (pq\ Q_{21} - \frac{1}{2} p^2\ Q_{22}) \sin\theta \quad . \tag{8.153}$$

The vertical deflection w of the rib plane at $x = p$ is found by substituting $x = p$ into (8.150):

$$w_p = \sin^2\theta\ (\frac{1}{3} p^3 Q_{22} - \frac{1}{2} p^2 q\ Q_{21}) + \frac{p}{4b\ Gh_s} \quad . \tag{8.154}$$

and so the vertical deflection w of the point x,y $(x>p)$ becomes

$$w = w_p + \left\{ (\varphi_x)_p y - (\varphi_y)_p (x-p) \right\} \sin\theta \quad . \tag{8.155}$$

or

$$w = \sin^2\theta\left(\frac{p^3}{3}Q_{22} - \frac{p^2q}{2}Q_{21}\right) + \frac{p}{4b\ Gh_s} +$$

$$+y(pq\ Q_{11} - \frac{1}{2}p^2Q_{12})\sin^2\theta - (x-p)(pq\ Q_{21} - \frac{1}{2}p^2Q_{22})\sin^2\theta \quad . \qquad (8.156)$$

By means of (8.151) and (8.156) for $x<p$ and $x>p$ respectively the deflections in the points 1,2...10 were calculated for vertical unit loads, placed in the points 1,2...10 (fig.7.1). For the values of Q_{ij} were taken those of (8.29).

Like in table 7.5, 7.10, 7.14 and 7.19, a matrix of influence coefficients C_{mn} is obtained (table 8.8), which is again symmetrical. Comparison of table 8.8 with the other tables shows that the success is only moderate.

In order to make a further comparison between displacements calculated with this elementary theory for the oblique beam and calculated by a more accurate theory, table 8.9 has been formed. This table deals with two loading cases which may be entitled as "shear force" and "torsion"[1] respectively, the loads being applied to the stations 2 and 3.

The flexibilities Q_{ij} used in the elementary theory are those based on the minimum principle for the stresses (with 4 types of supplementary stress systems). (These flexibilities are those of the hollow box beam, but in oblique coordinates). It is therefore reasonable to compare the displacements with those obtained by applying the minimum principle for the stresses and using 20 supplementary stress system.

Application of the formula's (8.143)...(8.156) for an infinite beam means in fact that of the infinitely long deformed beam, a certain (oblique) rib plane (which is distorted in and warped out of its original plane) is brought by a rigid body displacement in such a position that it coincides as good as possible with the (non-deformed) root plane (see for shear force loading fig.8.19).

Additional rotations of front and rear spar which are necessary to compensate for warping of the root rib (fig.8.19) are not included. Such rotations would increase the vertical deflections of the front spar (at least in the root region) and decrease the deflections of the rear spar.

Such differences in deflections are indeed observed if the deflect-

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1) In the sense of the theory of non swept beams. Thus not in the sense of the discussion at the end of section 8.2.2.2.

ions calculated by means of the elementary theory are compared with the deflections following from the minimum principle for the stresses (loading case A, table 8.9).

Similar considerations can be given for the torsional load. The elementary theory neglects the prevention of warping at the root plane. The additional rotations of front and rear spar which are necessary to keep the root section plane are opposite in sign to those of case A. In the root region these rotations will induce a decrease of the absolute values of the deflections. This explains the differences appearing in table 8.9 for loading case B.

The stresses following from the formulae of section 8.2 for the infinitely long beam deviate very much from those of section 7 more in particular at the concentrated load and, which is more important, near the clamped end at $x = 0$.

9 Stress analysis of a semi-infinite swept-back box for moment end loads.

9.1 Introduction.

The methods of section 7 have proved to predict adequately the stresses in the more concentrated structural elements, like spar booms and only shear-carrying spar and rib webs, as well as the stiffnesses to be used in aero-elastic problems. However, an important question remains with respect to the analysis of a swept beam of finite length, viz. whether it is necessary to apply the supplementary stress systems to all the cells. This would give rise to a great number of linear simultaneous equations. It is likely, however, that, in certain parts of a fairly long beam, some of or all the supplementary stress systems can be neglected or determined in a simplified way. For instance, it is certain that for cells far from the root the participation factors of the various supplementary stress systems can be found from the analysis of the infinite beam (chapter 8.2...8.4) where only a small number of simultaneous equations have to be solved (equal to the number of system types used), provided that no concentrated loads occur in this region. In the region near the root additional loads are required to keep the root section undeformed (compare the discussion at fig.8.19). The resultant of these loads is zero and it is well known that stresses stemming from such a load are confined to the root region itself.

So as to investigate this problem of root restrain, the box beam of section 8 is again analysed by means of the minimum principle for the stresses with the 5 supplementary stress systems given in section 8.3, but now one end is clamped whereas the external load, a moment $(M_x,0,0)$ and a moment $(0,M_y,0)$ resp., is applied in $x = \infty$.

Starting from the root, the participation factor of every type of supplementary stress system approaches the final value for the infinite beam as obtained in (8.74) and (8.76). This however will appear not to occur according to a simple law, but as

$$c_1(\lambda_1)^n + c_2(\lambda_2)^n + \ldots c_9(\lambda_9)^n$$

where n is the cell number, counted from the root. The coefficients $c_1 \ldots c_9$ are a set of constants depending on the load. The quantities $\lambda_1 \ldots \lambda_9$ are the roots (some of them complex) with modulus <1 of a reciprocal algebraic equation of the 18th degree.

9.2 Constant moment $(M_x=1,0,0)$.

Fig. 9.1 is a figure similar to fig. 8.1 and gives the numeration of the sets of supplementary stress systems which starts at the clamped root of the beam. The infinite set of simultaneous equations (8.7) now takes the form

$$B^{**}Z_0 + C^{*\prime}Z_1 + D'Z_2 = -E^* \qquad (9.1)$$

$$C^*Z_0 + B^*Z_1 + C'Z_2 + D'Z_3 = -E \qquad (9.2)$$

$$n \geq 3 \quad , \quad DZ_{n-2} + CZ_{n-1} + BZ_n + C'Z_{n+1} + D'Z_{n+2} = -E \ . \qquad (9.3)$$

The matrices Z_i are

$$Z_i = \left\| \begin{matrix} X_{1.i} \\ X_{2.i} \\ X_{3.i} \\ X_{4.i} \\ X_{5.i} \end{matrix} \right\| .$$

The matrices D,C,B,C' and D' are already given in table 8.2 and the matrix E by (8.72).

The equation (9.3) has the general form but the first two (9.1) and (9.2) are distorted by the structural irregularity caused by the presence of the clamped root. The matrices B^{**}, $C^{*\prime}$, C^* are given in table 9.1. As far as the first four types of supplementary stress systems are concerned, numerical values could be taken from tables 7.1, 7.7 and 7.12.

The matrix E^* is for the loading case (M_x=1,0,0)

$$E^* = \left\| \begin{matrix} \lambda_{0;1.0} \\ \lambda_{0;2.0} \\ \lambda_{0;3.0} \\ \lambda_{0;4.0} \\ \lambda_{0;5.0} \end{matrix} \right\| = \left\| \begin{matrix} -25.3672 \\ +10.4273 \\ 0 \\ 0 \\ -26.4088 \end{matrix} \right\| \times 10^{-6} . \qquad (9.4)$$

Except for the first two equations, the whole set of equations (9.1)...(9.3) is satisfied by the solution (8.74) but substitution of this solution into (9.1) and (9.2) yields $-E_1$, and $-E_2$ respctively instead of $-E^*$ and $-E$, where

$$-E_1 = \left\| \begin{matrix} +10.3193 \\ -13.0957 \\ +1.3765 \\ -1.7142 \\ 21.0759 \end{matrix} \right\| \times 10^{-6} \quad -E_2 = \left\| \begin{matrix} 0 \\ -18.1900 \\ 0 \\ 1.71469 \\ 26.4088 \end{matrix} \right\| \times 10^{-6} . \qquad (9.5)$$

Writing the solutions of the equations (9.1)...(9.3) in the form $Z_n = Z_n^* + Q_n$, where Z_n^* is the solution (8.74), the following set of equations for Q_n is obtained :

$$B^{**}Q_0 + C^{*\prime}Q_1 + D'Q_2 = E_1 - E^* = G_1 \qquad (9.6)$$

$$C^* Q_0 + B^*Q_1 + C'Q_2 + D'Q_3 = E_2 - E = G_2 \qquad (9.7)$$

$$D Q_0 + CQ_1 + BQ_2 + C'Q_3 + D'Q_4 = 0 \qquad (9.8)$$

$$DQ_1 + CQ_2 + BQ_3 + C'Q_4 + D'Q_5 = 0 \qquad (9.9)$$

$$DQ_{n-2} + CQ_{n-1} + BQ_n + C'Q_{n+1} + D'Q_{n+2} = 0 \qquad (9.10)$$

where

$$G_1 = \left\| \begin{matrix} +15.048 \\ +2.668 \\ -1.377 \\ +1.714 \\ +5.333 \end{matrix} \right\| \times 10^{-6} \quad G_2 = \left\| \begin{matrix} 0 \\ -2.665 \\ 0 \\ -1.715 \\ 0 \end{matrix} \right\| \times 10^{-6} . \qquad (9.11)$$

From equation (9.8) on $Q_n = 0$ satisfies of course the equations but in addition these homogeneous equations have a number of solutions, the characteristic functions

$$Q_n = R\lambda^n \tag{9.12}$$

where R is a column matrix and λ is a (real or complex) number, the characteristic value.

So as to obtain the characteristic values together with the corresponding column matrices R, (9.12) is substituted into (9.10)

$$(D + C\lambda + B\lambda^2 + C'\lambda^3 + D'\lambda^4)R = 0 \quad . \tag{9.13}$$

The existence of a non-zero column matrix R requires the following determinant to be zero :

$$\left| D + C\lambda + B\lambda^2 + C'\lambda^3 + D'\lambda^4 \right| = 0 \quad . \tag{9.14}$$

The determinant (9.14) is of the 5th order, so having 5! = 120 terms each consisting of a product of five 5-term polynomials. An equation in λ of the 20th degree can be expected

$$a_{20}\lambda^{20} + a_{19}\lambda^{19} + a_{18}\lambda^{18} + \ldots + a_2\lambda^2 + a_1\lambda + a_0 = 0 \quad . \tag{9.15}$$

This equation is reciprocal, i.e. $a_{20} = a_0$; $a_{19} = a_1$, etc. because if λ_1 is a root also $1/\lambda_1$ is a root. This latter conclusion is obtained by transposing the determinant (9.14) which leaves its value unaltered (note that B is a symmetrical matrix).

$$\left| D' + C'\lambda + B\lambda^2 + C\lambda^3 + D\lambda^4 \right| = 0$$

or (9.16)

$$\left| D + C\frac{1}{\lambda} + B\frac{1}{\lambda^2} + C'\frac{1}{\lambda^2} + D'\frac{1}{\lambda^4} \right| = 0 \quad .$$

Since $|D'| = |D| = 0$, the coefficients a_{20} and a_0 vanish. There remains an equation of the form

$$a_{19}\lambda^{18} + a_{18}\lambda^{17} + a_{17}\lambda^{16} + a_{16}\lambda^{15} + a_{15}\lambda^{14} + a_{14}\lambda^{13} + a_{13}\lambda^{12} +$$

$$a_{12}\lambda^{11} + a_{11}\lambda^{10} + a_{10}\lambda^9 +$$

$$a_{19} + a_{18}\lambda + a_{17}\lambda^2 + a_{16}\lambda^3 + a_{15}\lambda^4 + a_{14}\lambda^5 + a_{13}\lambda^6 + a_{12}\lambda^7 +$$

$$a_{11}\lambda^8 = 0 \quad . \tag{9.17}$$

The solution of the roots proceeds with the usual substitution for reciprocal equations $\xi = \lambda + 1/\lambda$[1]. Of course, half of the roots have moduli < 1. They are given in table 9.2, in the sequence of decreasing modulus, together with their characteristic solutions, the column matrices R. All columns R are normalized in such a way that the fifth element is 1. A state of stress S in the box beam, consisting of supplementary stress systems with participation factors according to the solution of (9.12) (or a proper combination of two of these solutions in order to avoid complex numbers for the stresses) claims to be compatible everywhere in the structure and should actually be so, if not a limited number of types of supplementary stress systems had been used. However, at the root of the beam they do not comply with the prescribed displacements and it is therefore necessary to combine the systems S so as to satisfy this condition. It is seen from (9.12) that the 9 roots of λ with modulus > 1 must have for long beams as a participation factor zero, since the stresses following from these solutions would increase exponentially with n.

It does not seem possible to attack a particular mechanical meaning to the column matrices R and the solution (9.12) following from them. This means that it seems impossible to predict which of the stress systems (9.12) will prevail when a clamped beam is loaded by a moment.

In order to obtain the required combination of the remaining 9 states of stress S, the 9 solutions (9.12) are substituted into (9.6) and (9.7)

$$B^{**}R + C^{*\prime}R\lambda + D'R\lambda^2 = G_1^* \qquad (9.18)$$

$$C^*R + B^*R\lambda + C'R\lambda^2 + D'R\lambda^3 = G_2^* \ . \qquad (9.19)$$

The 9 resulting column matrices G_1^* and G_2^* forming together the matrices V_1 and V_2 respectively are given in tables 9.3 and 9.4.

The multiplication factors $w_1 \ldots w_9$ of the 9 states of stress $S_1 \ldots S_9$ follow from the equations (9.6) and (9.7) :

$$\left\| \begin{matrix} V_1 \\ V_2 \end{matrix} \right\| \times \left\| \begin{matrix} w_1 \\ w_2 \\ \vdots \\ w_9 \end{matrix} \right\| = \left\| \begin{matrix} G_1 \\ G_2 \end{matrix} \right\| \ . \qquad (9.20)$$

1) A substitution $\xi = \lambda + 1/\lambda$ already in (9.14) is of no use, because $D \neq D'$ and $C \neq C'$.

The solution is

$$\left\| \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{matrix} \right\| = \left\| \begin{matrix} -612.842 - 361.438\,i \\ -612.842 + 361.438\,i \\ +524.700 + 103.255\,i \\ +524.700 - 103.255\,i \\ +359.369 \\ +\ 49.9688 \\ -\ 69.4195 \\ +\ \ 4.3882 \\ -\ \ 3.7185 \end{matrix} \right\| 10^{-6} = \text{matrix } W \ . \qquad (9.21)$$

Then the solution Q_n, satisfying all the equations (9.6), (9.7), (9.8)... is (R_i and λ_i from table 9.2).

$$Q_n = w_1 R_1 \lambda_1{}^n + w_2 R_2 \lambda_2{}^n + \ldots + w_9 R_9 \lambda_9{}^n \ . \qquad (9.22)$$

Q_n is a column matrix

$$Q_n = \left\| \begin{matrix} q_{1.n} \\ q_{2.n} \\ q_{3.n} \\ q_{4.n} \\ q_{5.n} \end{matrix} \right\| \qquad (9.23)$$

where e.g. $q_{3.n}$ is the participation factor of the supplementary stress *system type* 3 *(fig.*7.7) *in the n and* $(n+1)^{th}$ cell from the root (see fig.9.1), as far as root-perturbation stresses are concerned. The total participation factors

$$X_1 = \left\| \begin{matrix} x_{1.n} \\ x_{2.n} \\ x_{3.n} \\ x_{4.n} \\ x_{5.n} \end{matrix} \right\| \qquad (9.24)$$

are obtained by adding to the factors (9.23) the values (8.74) for the infinite beam under the same loading ($M_x = 1$, $M_y = 0$, $M_z = 0$).

Numerical values for (9.24) are given in table 9.5. Some stresses, following from the total participation factors (9.24) and the basic stress system fig. 8.5, have been plotted in fig. 9.2.

It is remarkable that stresses stemming from systems $Y_n = R\lambda^n$, which *rapidly tend to zero* (*the systems for the small values of* λ) have generally small participation factors (column matrix (9.21)), i.e. these stresses are already small in the first cell from the root.

From table 9.5 it appears that from the 5th cell of the root onward the stresses are almost those of the infinite beam. The stresses in the spars are already from the 4th cell on practically those of the infinite beam (fig.9.2).

The fact that in the root region the stresses in the front spar booms decrease and in the rear spar booms increase is confirmed by simple mechanical considerations like those given when discussing fig.8.19.

9.3 Constant moment $(0, M_y = 1, 0)$

This section follows quite the same lines as section 9.2.

$$E^* = \begin{Vmatrix} 0 \\ +11.3322 \\ 0 \\ 0 \\ -7.5548 \end{Vmatrix} \times 10^{-6} \quad . \tag{9.25}$$

The matrices E_1, E_2, G_1, G_2 and W are respectively

$$-E_1 = \begin{Vmatrix} +8.7131 \\ -10.8562 \\ -0.2466 \\ +0.3071 \\ +8.5123 \end{Vmatrix} \times 10^{-6} \qquad -E_2 = \begin{Vmatrix} 0 \\ -23.1458 \\ 0 \\ -0.3072 \\ +7.5598 \end{Vmatrix} \times 10^{-6} \tag{9.26}$$

$$G_1 = \begin{Vmatrix} -8.713 \\ -0.476 \\ +0.247 \\ -0.307 \\ -0.957 \end{Vmatrix} \times 10^{-6} \qquad G_2 = \begin{Vmatrix} 0 \\ +0.481 \\ 0 \\ +0.307 \\ 0 \end{Vmatrix} \times 10^{-6} \tag{9.27}$$

$$W = \begin{Vmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{Vmatrix} \begin{Vmatrix} 306.761 + 277.497\,i \\ 306.761 - 277.497\,i \\ -137.081 - 110.253\,i \\ -137.081 + 110.253\,i \\ +62.380 \\ -3.2666 \\ +36.1482 \\ -2.9200 \\ -0.5138 \end{Vmatrix} \times 10^{-6} \quad . \tag{9.28}$$

Final results are given in table 9.6 and fig. 9.3. With respect to the stresses near the root the same remarks as at the end of section 9.2 can be made.

There does not exist a linear combination of the two column matrices (9.21) and (9.28) such that the result is a zero column matrix. So there is no moment $(M_x, M_y, 0)$ which would not cause perturbation stresses at the root. Also the moment $(M_x, M_y = \xi M_x, 0)$ defined in section 8.2.6.3 will cause perturbation stresses near the root, which can be calculated only approximately, while the solution for cells far from the root is exact.

10 Final considerations.

The present investigations have shown that the methods used are adequate to analyse swept box structures. Of course some further adaptation may be desired in practical cases, for example if the shape of the skin panels is only nearly a parallelogram.

It seems that the use of the minimum principle for the stresses is to be preferred to the minimum principle for the strains. For a certain required accuracy the minimum principle for the stresses generally requires less unknowns to be solved. However, the amount of numerical work depends not only on the number of unknowns. If the calculation of the coefficients of the equations is automatized the number of structural elements and the number of the possibilities for their state of stress, respectively state of strain, attributed to these elements is possibly the most important factor. Also the type of computer used may affect the comparison. It be remembered that in the present work both principles are used, so as to compare results and to enclose results for displacements between bounds. The latter procedure may also have its importance for practical application.

It may be questioned whether the solutions obtained by the present methods will tend to the exact solution if the number of unknowns is evermore increased. It was already mentioned that the exact solution for the investigated schematized structure (spar booms and rib flanges that have no bending stiffness) will have stress singularities at the intersections of rib flanges with spar booms (corners). That means that stresses or their first derivatives may become infinite. In the corner, the two shear stressflows t of one skin panel acting on the spar boom and on the rib flange respectively need not to be equal or they may equally tend to infinity. If among the adapted supplementary stress (or strain) systems none contains the type of stress singularity which occurs the results obtained for the stresses near the corner by means of increasing the number of stress (or strain) systems will converge very badly.

Stress singularities occurring in corners of skin panels may be studied along the lines of ref.43 and 44. The stress singularities disappear or become less severe if bending stiffness in the plane of the panel is attributed to the spar booms (ref.45).

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12 Summary in Dutch.

Over de berekening van vleugelconstructies met pijlstelling

Het kenmerk van de vliegtuigvleugel met pijlstelling is, dat de voorzijde een grote hoek maakt met de langsas van het vliegtuig; in vliegrichting gezien is de vleugeltip achter de vleugelwortel. Hierdoor staan ook de liggers niet loodrecht of ongeveer loodrecht op het symmetrievlak van de vleugel. De invloed hiervan op de spanningsverdeling is groot.

Wat de ribben betreft zijn er twee mogelijkheden, deze kunnen nagenoeg loodrecht op de liggers staan of deze kunnen evenwijdig zijn aan de lengteas van het vliegtuig. In het laatste geval bevinden zich parallellogramvormige huidplaten tussen de ribben en de liggers. Deze verhandeling gaat over vleugelconstructies met ribben evenwijdig aan de langsas van het vliegtuig.

Bij vleugels zonder pijlstelling bestaan verschillende berekeningsmethoden. In de eerste plaats is er de methode welke uitgaat van de theorie van de prismatische balk belast door een buigend moment, een wringend moment en een dwarskracht. Een specialisatie voor dunwandige cylinders belast door een wringend moment is bekend. De dunwandige cylinders kunnen hierbij uit meer dan één cel bestaan.[1] Van der Neut (lit.4) behandelt zulk een dunwandige cylinder waarin zich een continuum van in hun vlak oneindig stijve ribben bevindt en welke cylinder door een dwarskracht belast wordt. Deze ribben zijn geplaatst loodrecht op de lengterichting van de cylinder. Koiter (lit.6) behandelt eveneens dit geval terwijl het continuum van ribben ook afwezig kan zijn. Verder is het mogelijk rekening te houden met geringe tapsheid, zowel van huiddikte als van de vleugelafmetingen, geringe pijlstelling en belemmering van welving. Er zijn ook uitbreidingen voor kegelvormige schalen (lit.7).

In de tweede plaats bestaat er de methode waarbij de constructie wordt geacht te bestaan uit elementen die meestal één soort of een beperkt aantal soorten van belastingen kunnen opnemen. Zo wordt van ligger- en riblijfplaten meestal verondersteld dat deze langs hun randen slechts schuifspanningen opnemen. De liggergordingen en de ribflenzen worden dan geacht slechts normaalspanningen te kunnen opnemen. Ook van de ongeveer rechthoekige huidvelden wordt dan verondersteld dat zij langs hun randen slechts schuifspanningen opnemen. Het normaal-spanningsopnemend vermogen van ligger- en riblijfplaten zomede van huidvelden en eventuele langsverstijvers kan gevoegd worden bij de aangrenzende liggergordingen of ribflenzen. De aldus geïdealiseerde constructie is eindig-voudig statisch onbepaald. Iedere oplosmethode die deze statische onbepaaldheid volledig in rekening brengt levert dan hetzelfde resultaat op, terwijl dit resultaat voor de geïdealiseerdeconstructie exact is. Specialisaties gaven bij voorbeeld Van der Neut en Plantema (lit.13 en 14).

In de literatuur zijn beide methoden uitgebreid voor vleugels met pijlstelling. Er is werk waarbij wederom een continuum van ribben oneindig stijf in hun vlak in cylinders geplaatst wordt. Deze ribben staan dan niet meer loodrecht op de lengte-as van de cylinder. Specialisatie voor kegelschalen

1) Zulke cellen zijn gescheiden door wanden in langsrichting van de cylinder. Elders wordt in dit werk met cellen meestal bedoeld de ruimte tussen twee opeenvolgende ribben.

zijn ook bekend. In deze verhandeling wordt aangetoond dat dit uitgangspunt tot foutieve resultaten leidt.

Ook van de thans parallellogramvormige huidvelden tussen liggers en ribben is wel verondersteld dat zij langs hun randen slechts schuifspanningen opnemen (Levy, lit.23). Deze veronderstelling leidt eveneens tot foutieve resultaten.

In het thans ondernomen werk worden de spanning-rek relaties van de parallellogramvormige huidvelden niet vereenvoudigd. Slechts de langsverstijvers van de huid worden continu verdeeld geacht. De constructie blijft daarmede oneindigvoudig statisch onbepaald en de principes van de variatie-rekening zijn nodig om tot practisch bruikbare oplosmethoden te komen.

De twee gebruikte variatie-principes luiden als volgt:

"Van alle spannings-systemen die aan de evenwichtsvoorwaarden voldoen, is die de juiste, welke de complementaire energie minimaal maakt" en

"Van alle rek-systemen die aan de aansluitingsvoorwaarden voldoen is die de juiste welke de potentiële energie minimaal maakt."

Het gebruik van ieder dezer principes afzonderlijk leidt tot een oplossing. Het gebruik van deze principes is welbekend. De onderhavige constructie vraagt echter speciale vormen van toepassing welke in dit rapport behandeld zijn. Hierbij is gebruik gemaakt van scheefhoekige coördinaten zoals ze door Hemp (lit.22) bij de behandeling van scheve vleugelconstructies ingevoerd zijn. Voor zulk een scheefhoekig coördinatenstelsel zijn de spanningen en rekken zodanig gedefinieerd dat veel relaties, geldig bij orthogonale coördinaten, hun vorm behouden. Het equivalent van een normaalrek is thans evenwel niet alleen afhankelijk van het equivalent van twee normaalspanningen, maar ook van het equivalent van de schuifspanning.

Het vergelijken van de resultaten met behulp van de beide principes verkregen geeft een fraaie gelegenheid om de mate van nauwkeurigheid te beoordelen. Voor belangrijke uitkomsten kunnen zelfs onderste en bovenste grenzen worden aangegeven waartussen het exact antwoord van de nog altijd enigszins geidealiseerde constructie met zekerheid moet liggen.

Numeriek wordt behandeld een aan één zijde ingeklemde doosbalk met 45° pijlstelling met 5 cellen. Hierbij is onder een cel te verstaan de ruimte tussen 2 opeenvolgende ribben of tussen een rib en de inklemming. De belasting bestaat uit verticale krachten ter plaatse van de bevestigingen van de ribben aan de liggers (de punten 1 t/m 10, fig. 7.1).

Bij het toepassen van de methode waarbij de complementaire energie minimaal gemaakt wordt is een "basis spanningssysteem" aangenomen, hetwelk in evenwicht is met de uitwendige belasting, op welk systeem worden gesuperponeerd "aanvullende spanningssystemen", die geen uitwendige belasting vereisen en voldoen aan de evenwichtsvoorwaarden. Onbekende multiplicatoren van laatstgenoemde systemen zijn de onbekenden welke moeten worden opgelost door te eisen dat de complementaire energie minimaal is.

Voor de doosbalk werden achtereevolgens oplossingen verkregen waarbij gebruik werd gemaakt van 2, 3 en 4 types van "aanvullende spanningssystemen" per cel, waarbij dus achtereenvolgens 10, 15 en 20 onbekenden op te lossen waren.

Het toepassen van de methode waarbij de potentiële energie minimaal gemaakt wordt geschiedde op overeenkomstige wijze door invoering van een "basis reksysteem" en van "aanvullende reksystemen". Ten einde tot ongeveer dezelfde nauwkeurigheid te komen als bij de eerstgenoemde berekeningen werd het nodig geoordeeld 10 types van "aanvullende reksystemen" per cel te gebruiken; het vraagstuk leverde dus hier 50 op te lossen onbekenden op. De onderlinge overeenstemming tussen de diverse uitkomsten van de vier verkregen oplossingen was bevredigend.

Beide methoden werden ook toegepast op de naar beide zijden oneindig lange doosbalk (met overigens dezelfde afmetingen als van de 5-cellige doosbalk) voor de belastingsgevallen constant moment en constante dwarskracht. Hierbij werden 5 types van "aanvullende spanningssystemen" per cel gebruikt en weer 10 types "aanvullende reksystemen". Deze analyse voert tot relaties (in scheefhoekige coördinaten) tussen moment en specifieke hoekverdraaiing van de balk. Voor de coëfficiënten, voorkomende in deze relaties worden onderste en bovenste grenzen bepaald welke betrekkelijk dicht bij elkaar liggen. Bedoelde relaties zouden kunnen dienen tot een equivalent van de zogenaamde elementaire balkentheorie voor balken zonder pijlstelling. Dit komt neer op het geldig verklaren van bedoelde relaties in gevallen waarbij de dwarskracht, haar werklijn ten opzichte van de doorsnede (eventueel wringend moment) of de afmetingen in lengterichting veranderen, of wanneer de balkeinden niet vrij kunnen welven. Een poging om op basis van zulk een elementaire theorie de vijfcellige doosbalk te behandelen had slechts een matig succes, doordat de afwijkingen van die theorie in de buurt van de inklemming van de balk te ernstig zijn.

Om de uitgestrektheid van de stoorspanningen in de nabijheid van een inklemming te onderzoeken werd, uitsluitend met minimalisatie van de complementaire energie, een half-oneindige doosbalk (overigens weer met dezelfde afmetingen) onderzocht voor het belastingsgeval constant moment. De stoorspanningen dempen vanaf de inklemming snel uit (niet volgens een eenvoudige wet) en zijn van de vierde cel af verwaarloosbaar klein.

De gevolgde methoden worden geacht voor de practijk goede resultaten te leveren. Ze zijn evenwel moeilijk tot convergentie naar een exact resultaat te krijgen. Dit zou studie vragen van de bijzondere spanningstoestand in de hoeken van de parallellogramvormige platen. Dit heeft slechts zin wanneer de liggergordingen en ribflenzen minder sterk geidealiseerd zouden worden. Want juist in de hoeken krijgt de eigen buigstijfheid en afschuifstijfheid van zulke elementen betekenis.

Table 7.1 Values of $10^6 \lambda_{ij}$ of the 10 supplementary stress systems $\lambda_{ij} = \lambda_{ji}$

The λ_{ij} are the "coefficients of the unknowns".

	i=1	2	3	4	5	6	7	8	9	10
j=1	7721	-4952	494	0	0	-1502	-1502	0	0	0
2		16926	-5941	494	0	1502	0	-1502	0	0
3			17420	-5941	494	0	1502	0	-1502	0
4				17420	-5941	0	0	1502	0	-1502
5					17420	0	0	0	1502	0
6						6216	-3272	811	0	0
7							14866	-4895	811	0
8								15677	-4895	811
9									15677	-4895
10										15677

Table 7.2 Values of $-10^6 \lambda_{0i}$ of the basic stress systems and the 10 supplementary stress systems for the m loading cases $(-10^6 \lambda_{m0i})$.

The λ_{0i} are the "known terms" of the equations (6.6).

	m=1	2	3	4	5	6	7	8	9	10
i=1	-692.868	+692.868	-556.880	+556.880	-420.894	+420.894	-284.906	+284.906	-148.920	+148.920
2	-1087.895	+1087.895	-815.921	+815.921	-543.947	+543.947	-271.974	+271.974	+ 12.934	- 12.934
3	-815.921	+815.921	-543.947	+543.947	-271.974	+271.974	+ 12.934	- 12.934	0	0
4	-543.947	+543.947	-271.974	+271.974	+ 12.934	- 12.934	0	0	0	0
5	-271.974	+271.974	+ 12.934	- 12.934	0	0	0	0	0	0
6	+448.735	+448.735	+352.577	+352.577	+256.420	+256.420	+160.262	+160.262	+ 64.105	+ 64.105
7	+769.259	+769.259	+576.944	+576.944	+384.629	+384.629	+192.315	+192.315	+ 32.052	+ 32.052
8	+576.944	+576.944	+384.629	+384.629	+192.315	+192.315	+ 32.052	+ 32.052	0	0
9	+384.629	+384.629	+192.315	+192.315	+ 32.052	+ 32.052	0	0	0	0
10	+192.315	+192.315	+ 32.052	+ 32.052	0	0	0	0	0	0

Table 7.3 Values of $10^5 X_{mj}$. The values X_{mj} are the solutions X_j of the 10 unknowns for the m loading cases.

	m=1	2	3	4	5	6	7	8	9	10
j=1	−13714	+24338	−10900	+18793	−8055	+13257	−5135	+7835	−2190	+2959
2	−15202	+17027	−11416	+12468	−7603	+ 7985	−3846	+3824	− 714	+ 641
3	−13006	+11658	− 8890	+ 7540	−4838	+ 3688	−1395	+ 722	− 218	+ 33
4	− 9174	+ 7134	− 5145	+ 3500	−1671	+ 629	− 422	− 13	− 52	− 40
5	− 4627	+ 3081	− 1549	+ 561	− 434	− 27	− 88	− 58	− 7	− 18
6	+11814	+14258	+ 8894	+11336	+5969	+ 8368	+3125	+5274	+ 771	+2118
7	+ 9670	+12419	+ 6868	+ 9563	+4119	+ 6644	+1674	+3667	+ 167	+1076
8	+ 6561	+ 9696	+ 3974	+ 6886	+1652	+ 4051	+ 185	+1563	− 68	+ 305
9	+ 3548	+ 6759	+ 1390	+ 3998	+ 8	+ 1591	− 188	+ 385	− 56	+ 43
10	+ 1118	+ 3520	− 60	+ 1432	− 242	+ 347	− 109	+ 38	− 19	− 6

Table 7.4 Values of the symmetrical matrix $\int S_{m0} \cdot R_{n0} dv$ multiplied by 10^6.

	m=1	2	3	4	5	6	7	8	9	10
n=1	992.905	0	703.666	0	437.091	0	215.846	0	62.594	0
2		992.905	0	703.666	0	437.091	0	215.846	0	62.594
3			522.350	0	335.102	0	170.517	0	51.261	0
4				522.350	0	335.102	0	170.517	0	51.261
5					233.111	0	125.188	0	39.929	0
6						233.111	0	125.188	0	39.929
7							79.860	0	28.597	0
8								79.860	0	28.597
9									17.265	0
10										17.265

Table 7.5 Influence coefficients C_{mn} ($C_{mn} = C_{nm}$) of the swept box multiplied by 10^6 and expressed in inch/lb. Computed with the minimum principle for the stresses with 10 supplementary stress systems.

	m=1	2	3	4	5	6	7	8	9	10
n=1	382.86	247.95	278.30	165.64	181.25	93.35	97.48	39.10	33.47	7.783
2		248.52	183.54	173.45	120.70	106.11	64.30	51.87	20.92	15.43
3			217.44	127.07	146.30	74.17	80.83	32.22	28.60	6.852
4				135.74	86.60	87.76	47.71	45.28	16.04	14.56
5					110.58	53.88	64.17	24.99	23.77	5.867
6						68.62	31.53	38.69	11.22	13.73
7							46.66	16.63	18.91	4.609
8								31.23	6.618	12.87
9									13.55	2.622
10										11.24

Table 7.6 Values of stresses in lbs/inch^2 for the m loading cases.

Computed with the minimum principle for the stresses with 10 supplementary stress systems.

		m=1	2	3	4	5	6	7	8	9	10
Normal stresses in upper spar booms at station	1	0	0	0	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0	0
	3	+ 1.735	+0.352	-0.896	-0.269	-0.156	-0.162	-0.007	-0.051	+0.004	-0.008
	4	+ 2.282	+1.644	+0.946	-0.936	+0.360	-0.130	+0.098	+0.019	+0.012	+0.014
	5	+ 2.965	+1.400	+1.312	+0.400	-0.998	-0.295	-0.172	-0.170	-0.008	-0.042
	6	+ 3.966	+2.828	+2.476	+1.193	+0.990	-1.043	+0.330	-0.154	+0.055	+0.006
	7	+ 4.376	+2.857	+2.955	+1.589	+1.385	+0.490	-0.907	-0.228	-0.101	-0.109
	8	+ 4.978	+3.858	+3.617	+2.533	+2.183	+1.059	+0.751	-1.087	+0.158	-0.148
	9	+ 5.718	+4.905	+5.192	+3.395	+3.659	+1.956	+1.965	+0.732	-0.495	-0.071
	10	+ 4.976	+4.478	+3.902	+3.433	+2.790	+2.370	+1.585	+1.141	+0.355	-0.834
	11	+ 11.660	+8.482	+9.605	+6.412	+7.570	+4.367	+5.547	+2.443	+3.330	+0.869
	12	+ 3.188	+4.312	+2.744	+3.882	+2.282	+3.466	+1.740	+3.035	+0.978	+2.306
shear stresses in spar webs, between the stations	1-3	+ 2.361	+0.604	-0.304	+0.110	-0.085	-0.005	-0.017	-0.011	-0.001	-0.004
	2-4	+ 0.907	+2.664	+0.304	-0.110	+0.085	+0.005	+0.017	+0.011	+0.001	+0.004
	3-5	+ 2.376	+0.795	+2.563	+0.576	-0.243	+0.129	-0.065	+0.009	-0.009	-0.004
	4-6	+ 0.891	+2.473	+0.705	+2.692	+0.243	-0.129	+0.065	-0.009	+0.009	+0.004
	5-7	+ 2.517	+0.887	+2.534	+0.792	+2.647	+0.600	-0.191	+0.144	-0.033	+0.014
	6-8	+ 0.751	+2.381	+0.734	+2.476	+0.621	+2.668	+0.191	-0.144	+0.033	-0.014
	7-9	+ 2.837	+1.053	+2.773	+0.966	+2.726	+0.843	+2.787	+0.608	-0.097	+0.119
	8-10	+ 0.431	+2.215	+0.495	+2.301	+0.542	+2.425	+0.481	+2.660	+0.097	-0.119
	9-11	+ 3.560	+1.434	+3.369	+1.240	+3.179	+1.033	+3.016	+0.786	+2.979	+0.454
	10-12	- 0.292	+1.834	-0.101	+2.028	+0.089	+2.235	+0.252	+2.482	+0.289	+2.814
shear stresses in rib webs between the stations	1-2	+ 0.907	-0.604	+0.304	-0.110	+0.085	+0.005	+0.017	+0.011	+0.001	+0.004
	3-4	- 0.016	-0.191	+0.402	-0.465	+0.158	-0.134	+0.048	-0.020	+0.008	+0.001
	5-6	- 0.140	-0.093	+0.029	-0.216	+0.378	-0.471	+0.125	-0.135	+0.024	-0.019
	7-8	- 0.321	-0.156	-0.239	-0.174	-0.079	-0.243	+0.290	-0.464	+0.065	-0.105
	9-10	- 0.723	-0.381	-0.596	-0.274	-0.453	-0.191	-0.229	-0.178	+0.192	-0.335

Table 7.7 Additional values of $10^6 \lambda_{ij}$ of the 15 supplementary stress systems $\lambda_{ij} = \lambda_{ji}$. For the other values of of λ_{ij} see table 7.1.

	i=11	12	13	14	15
j=1	-1282	- 641	0	0	0
2	- 641	-2564	- 641	0	0
3	0	- 641	- 2564	- 641	0
4	0	0	- 641	- 2564	- 641
5	0	0	0	- 641	- 2564
6	1770	1770	0	0	0
7	-1770	0	1770	0	0
8	0	-1770	0	1770	0
9	0	0	- 1770	0	1770
10	0	0	0	- 1770	0
11	4994	482	80	0	0
12		10230	321	80	0
13			10310	321	80
14				10310	321
15					10310

Table 7.8 Additional values of $-10^6 \lambda_{0i}$ of the basic stress system and the 15 supplementary stress systems for the m loading cases ($-10^6 \lambda_{m0i}$). For the other values see table 7.2.

	m=1	2	3	4	5	6	7	8	9	10
i=11	+373.944	-373.944	+293.813	-293.813	+213.682	-213.682	+133.551	-133.551	+53.421	-53.421
12	+641.046	-641.046	+480.785	-480.785	+320.523	-320.523	+160.262	-160.262	+26.710	-26.710
13	+480.785	-480.785	+320.523	-320.523	+160.262	-160.262	+ 26.710	- 26.710	0	0
14	+320.523	-320.523	+160.262	-160.262	+ 26.710	- 26.710	0	0	0	0
15	+160.262	-160.262	+ 26.710	- 26.710	0	0	0	0	0	0

Table 7.9 Values of X_{mj} ($j = 1...15$).

The values X_{mj} are the solutions of the 15 unknowns for the m loading cases.

	m=1	2	3	4	5	6	7	8	9	10
j=1	−13343	+24301	−10682	+18566	−7971	+12862	−5130	+7388	−2193	+ 2703
2	−15230	+16617	−11555	+11912	−7814	+7354	−4042	+3289	− 790	+ 397
3	−13561	+10876	− 9445	+ 6789	−5330	+3073	−1712	+ 355	− 304	− 103
4	− 9852	+ 6247	− 5763	+ 2842	−2127	+ 237	− 632	− 192	− 91	− 93
5	− 5015	+ 2497	− 1892	+ 236	− 629	− 177	− 149	− 113	− 13	− 30
6	+11718	+14650	+ 8949	+11912	+6129	+9063	+3292	+5902	+ 838	+ 2423
7	+10232	+12906	+ 7352	+ 9940	+4495	+6854	+1901	+3682	+ 231	+ 1028
8	+ 6954	+10008	+ 4268	+ 6979	+1808	+3931	+ 194	+1370	− 95	+ 206
9	+ 3479	+ 6728	+ 1295	+ 3779	− 162	+1331	− 348	+ 214	− 103	− 25
10	+ 843	+ 3207	− 311	+ 1144	− 459	+ 166	− 215	− 51	− 39	− 35
11	+ 1611	+ 353	+ 1139	− 175	+ 707	− 690	+ 398	−1040	+ 218	− 784
12	− 82	− 656	− 270	− 913	− 396	−1104	− 401	− 973	− 259	− 340
13	− 1386	− 1511	− 1311	− 1485	−1150	−1206	− 824	− 527	− 180	− 169
14	− 1475	− 1789	− 1304	− 1340	− 983	− 613	− 316	− 251	− 25	− 65
15	− 846	− 1633	− 741	− 619	− 221	− 229	0	− 65	+ 11	− 6

Table 7.10 Influence coefficients C_{mn} ($C_{mn} = C_{nm}$) of the swept-box multiplied by 10^6 and expressed in inch/lb. Computed with the minimum principle for the stresses with 15 supplementary stress systems.

	m=1	2	3	4	5	6	7	8	9	10
n=1	377.73	242.34	273.83	161.73	178.02	91.28	95.83	38.64	33.06	7.948
2		240.93	178.42	168.09	117.07	102.79	62.50	50.34	20.50	15.04
3			213.45	123.35	143.38	72.02	79.29	31.53	28.22	6.856
4				131.34	83.66	84.61	46.06	43.57	15.62	13.99
5					108.27	51.98	62.87	24.23	23.43	5.763
6						65.95	30.30	36.96	10.86	13.06
7							45.81	16.06	18.66	4.503
8								29.84	6.429	12.24
9									13.44	2.601
10										10.90

Table 7.11 Values of stresses in $lbs/inch^2$ for the m loading cases.

Computed with the minimum principle for the stresses with 15 supplementary stress systems.

		m=1	2	3	4	5	6	7	8	9	10
normal stresses in upper spar booms at station	1	0	0	0	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0	0
	3	+ 1.916	+0.709	-0.735	-0.124	-0.104	-0.095	+0.002	-0.023	+0.005	-0.001
	4	+ 2.332	+1.551	+0.996	-0.839	+0.490	-0.045	+0.179	+0.066	+0.028	+0.030
	5	+ 3.108	+1.512	+1.442	+0.570	-0.852	-0.203	-0.119	-0.117	-0.002	-0.022
	6	+ 3.881	+2.741	+2.426	+1.207	+0.988	-0.916	+0.411	-0.064	+0.088	+0.043
	7	+ 4.366	+2.790	+2.960	+1.623	+1.430	+0.597	-0.811	-0.181	-0.077	-0.088
	8	+ 4.557	+3.662	+3.364	+2.421	+2.007	+1.053	+0.647	-0.972	+0.157	-0.085
	9	+ 6.495	+4.686	+5.000	+3.225	+3.515	+1.878	+1.894	+0.750	-0.476	-0.077
	10	+ 4.727	+4.286	+3.687	+3.285	+2.619	+2.270	+1.464	+1.111	+0.282	-0.788
	11	+11.356	+8.171	+9.312	+6.096	+7.305	+4.081	+5.340	+2.277	+3.224	+0.863
	12	+ 3.573	+4.292	+2.990	+3.713	+2.412	+3.167	+1.807	+2.673	+1.029	+2.056
Shear stresses in spar webs, between the stations	1-3	+ 2.285	+0.489	-0.371	+0.046	-0.123	-0.035	-0.029	-0.022	-0.002	-0.006
	2-4	+ 0.983	+2.779	+0.371	-0.046	+0.123	+0.035	+0.029	+0.022	+0.002	+0.006
	3-5	+ 2.320	+0.735	+2.509	+0.511	-0.294	+0.081	-0.095	-0.016	-0.015	-0.012
	4-6	+ 0.948	+2.533	+0.759	+2.757	+0.294	-0.081	+0.095	+0.016	+0.015	+0.012
	5-7	+ 2.541	+0.908	+2.546	+0.774	+2.640	+0.556	-0.212	+0.107	-0.042	-0.002
	6-8	+ 0.727	+2.360	+0.722	+2.494	+0.628	+2.712	+0.212	-0.107	+0.042	+0.002
	7-9	+ 2.941	+1.126	+2.854	+1.005	+2.781	+0.839	+2.811	+0.575	-0.095	+0.098
	8-10	+ 0.327	+2.142	+0.414	+2.263	+0.487	+2.429	+0.457	+2.693	+0.095	-0.098
	9-11	+ 3.638	+1.507	+3.439	+1.305	+3.237	+1.080	+3.055	+0.803	+2.993	+0.452
	10-12	- 0.370	+1.761	-0.171	+1.963	+0.031	+2.188	+0.213	+2.465	+0.275	+2.816
Shear stresses in rib webs between the stations	1-2	+ 0.983	-0.489	+0.371	-0.046	+0.123	+0.035	+0.029	+0.022	+0.002	+0.006
	3-4	- 0.035	-0.246	+0.388	-0.464	+0.170	-0.116	+0.066	-0.007	+0.013	+0.007
	5-6	- 0.221	-0.173	-0.037	-0.263	+0.334	-0.475	+0.117	-0.123	+0.026	-0.011
	7-8	- 0.400	-0.218	-0.308	-0.231	-0.141	-0.283	+0.245	-0.468	+0.054	-0.100
	9-10	- 0.697	-0.381	-0.585	-0.300	-0.457	-0.241	-0.244	-0.228	+0.180	-0.354

Table 7.12 Additional values of $10^6 \lambda_{ij}$ of the 20 supplementary stress systems $\lambda_{ij} = \lambda_{ji}$. For the other values of λ_{ij} see table 7.1 and 7.7.

	i=16	17	18	19	20
j=1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0
5	0	0	0	0	0
6	+2262	-2784	+ 522	0	0
7	-2784	+6089	- 3827	+ 522	0
8	+ 522	-3827	+ 6611	- 3827	+ 522
9	0	+ 522	- 3827	+ 6611	- 3827
10	0	0	+ 522	- 3827	+ 6611
11	0	-2123	0	0	0
12	+2123	0	- 2123	0	0
13	0	+2123	0	- 2123	0
14	0	0	+ 2123	0	- 2123
15	0	0	0	+ 2123	0
16	+4388	- 913	+ 348	0	0
17		+9819	- 1609	+ 348	0
18			+10167	- 1609	+ 348
19				+10167	-1609
20					+10167

Table 7.13 Values of $10^6 \ X_{mj}$ (j= 1...20)

The values X_{mj} are the solutions of the 20 unknowns for the m loading cases.

	m=1	2	3	4	5	6	7	8	9	10
j= 1	-132900	+243344	-106305	+185753	-79181	+128618	-50837	+74036	-21780	+27236
2	-152072	+166197	-115245	+119139	-77843	+73658	-40280	+33062	- 7881	+ 4075
3	-135396	+108710	- 94176	+ 67995	-53088	+30936	-16980	+ 3819	- 3005	- 791
4	- 98570	+ 62503	- 57648	+ 28434	-21234	+ 2495	- 6329	- 1724	- 965	- 839
5	- 50312	+ 24979	- 19065	+ 2325	- 6378	- 1724	- 1603	- 1114	- 162	- 307
6	+119161	+147969	+ 91152	+120187	+62817	+91240	+34727	+59816	+ 9700	+26220
7	+101654	+128705	+ 72965	+ 98912	+44680	+68211	+18567	+37077	+ 1336	+ 9345
8	+ 68702	+ 99280	+ 42206	+ 69365	+17424	+39604	+ 757	+12726	- 695	+1736
9	+ 35764	+ 66988	+ 13487	+ 38531	- 1818	+12765	- 2500	+ 2214	- 808	+ 35
10	+ 7998	+ 34392	- 3776	+ 10987	- 3706	+ 1690	- 1906	- 177	- 423	- 256
11	+ 16192	+ 3540	+ 11305	- 2174	+ 6860	- 7917	+ 3922	-11974	+ 2453	- 8821
12	+ 837	- 5474	- 1085	- 8488	- 2238	-10449	- 2405	- 8311	- 2199	- 2061
13	- 14004	- 15484	- 12810	- 15018	-11061	-11338	- 8555	- 4392	- 2239	- 1515
14	- 15316	- 19326	- 13286	- 13426	-10691	- 5811	- 4016	- 2733	- 385	- 801
15	- 8677	- 16168	- 7932	- 6008	- 2751	- 2599	+ 44	- 864	+ 182	- 76
16	- 5368	- 3709	- 5157	- 3333	- 5161	- 3656	- 5008	- 5608	- 2496	- 6823
17	+ 2379	+ 1571	+ 1622	+ 382	+ 999	- 1480	+ 1769	- 3031	+ 2529	+ 279
18	+ 4427	+ 3200	+ 3799	+ 836	+ 4348	- 836	+ 4511	+ 2175	+ 200	+ 1071
19	+ 126	- 274	+ 1914	- 1534	+ 2796	+ 1960	- 1044	+ 1062	- 551	- 107
20	+ 1405	- 6433	+ 2763	+ 723	- 1108	+ 798	- 899	- 182	- 168	- 130

In solving the unknowns X_{mj} the matrix $- \lambda_{m0i}$ is used which follows from tables 7.2 and 7.8 and by adding 5 rows of zeros.

Table 7.14 Influence coefficients C_{mn} ($C_{mn}=C_{nm}$) of the swept-box multiplied by 10^6 and expressed in inch/lb. Computed with the minimum principle for the stresses with 20 supplementary stress systems.

	m=1	2	3	4	5	6	7	8	9	10
n= 1	377.452	242.251	273.566	161.657	177.799	91.267	95.601	38.549	32.985	7.816
2		240.524	178.412	168.061	116.894	102.826	62.295	50.251	20.435	14.938
3			213.167	123.301	143.158	71.972	79.117	31.415	28.163	6.734
4				131.282	83.626	84.600	45.990	43.510	15.588	13.922
5					108.002	51.938	62.663	24.088	23.378	5.633
6						65.858	30.302	36.865	10.867	12.991
7							45.580	15.960	18.583	4.372
8								29.651	6.436	12.115
9									13.387	2.549
10										10.755

Table 7.15 Values of stresses in $lbs/inch^2$ for the m loading cases.

Computed with the minimum principle for the stresses with 20 supplementary stress systems.

		m=1	2	3	4	5	6	7	8	9	10
Normal stresses in upper spar booms at station	1	0	0	0	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0	0
	3	+ 1.932	+0.606	-0.697	-0.113	-0.127	-0.083	-0.017	-0.029	+0.002	-0.005
	4	+ 2.353	+1.458	+1.015	-0.811	+0.439	-0.060	+0.177	+0.043	+0.034	+0.026
	5	+ 3.084	+1.578	+1.427	+0.541	-0.812	-0.185	-0.131	-0.100	-0.010	-0.023
	6	+ 3.823	+2.702	+2.397	+1.175	+0.965	-0.889	+0.341	-0.086	+0.078	+0.020
	7	+ 4.419	+2.834	+2.986	+1.654	+1.454	+0.572	-0.742	-0.154	-0.071	-0.067
	8	+ 4.674	+3.687	+3.378	+2.426	+2.038	+1.054	+0.678	-0.917	+0.130	-0.079
	9	+ 6.478	+4.665	+4.986	+3.225	+3.483	+1.879	+1.867	+0.699	-0.448	-0.078
	10	+ 4.800	+4.337	+3.749	+3.327	+2.673	+2.298	+1.529	+1.140	+0.336	-0.708
	11	+11.302	+8.129	+9.275	+6.071	+7.280	+4.091	+5.293	+2.308	+3.168	+0.827
	12	+ 3.462	+4.211	+2.886	+3.647	+2.309	+3.106	+1.701	+2.575	+0.974	+1.925
Shear stresses in spar webs, between the stations	1-3	+ 2.281	+0.490	-0.374	+0.046	-0.125	-0.034	-0.031	-0.022	-0.003	-0.006
	2-4	+ 0.987	+2.778	+0.374	-0.046	+0.125	+0.034	+0.031	+0.022	+0.003	+0.006
	3-5	+ 2.322	+0.736	+2.511	+0.512	-0.291	+0.083	-0.093	-0.012	-0.016	-0.010
	4-6	+ 0.946	+2.532	+0.757	+2.756	+0.291	-0.083	+0.093	+0.012	+0.016	+0.010
	5-7	+ 2.546	+0.906	+2.552	+0.776	+2.643	+0.558	-0.209	+0.109	-0.040	+0.001
	6-8	+ 0.722	+2.362	+0.716	+2.492	+0.625	+2.710	+0.209	-0.109	+0.040	-0.001
	7-9	+ 2.941	+1.127	+2.855	+1.003	+2.783	+0.838	+2.811	+0.573	-0.096	+0.095
	8-10	+ 0.327	+2.141	+0.413	+2.265	+0.485	+2.430	+0.457	+2.695	+0.096	-0.095
	9-11	+ 3.644	+1.513	+3.443	+1.306	+3.242	+1.078	+3.061	+0.803	+2.995	+0.454
	10-12	- 0.376	+1.755	-0.175	+1.962	+0.026	+2.190	+0.207	+2.465	+0.273	+2.814
Shear stresses in rib webs between the stations	1-2	+ 0.987	-0.490	+0.374	-0.046	+0.125	+0.034	+0.031	+0.022	+0.003	+0.006
	3-4	- 0.040	-0.246	+0.383	-0.466	+0.166	-0.117	+0.061	-0.010	+0.013	+0.004
	5-6	- 0.224	-0.170	-0.040	-0.264	+0.333	-0.475	+0.116	-0.121	+0.024	-0.011
	7-8	- 0.395	-0.221	-0.303	-0.227	-0.139	-0.280	+0.248	-0.465	+0.056	-0.094
	9-10	- 0.703	-0.385	-0.588	-0.303	-0.459	-0.240	-0.250	-0.230	+0.177	-0.359

Table 7.16 Vertical displacements in the stations 1...10 belonging to the 50 supplementary strain systems.

Also: Values of $\int_k \vec{k}.\vec{u}_i \, df$ for the m loading cases.

The values $\int_k \vec{k}.\vec{u}_i \, df$ are the "known terms" of the equations (6.29).

Vertical displacements in inches at station: m (fig. 7.1) or: values of $\int_k \vec{k}.\vec{u}_i \, df$ for the m loading cases.

supplementary strain system nr. i	m=1	2	3	4	5	6	7	8	9	10
i=1	0	0	0	0	0	0	0	0	0	0
'	'	'	'	'	'	'	'	'	'	'
'	'	'	'	'	'	'	'	'	'	'
'	'	'	'	'	'	'	'	'	'	'
10	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0
'	'	'	'	'	'	'	'	'	'	'
'	'	'	'	'	'	'	'	'	'	'
20	0	0	0	0	0	0	0	0	0	0
21	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120
22	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0	0
23	0.120	0.120	0.120	0.120	0.120	0.120	0	0	0	0
24	0.120	0.120	0.120	0.120	0	0	0	0	0	0
25	0.120	0.120	0	0	0	0	0	0	0	0
26	0.648	0.648	0.504	0.504	0.360	0.360	0.216	0.216	0.072	0.072
27	0.504	0.504	0.360	0.360	0.216	0.216	0.072	0.072	0	0
28	0.360	0.360	0.216	0.216	0.072	0.072	0	0	0	0
29	0.216	0.216	0.072	0.072	0	0	0	0	0	0
30	0.072	0.072	0	0	0	0	0	0	0	0
31	0.120	-0.120	0.120	-0.120	0.120	-0.120	0.120	-0.120	0.120	-0.120
32	0.120	-0.120	0.120	-0.120	0.120	-0.120	0.120	-0.120	0	0
33	0.120	-0.120	0.120	-0.120	0.120	-0.120	0	0	0	0
34	0.120	-0.120	0.120	-0.120	0	0	0	0	0	0
35	0.120	-0.120	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0
'	'	'	'	'	'	'	'	'	'	'
50	0	0	0	0	0	0	0	0	0	0

Table 7.17 Matrix of the values of λ_{ij} of the 50 supplementary strain systems

$\lambda_{ij} = \lambda_{ji}$. The λ_{ij} are the "coefficients of the unknowns".

	i= 1-5	6-10	11-15	16-20	21-25	26-30	31-35	36 37 38 39 40	41 42 43 44 45	46 47 48 49 50
j=1-5	aI	-bI	0	0	cI	0	0	0	0	1105.9 K
6-10		dI	0	0	0	0	0	0	0	-1203.1 K
11-15			eI	-fI	0	gI	-hI	-1048.7 J	-1218.1 K	850.7 J
16-20				iI	0	-jI	kI	1203.1 J	2552.3 K	-2406.1 J
21-25					ℓI	0	0	0	0	0
26-30						mI	-nI	-1276.1 J	-2707.1 K	1658.8 J
31-35							pI	249.2 J	3045.2 K	-2126.9 J

j	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
36	q	-r	o	o	o	o	s	o	o	o	-t	-u	o	o	o
37		q	-r	o	o	-s	o	s	o	o	-u	-t	-u	o	o
38			q	-r	o	o	-s	o	s	o	o	-u	-t	-u	o
39				q	-r	o	o	-s	o	s	o	o	-u	-t	-u
40					1/2q	o	o	o	-s	s	o	o	o	-u	-1/2t
41						v	-z	o	o	o	o	α	o	o	o
42							v	-z	o	o	-α	o	α	o	o
43								v	-z	o	o	-α	o	α	o
44									v	-z	o	o	-α	o	α
45										w	o	o	o	-α	-α
46											β	δ	o	o	o
47												β	δ	o	o
48													β	δ	o
49														β	δ
50															γ

$$I = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$J = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{Vmatrix} \qquad O = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

$$K = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{Vmatrix}$$

a = 8731.9	f = 3609.4	k = 3007.9	p = 6554.6	u = 283.6	β = 6504.3
b = 3609.4	g = 1276.1	ℓ = 2965.9	q = 5016.8	v = 8187.7	γ = 3470.3
c = 1186.3	h = 249.2	m = 6193.0	r = 410.9	w = 5603.8	δ = 368.6
d = 3445.2	i = 8676.8	n = 3190.3	s = 1218.1	z = 2583.9	
e = 5427.7	j = 2345.9	o = zero	t = 2835.8	α = 1804.7	

Table 7.18 Values of 10^6 Y_{mi}. The values Y_i are the solutions of the 50 unknowns for the m loading cases.

	m=1	2	3	4	5	6	7	8	9	10	
i= 1	- 9.2747	-11.2387	- 9.4322	-11.3820	- 9.5930	-11.4958	- 9.8094	-11.5030	-10.3409	-11.1265	1
2	-10.351	-10.7036	-10.4669	-10.7677	-10.6247	-10.7532	-11.0412	-10.4398	- 0.2928	+ 0.3302	2
3	-10.7455	-10.8401	-10.8756	-10.7605	-11.2421	-10.3931	- 0.4354	+ 0.4167	- 0.0681	+ 0.0664	3
4	-10.9177	-10.8650	-11.3233	-10.4388	- 0.5066	+ 0.4014	- 0.1157	+ 0.0729	- 0.0165	+ 0.0075	4
5	-11.0590	-10.7107	- 0.4573	+ 0.3415	- 0.1182	+ 0.0612	- 0.0275	+ 0.0074	- 0.0034	- 0.0001	5
6	+ 4.9350	-16.9868	+ 3.1771	-18.5865	+ 1.3828	-19.8566	- 1.0323	-19.9361	- 0.9651	-15.7343	6
7	- 7.0796	-11.0136	- 8.3719	-11.7288	-10.1326	-11.5672	-14.7818	- 8.0693	- 3.2678	+ 3.6860	7
8	-11.4815	-12.5372	-12.9335	-11.6485	-17.0247	- 7.5483	- 4.8597	+ 4.6508	- 0.7598	+ 0.7408	8
9	-13.4041	-12.8157	-17.9312	- 8.0585	- 5.6551	+ 4.4805	- 1.2916	+ 0.8133	- 0.1843	+ 0.0836	9
10	-14.9808	-11.0933	- 5.1042	+ 3.8112	- 1.3192	+ 0.6826	- 0.3072	+ 0.0828	- 0.0376	- 0.0012	10
11	-24.8997	-13.3870	-21.0372	- 9.4683	-17.1506	- 5.3912	-13.3675	- 0.6959	-10.8322	+ 5.9600	11
12	-10.4614	- 4.9153	- 8.8556	- 3.0037	- 7.2848	- 0.4270	- 6.8954	+ 4.4702	+ 1.8612	- 1.5906	12
13	- 4.0816	- 0.5511	- 3.7758	+ 1.1695	- 4.4093	+ 5.1685	+ 3.3972	- 1.6186	+ 0.4595	- 0.1001	13
14	- 1.1109	+ 1.6857	- 3.0249	+ 5.6481	+ 4.2387	- 1.4030	+ 0.9621	- 0.0849	+ 0.1072	+ 0.0328	14
15	+ 0.2362	+ 5.0916	+ 5.2525	- 1.3220	+ 1.3225	- 0.0307	+ 0.2642	+ 0.0354	+ 0.0252	+ 0.0154	15
16	- 8.2122	+ 5.6227	- 7.8933	+ 5.9707	- 7.6106	+ 6.3466	- 7.5353	+ 6.7865	- 8.1092	+ 7.6652	16
17	- 5.9977	+ 0.5164	- 5.3080	+ 1.3800	- 4.8429	+ 2.4315	- 5.3657	+ 4.3058	+ 0.9797	- 1.2148	17
18	- 5.0366	+ 0.7518	- 4.5938	+ 1.8482	- 5.1067	+ 3.7575	+ 1.2419	- 1.7097	+ 0.2148	- 0.3343	18
19	- 4.3162	+ 1.8377	- 4.8984	+ 3.7861	+ 1.3906	- 1.7107	+ 0.3077	- 0.3795	+ 0.0544	- 0.0553	19
20	- 3.9376	+ 2.6679	+ 1.7277	- 1.9154	+ 0.4549	- 0.3927	+ 0.1165	- 0.0571	+ 0.0160	- 0.0030	20
21	+44.1696	+44.9552	+44.2326	+45.0125	+44.2969	+45.0580	+44.3834	+45.0608	+44.5960	+44.9103	21
22	+44.6001	+44.7411	+44.6464	+44.7667	+44.7095	+44.7609	+44.8761	+44.6356	+ 0.1171	- 0.1321	22
23	+44.7579	+44.7957	+44.8099	+44.7639	+44.9565	+44.6169	+ 0.1741	- 0.1667	+ 0.0272	- 0.0265	23
24	+44.8268	+44.8057	+44.9890	+44.6352	+ 0.2026	- 0.1606	+ 0.0463	- 0.0291	+ 0.0066	- 0.0030	24
25	+44.8833	+44.7440	+ 0.1829	- 0.1366	+ 0.0473	- 0.0245	+ 0.0110	- 0.0030	+ 0.0013	+ 0.0000	25

Table 7.18 continued

	m=1	2	3	4	5	6	7	8	9	10	
i= 26	+183.070	+151.680	+145.172	+113.866	+107.335	+76.3160	+69.4229	+39.8183	+29.6518	+ 6.5375	26
27	+162.765	+124.639	+120.718	+ 83.0394	+ 78.7242	+42.5118	+35.1611	+ 5.6367	+ 2.7666	- 2.1716	27
28	+128.958	+ 87.6641	+ 84.7553	+ 45.6111	+ 39.3918	+ 7.1419	+ 5.3284	- 1.9683	+ 0.6968	- 0.0052	28
29	+ 89.2904	+ 46.5224	+ 41.7152	+ 7.9690	+ 6.7294	- 1.5895	+ 1.5314	+ 0.0446	+ 0.1604	+ 0.0857	29
30	+ 46.9481	+ 7.5627	+ 8.3364	- 1.3430	+ 2.0889	+ 0.1465	+ 0.4043	+ 0.0913	+ 0.0361	+ 0.0286	30
31	+110.357	+ 37.4443	+ 93.0423	+ 20.1587	+ 75.7056	+ 2.9358	+58.3664	-13.8629	+41.4855	-27.9312	31
32	+127.352	+ 33.4612	+103.635	+ 10.0825	+ 79.9614	-12.3630	+56.1971	-30.1345	+ 3.1719	+ 0.3592	32
33	+111.319	+ 12.5147	+ 85.9899	- 11.0577	+ 60.8362	-29.9583	+ 6.3586	- 0.1747	+ 0.6529	+ 0.1123	33
34	+ 89.9128	- 11.2550	+ 62.6515	- 29.7731	+ 7.4247	- 0.1701	+ 1.3249	- 0.0168	+ 0.1335	+ 0.0197	34
35	+ 74.7287	- 32.5072	+ 10.7011	- 0.1664	+ 2.1808	+ 0.0563	+ 0.3808	+ 0.0126	+ 0.0371	+ 0.0076	35
36	+106.348	+ 54.7760	+ 83.9358	+ 32.6798	+ 61.6972	+11.5583	+39.3972	- 5.1939	+12.8844	- 5.2262	36
37	+117.931	+ 50.5534	+ 88.3158	+ 22.7441	+ 59.0101	- 0.4487	+25.4117	- 5.3852	+ 3.0099	+ 0.2759	37
38	+ 99.6441	+ 26.3131	+ 66.5312	+ 2.0705	+ 30.6966	- 4.3789	+ 6.4078	+ 0.1978	+ 0.6570	+ 0.2905	38
39	+ 79.2669	+ 0.9331	+ 34.9577	- 3.8155	+ 8.1353	+ 0.4158	+ 1.5251	+ 0.2460	+ 0.1406	+ 0.0828	39
40	+ 68.2876	- 9.8025	+ 16.9016	+ 0.8351	+ 3.5876	+ 0.5486	+ 0.6147	+ 0.1291	+ 0.0533	+ 0.0304	40
41	+ 20.3371	+ 17.3471	+ 16.5527	+ 13.8238	+ 12.8362	+10.9449	+ 8.1220	+10.0428	- 5.0173	+12.7516	41
42	+ 9.0003	+ 6.5132	+ 7.2094	+ 6.0227	+ 4.6458	+ 7.4854	- 6.6133	+12.5786	- 1.5095	+ 1.5478	42
43	+ 3.9601	+ 2.5223	+ 1.4124	+ 5.2504	- 9.0496	+11.3734	- 2.9559	+ 1.2343	- 0.4062	- 0.0314	43
44	+ 1.7526	+ 4.5365	- 9.7285	+ 10.9575	- 3.6946	+ 0.9964	- 0.9117	- 0.0875	- 0.0936	- 0.0708	44
45	- 20.5514	+ 11.2828	- 6.6185	+ 0.4142	- 1.6496	- 0.2848	- 0.3079	- 0.1027	- 0.0253	- 0.0263	45
46	+ 41.9569	- 14.9263	+ 37.3954	- 19.0772	+ 32.7394	-22.3728	+26.4727	-22.5792	+11.0783	-11.6764	46
47	+ 52.7380	- 14.3532	+ 44.8233	- 20.3601	+ 35.5986	-23.2362	+17.2681	-14.3661	+ 2.5988	- 2.1119	47
48	+ 52.0969	- 17.7335	+ 40.4145	- 21.4343	+ 20.5740	-13.6713	+ 4.6580	- 2.2982	+ 0.6273	- 0.1898	48
49	+ 46.4671	- 21.8365	+ 23.0376	- 13.1933	+ 5.9001	- 2.0453	+ 1.3065	- 0.1879	+ 0.1490	+ 0.0272	49
50	+ 36.7461	- 21.4703	+ 9.7932	- 3.3038	+ 2.4771	- 0.2740	+ 0.5094	+ 0.0271	- 0.0514	+ 0.0242	50

Table 7.19 Influence coefficients C_{mn} ($C_{mn} = C_{nm}$) of the swept box multiplied by 10^6 and expressed in inch/lb. Computed with the minimum principle for the strains with 50 supplementary strain systems.

	m=1	2	3	4	5	6	7	8	9	10
n=1	358.18	234.90	259.22	153.88	168.25	84.49	90.44	33.39	31.72	5.238
2		225.38	173.78	156.46	113.99	93.97	61.01	43.99	20.81	11.82
3			200.82	117.94	134.40	66.56	74.32	27.11	26.92	4.595
4				120.48	80.66	76.05	44.98	37.72	16.02	11.181
5					100.54	48.58	58.21	20.85	22.13	3.959
6						58.03	29.19	31.45	11.25	10.55
7							41.99	14.49	17.33	3.321
8								25.05	6.611	9.938
9									12.465	2.508
10										9.212

Table 7.20 Values of stresses in $lbs/inch^2$ for the m loading cases. Computed with the minimum principle for the strains with 50 supplementary strain systems. Compare table 7.21.

		m=1	2	3	4	5	6	7	8	9	10
normal stresses in upper spar booms, immediately at tip side of station	1	-	-	-	-	-	-	-	-	-	-
	2	-	-	-	-	-	-	-	-	-	-
	3	+ 1.930	+0.367	-0.419	+0.117	-0.106	+0.005	-0.021	-0.002	-0.002	-0.001
	4	+ 2.421	+1.459	+1.001	-0.245	+0.252	-0.004	+0.050	+0.007	+0.005	+0.003
	5	+ 3.143	+1.511	+1.555	+0.429	-0.497	+0.114	-0.108	+0.003	-0.012	-0.004
	6	+ 3.858	+2.789	+2.500	+1.388	+0.985	-0.264	+0.219	-0.010	+0.024	+0.009
	7	+ 4.612	+2.970	+3.135	+1.607	+1.632	+0.472	-0.418	+0.131	-0.052	+0.002
	8	+ 4.866	+3.919	+3.575	+2.622	+2.266	+1.288	+0.808	-0.307	+0.105	-0.011
	9	+ 6.682	+4.821	+5.112	+3.275	+3.552	+1.791	+1.944	+0.550	-0.219	+0.127
	10	+ 4.877	+4.379	+3.812	+3.314	+2.747	+2.242	+1.662	+1.121	+0.430	-0.305
	11	+10.153	+7.480	+8.255	+5.587	+6.362	+3.711	+4.474	+1.914	+2.539	+0.422
	12	+ 2.549	+3.492	+2.079	+3.021	+1.609	+2.546	+1.135	+2.044	+0.633	+1.392
Shear stresses in spar webs, between the stations:	1-3	+ 2.272	+0.547	-0.321	+0.020	-0.080	-0.014	-0.015	-0.005	-0.001	-0.001
	2-4	+ 0.996	+2.721	+0.321	-0.020	+0.080	+0.014	+0.015	+0.005	+0.001	+0.001
	3-5	+ 2.357	+0.767	+2.476	+0.551	-0.259	+0.034	-0.059	-0.009	-0.006	-0.005
	4-6	+ 0.911	+2.501	+0.792	+2.717	+0.259	-0.034	+0.059	+0.009	+0.006	+0.005
	5-7	+ 2.549	+0.889	+2.544	+0.805	+2.570	+0.586	-0.202	+0.051	-0.025	-0.006
	6-8	+ 0.719	+2.379	+0.724	+2.462	+0.697	+2.682	+0.202	-0.051	+0.025	+0.006
	7-9	+ 2.985	+1.204	+2.894	+1.097	+2.796	+0.948	+2.745	+0.660	-0.099	+0.069
	8-10	+ 0.283	+2.064	+0.374	+2.171	+0.472	+2.320	+0.523	+2.608	+0.099	-0.069
	9-11	+ 3.971	+2.045	+3.596	+1.767	+3.418	+1.479	+3.139	+1.147	+2.926	+0.687
	10-12	- 0.703	+1.223	-0.428	+1.501	-0.150	+1.789	+0.129	+2.121	+0.342	+2.581
Shear stresses in rib webs between the stations:	1-2	+ 0.996	-0.547	+0.321	-0.020	+0.080	+0.014	+0.015	+0.005	+0.001	+0.001
	3-4	- 0.085	-0.220	+0.471	-0.531	+0.179	-0.048	+0.044	+0.004	+0.005	+0.003
	5-6	- 0.192	-0.122	-0.068	-0.254	+0.439	-0.551	+0.143	-0.060	+0.020	+0.002
	7-8	- 0.436	-0.316	-0.349	-0.292	-0.225	-0.363	+0.320	-0.610	+0.073	-0.075
	9-10	- 0.986	-0.841	-0.802	-0.670	-0.622	-0.530	-0.394	-0.487	+0.243	-0.618

Table 7.21 Values of normal stresses in lbs/inch^2 in spar booms *for the m* loading cases. Computed with the minimum principle for the strains with 50 supplementary strain systems. Compare table 7.20.

		m=1	2	3	4	5	6	7	8	9	10
Normal stresses in upper spar booms, immediately at root side of station:	1	+0.566	-0.341	+0.186	-0.006	+0.046	+0.009	+0.009	+0.003	+0.001	+0.001
	2	+0.998	-0.532	+0.282	-0.035	+0.071	+0.008	+0.013	+0.004	+0.001	+0.001
	3	+1.627	+0.354	-0.253	-0.174	-0.027	-0.012	-0.001	+0.002	0.000	+0.001
	4	+2.623	+1.208	+1.454	-0.639	+0.387	-0.037	+0.083	+0.010	+0.008	+0.005
	5	+2.744	+1.535	+1.289	+0.398	-0.340	-0.187	-0.045	-0.020	-0.003	-0.002
	6	+4.026	+2.522	+2.680	+1.119	+1.405	-0.673	+0.325	-0.051	+0.038	+0.010
	7	+4.059	+2.853	+2.677	+1.539	+1.295	+0.383	-0.316	-0.203	-0.021	-0.032
	8	+4.891	+3.650	+3.609	+2.335	+2.326	+0.941	+1.140	-0.758	+0.159	-0.063
	9	+5.847	+4.377	+4.416	+2.959	+2.992	+1.583	+1.554	+0.377	-0.129	-0.229
	10	+4.518	+3.763	+3.541	+2.780	+2.562	+1.777	+1.584	+0.683	+0.694	-0.761
	11	-	-	-	-	-	-	-	-	-	-
	12	-	-	-	-	-	-	-	-	-	-

Table 7.22 Comparison of the main diagonal elements C_{mm} of the matrices of influence coefficients of tables 7.5, 7.10, 7.14 and 7.19. Elements multiplied by 10^6 and expressed in inch/lbs.

	From minimum theorem for the stresses			From min. ther. for the strains
	10 unkn.	15 unkn.	20 unkn.	50 unkn.
	table 7.5	table 7.10	table 7.14	table 7.19
element 1.1	382.86	377.73	377.452	358.18
2.2	248.52	240.93	240.524	225.38
3.3	135.74	131.34	131.282	120.48
4.4	110.58	108.27	108.002	100.54
5.5	217.44	213.45	213.167	200.82
6.6	68.62	65.95	65.858	58.03
7.7	46.66	45.81	45.580	41.99
8.8	31.23	29.84	29.651	25.05
9.9	13.55	13.44	13.387	12.465
10.10	11.24	10.90	10.775	9.212

Table 7.23 Values of stresses in lbs/inch^2 at several stations for vertical loads of 1 lb in all stations.

		(1)	(2)	(3)	(4)	(5)
Normal stresses in upper spar booms at station:	1	0	0	0	-	0.474
	2	0	0	0	-	0.811
	3	0.542	1.550	1.469	1.868	1.517
	4	4.309	4.788	4.674	4.949	5.102
	5	4.392	5.317	5.369	6.134	5.369
	6	10.647	10.805	10.526	11.498	11.501
	7	12.307	12.609	12.885	14.091	12.234
	8	17.902	16.911	17.069	19.131	18.230
	9	27.956	26.890	26.756	27.635	23.747
	10	24.196	22.943	23.481	24.279	21.141
	11	60.285	58.025	57.744	50.897	-
	12	27.933	27.712	26.796	20.500	-
Shear stress in spar webs, between the stations:	1-3	2.648	2.232	2.222	2.402	
	2-4	3.888	4.304	4.314	4.134	
	3-5	6.127	5.724	5.742	5.847	
	4-6	6.944	7.348	7.330	7.225	
	5-7	9.911	9.816	9.842	9.761	
	6-8	9.697	9.792	9.766	9.845	
	7-9	14.615	14.935	14.930	15.299	
	8-10	11.528	11.209	11.214	10.845	
	9-11	21.050	21.509	21.539	24.275	
	10-12	11.630	11.171	11.141	8.405	
Shear stresses in rib webs between the stations	1-2	+0.620	+1.036	+1.046	+0.866	
	3-4	-0.210	-0.224	-0.252	-0.178	
	5-6	-0.518	-0.826	-0.832	-0.643	
	7-8	-1.436	-1.850	-1.820	-2.273	
	9-10	-3.168	-3.307	-3.340	-5.707	

Computed with:

(1) minimum principle for the stresses, 10 unknowns.

(2) id , 15 unknowns

(3) id , 20 unknowns

(4) minimum principle for the strains, 50 unknowns normal stresses in upper spar booms, immediately at tip side of stations.

(5) minimum principle for the strains, 50 unknowns normal stresses in upper booms, immediately at root side of stations.

Table 7.24 Values of stresses in $lbs/inch^2$ at several stations for vertical loads of 1 lb at stations 1, 3, 5, 7, 9 and of -1 lb at stations 2, 4, 6, 8, 10.

		(1)	(2)	(3)	(4)	(5)
Normal stresses in upper spar booms at station:	1	0	0	0	-	1.142
	2	0	0	0	-	1.919
	3	0.818	0.618	0.717	0.896	1.175
	4	3.087	3.262	3.362	2.509	4.008
	5	1.806	1.837	1.747	2.028	1.921
	6	4.987	4.783	4.682	3.674	5.447
	7	3.109	3.127	3.207	3.727	3.154
	8	5.472	4.753	4.727	4.109	6.020
	9	6.122	5.966	5.976	6.507	5.613
	10	3.020	2.615	2.693	2.777	4.657
	11	+15.139	+15.049	14.892	12.669	-
	12	-6.069	-4.090	-4.132	-4.490	-
Shear stresses in spar webs, between the stations:	1-3	+1.260	+1.288	+1.274	+1.308	
	2-4	-1.260	-1.288	-1.274	-1.308	
	3-5	+3.117	+3.126	+3.124	+3.171	
	4-6	-3.118	-3.126	-3.124	-3.171	
	5-7	+5.037	+5.130	+5.142	+5.111	
	6-8	-5.037	-5.130	-5.142	-5.111	
	7-9	+7.437	+7.649	+7.658	+7.343	
	8-10	-7.436	-7.649	-7.658	-7.343	
	9-11	+11.156	+11.215	+11.231	+10.025	
	10-12	-11.156	-11.215	-11.231	-10.025	
Shear stresses in rib webs between the stations	1-2	+2.008	+1.980	+1.994	+1.960	
	3-4	+1.410	+1.428	+1.418	+1.406	
	5-6	+1.350	+1.264	+1.250	1.327	
	7-8	+0.868	+0.750	+0.754	+1.039	
	9-10	-0.450	-0.299	-0.306	+0.585	

Computed with:

(1) minimum principle for the stresses, 10 unknowns.

(2) id , 15 unknowns.

(3) id , 20 unknowns.

(4) minimum principle for the strains, 50 unknowns
normal stresses in upper spar booms, immediately at tip side of stations.

(5) minimum principle for the strains, 50 unknowns
normal stresses in upper booms, immediately at root side of stations.

Table 8.1 Numerical values of $10^6 \lambda_{pq}$ (Three types of supplementary stress systems).

	p=1.(n-2)	2.(n-2)	3.(n-2)	
q = 1.n	+494	0	0	= 10^6 × matrix D
2.n	0	+811	0	
3.n	0	0	+80	

	p=1.(n-1)	2.(n-1)	3.(n-1)	
q = 1.n	-5941	+1502	-641	= 10^6 × matrix C
2.n	-1502	-4895	-1770	
3.n	- 641	+1770	+ 321	

	p=1.n	2.n	3.n	
q = 1.n	+17420	0	-2564	= 10^6 × matrix B
2.n	0	+15677	0	
3.n	-2564	0	+10310	

	p=1.(n+1)	2.(n+1)	3.(n+1)	
q = 1.n	-5941	-1502	-641	= 10^6 × matrix C'
2.n	+1502	-4895	+1770	
3.n	-641	-1770	+321	

	p=1.(n+2)	2.(n+2)	3.(n+2)	
q = 1.n	+494	0	0	= 10^6 × matrix D'
2.n	0	+811	0	
3.n	0	0	+80	

Table 8.2

Numerical values of $10^6 \lambda_{pq}$.(Five types of supplementary stress systems).

	p=1.(n-2)	2.(n-2)	3.(n-2)	4.(n-2)	5.(n-2)	
q = 1.n	+494	0	0	0	0	
2.n	0	+811	0	+522	-1623	$=10^6 \times$ matrix D
3.n	0	0	+80	0	0	
4.n	0	+522	0	+384	-1043	
5.n	0	0	0	0	0	

	p=1.(n-1)	2.(n-1)	3.(n-1)	4.(n-1)	5.(n-1)	
q = 1.n	-5941	+1502	-641	0	-3803	
2.n	-1502	-4895	-1770	-3827	-125	$=10^6 \times$ matrix C
3.n	-641	+1770	+321	+2123	-1840	
4.n	0	-3827	-2123	-1609	+746	
5.n	0	-1623	0	-1043	+3245	

	p=1.n	2.n	3.n	4.n	5.n	
q = 1.n	+17420	0	-2564	0	+3803	$=10^6 \times$ matrix B
2.n	0	+15677	0	+6611	-125	
3.n	-2564	0	+10310	0	+1840	
4.n	0	+6611	0	+10167	+746	
5.n	+3803	-125	+1840	+746	+15507	

	p=1.(n+1)	2.(n+1)	3.(n+1)	4.(n+1)	5.(n+1)	
q = 1.n	-5941	-1502	-641	0	0	
2.n	+1502	-4895	+1770	-3827	-1623	$=10^6 \times$ matrix C'
3.n	-641	-1770	+321	-2123	-1043	
4.n	0	-3827	+2123	-1609	-1043	
5.n	-3803	-125	-1840	+746	+3245	

	p=1.(n+2)	2.(n+2)	3.(n+2)	4.(n+2)	5.(n+2)	
p = 1.n	+494	0	0	0	0	
2.n	0	+811	0	+522	0	$=10^6 \times$ matrix D'
3.n	0	0	+80	0	0	
4.n	0	+522	0	+348	0	
5.n	0	-1623	0	-1043	0	

Table 8.3 Numerical values of $10^6 \lambda_{pq}$ (10 types of supplementary strain systems). Matrix $T \times 10^5$.

q= \ p=	1.(n-1)	2.(n-1)	3.(n-1)	4.(n-1)	5.(n-1)	6.(n-1)	7.(n-1)	8.(n-1)	9.(n-1)	10.(n-1)
1.n	0	0	0	0	0	0	0	0	0	-1105.9
2.n	0	0	0	0	0	0	0	0	0	+1203.1
3.n	0	0	0	0	0	0	0	-1048.7	+1218.1	+ 850.7
4.n	0	0	0	0	0	0	0	+1203.1	-2552.3	-2406.1
5.n	0	0	0	0	0	0	0	0	0	0
6.n	0	0	0	0	0	0	0	-1276.1	+2707.1	+1658.8
7.n	0	0	0	0	0	0	0	+ 249.2	-3045.2	-2126.9
8.n	0	0	0	0	0	0	0	- 410.9	-1218.1	- 283.6
9.n	0	0	0	0	0	0	0	+1218.1	-2583.9	-1804.7
10.n	0	0	0	0	0	0	0	- 283.6	+1804.7	+ 368.6

Table 8.4 Numerical values of $10^6 \lambda_{pq}$ (10 supplementary strain systems II). Matrix $V \times 10^6$.

q= \ p=	1.n	2.n	3.n	4.n	5.n	6.n	7.n	8.n	9.n	10.n
1.n	+8731.9	-3609.4	0	0	+1186.3	0	0	0	0	+1105.9
2.n	-3609.4	+3445.2	0	0	0	0	0	0	0	-1203.1
3.n	0	0	+5427.7	-3609.4	0	+1276.1	- 249.2	-1048.7	-1218.1	+ 850.7
4.n	0	0	-3609.4	+8676.8	0	-2345.9	+3007.9	+1203.1	+2552.3	-2406.1
5.n	+1186.3	0	0	0	+2965.9	0	0	0	0	0
6.n	0	0	+1276.1	-2345.9	0	+6193.0	-3190.3	-1276.1	-2707.1	+1658.8
7.n	0	0	- 249.2	+3007.9	0	-3190.3	+6554.6	+ 249.2	+3045.2	-2126.9
8.n	0	0	-1048.7	+1203.1	0	-1276.1	+ 249.2	+5016.8	0	-2835.8
9.n	0	0	-1218.1	+2552.3	0	-2707.1	+3045.2	0	+8187.7	0
10.n	+1105.9	-1203.1	+ 850.7	-2406.1	0	+1658.8	-2126.9	-2835.8	0	+6504.3

Table 8.5 Numerical values of $10^6 \lambda_{pq}$ (10 supplementary strain systems III). Matrix $T' \times 10^6$.

q= \ p=	1.(n+1)	2.(n+1)	3.(n+1)	4.(n+1)	5.(n+1)	6.(n+1)	7.(n+1)	8.(n+1)	9.(n+1)	10.(n+1)
1.n	0	0	0	0	0	0	0	0	0	0
2.n	0	0	0	0	0	0	0	0	0	0
3.n	0	0	0	0	0	0	0	0	0	0
4.n	0	0	0	0	0	0	0	0	0	0
5.n	0	0	0	0	0	0	0	0	0	0
6.n	0	0	0	0	0	0	0	0	0	0
7.n	0	0	0	0	0	0	0	0	0	0
8.n	0	0	−1048.7	+1203.1	0	−1276.1	+ 249.2	− 410.9	+1218.1	− 283.6
9.n	0	0	+1218.1	−2552.3	0	+2707.1	−3045.2	−1218.1	−2583.9	+1804.7
10.n	−1105.9	+1203.1	+ 850.7	−2406.1	0	+1658.8	−2126.9	− 283.6	−1804.7	+ 368.6

Table 8.6 Matrix $H \times 10^6$.

H = T + V + T' from tables 8.3, 8.4 and 8.5 respectively.

+8731.9	-3609.4	0	0	+1186.3	0	0	0	0	0
-3609.4	+3445.2	0	0	0	0	0	0	0	0
0	0	+5427.7	-3609.4	0	+1276.1	- 249.2	-2097.5	0	+1701.5
0	0	-3609.4	+8676.8	0	-2345.9	+3007.9	+2406.2	0	-4812.4
+1186.3	0	0	0	+2965.9	0	0	0	0	0
0	0	+1276.1	-2345.9	0	+6193.0	-3190.3	-2552.2	0	+3317.6
0	0	- 249.2	+3007.9	0	-3190.3	+6554.6	+ 498.4	0	-4253.8
0	0	-2097.5	+2406.2	0	-2552.2	+ 498.4	+4195.0	0	-3403.0
0	0	0	0	0	0	0	0	+3019.9	0
0	0	+1701.5	-4812.4	0	+3317.6	-4253.8	-3403.0	0	+7241.5

Table 8.7 10^6 Matrix T - T' = matrix table 8.3 - matrix table 8.5

0	0	0	0	0	0	0	0	0	-1105.9
0	0	0	0	0	0	0	0	0	+1203.1
0	0	0	0	0	0	0	-1048.7	+1218.1	+ 850.7
0	0	0	0	0	0	0	+1203.1	-2552.3	-2406.1
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-1276.1	+2707.1	+1658.8
0	0	0	0	0	0	0	+ 249.2	-3045.2	-2126.9
0	0	+1048.7	-1203.1	0	+1276.1	- 249.2	0	-2436.2	0
0	0	-1218.1	+2552.3	0	-2707.1	+3045.2	+2436.2	0	-3609.4
+1105.9	-1203.1	- 850.7	+2406.1	0	-1658.8	+2126.9	0	+3609.4	0

Table 8.8 Influence coefficients C_{mn} ($C_{mn} = C_{nm}$) of the 5 cell swept-box multiplied by 10^6 and expressed in inch/lb. Computed with the simplifying assumptions of section 8.6. Compare table 7.14 and 7.19.

	m = 1	2	3	4	5	6	7	8	9	10
n = 1	404.79	257.85	298.20	167.83	198.02	90.630	110.65	32.653	42.511	0.307
2		244.60	193.46	170.05	129.08	101.90	71.100	46.576	25.938	10.472
3			234.12	129.38	159.57	71.406	91.430	26.244	36.103	0.307
4				131.60	90.630	82.679	51.876	40.168	19.530	10.472
5					121.13	52.183	72.207	19.837	29.696	0.307
6						63.455	32.652	33.760	13.122	10.472
7							52.983	13.429	23.288	0.307
8								27.352	6.715	10.472
9									16.880	0.307
10										10.472

Table 8.9

Comparison between displacements established with an elementary theory for the oblique beam (section 8.6) and with the minimum principle for the stresses (with 20 unknowns).

Loading case A, vertical downwardly directed loads of 1 lb in station 2 and station 3 (fig.7.1) (shear force)

Station		Elementary theory	Minimum principle for the stresses
rear spar	1	556.05	515.82
	3	427.58	391.58
	5	288.65	260.05
	7	162.53	141.41
	9	62.04	48.60
front spar	2	438.06	418.94
	4	299.43	291.36
	6	173.31	174.78
	8	72.82	81.67
	10	10.78	21.67
Derived from table		8.8	7.14

Loading case B, vertical downwardly directed load of 1 lb in station 2 and upwardly directed load of 1 lb in station 3 (fig.7.1) (torsion)

Station		Elementary theory	Minimum principle for the stresses
rear spar	1	-40.35	-31.32
	3	-40.67	-34.76
	5	-30.49	-26.26
	7	-20.33	-16.82
	9	-10.16	- 7.73
front spar	2	51.14	62.11
	4	40.67	44.76
	6	30.49	30.85
	8	20.33	18.84
	10	10.16	8.20
Derived from table		8.8	7.14

Displacements in 10^{-6} inch

Table 9.1 Numerical values of $10^6 \lambda_{pq}$ for the structure near the clamped root (fig.9.1).

	p = 1.0	2.0	3.0	4.0	5.0	
q = 1.0	+7721	-1502	-1282	0	+3803	
2.0	-1502	+6216	+1770	+2262	-3370	
3.0	-1282	+1770	+4994	0	+1840	= 10^6 x matrix B^{**}
4.0	0	+2262	0	+4388	-1340	
5.0	+3803	-3370	+1840	-1340	+12262	

	p = 1.1	2.1	3.1	4.1	5.1	
q = 1.0	+4952	-1502	- 640	0	0	
2.0	+1502	-3272	+1770	-2784	-1623	= 10^6 x matrix $C^{*\prime}$
3.0	- 641	-1770	+ 482	-2123	0	
4.0	0	-2784	+2123	- 913	-1043	
5.0	-3803	+1498	-1840	+1789	+3245	

	p = 1.0	2.0	3.0	4.0	5.0	
q = 1.1	-4952	+1502	- 641	0	-3803	
2.1	-1502	-3272	-1770	-2784	+1498	
3.1	- 641	+1770	+ 482	+2123	-1840	= 10^6 x matrix C^*
4.1	0	-2784	-2123	- 913	+1789	
5.1	0	-1623	0	-1043	+3245	

	p = 1.1	2.1	3.1	4.1	5.1	
q = 1.1	+16926	0	-2564	0	+3803	
2.1	0	+14866	0	+6089	- 125	
3.1	-2564	0	+10230	0	+1840	= 10^6 x matrix B^*
4.1	0	+6089	0	+9819	+ 746	
5.1	+3803	- 125	+ 1840	+ 746	+15507	

Table 9.2. Roots of λ in (9.13) together with their corresponding column matrices R.

$\lambda_{1,2} = 0.19708 \pm 0.17719\,i$ $= 0.265022 \exp \pm 0.73230\,i$	$\lambda_{3,4} = 0.010259 \pm 0.18617\,i$ $= 0.186452 \exp \pm 1.51578\,i$
$R_{1,2}$	$R_{3,4}$
$- 0.792771 \pm 2.74323\,i$ $+ 2.29316 \pm 2.11288\,i$ $- 2.00890 \pm 0.205721\,i$ $+ 0.141688 \pm 0.858861\,i$ $+ 1.0$	$+ 0.128175 \pm 0.478017\,i$ $+ 2.10405 \pm 0.274173\,i$ $- 2.48997 \pm 0.305342\,i$ $- 0.707807 \pm 1.53990\,i$ $+ 1.0$

$\lambda_5 =$ -0.0990664	$\lambda_6 =$ $+0.0863584$	$\lambda_7 =$ -0.0670232	$\lambda_8 =$ -0.00986230	$\lambda_9 =$ -0.000736414
R_5	R_6	R_7	R_8	R_9
-0.253097 +0.734260 +0.329722 +0.538510 +1.0	-1.172421 +2.216659 -1.872067 +0.368596 +1.0	+0.354021 +5.245274 -2.860612 -5.479570 +1.0	+0.0079897 +5.35387 -0.471302 -5.31741 +1.0	-0.0009523 +2.110645 +0.0138926 -0.183812 +1.0

Table 9.3 Values of the 10 column matrices $G_1^{\bullet}$ from (9.18) after substitution of the 9 sets λ_i, R_i (i = 1...10) given in table 9.2. Multiplied by 10^6.

λ_1, R_1	λ_2, R_2	λ_3, R_3	λ_4, R_4	λ_5, R_5	λ_6, R_6	λ_7, R_7	λ_8, R_8	λ_9, R_9
+56.929 + 14710.8 i	+ 56.929 − 14710.8 i	+5279.77 + 3226.62 i	+5279.77 − 3226.62 i	+ 328.079	+5865.53	+2848.72	−3495.90	+609.98
+6745.48 + 7993 .43 i	+ 6745.48 − 7993 .43 i	+4279.62 + 1845.74 i	+4279.62 − 1845.74 i	+3911.79	+8411.42	+11791.4	+17086.5	+9365.93
−2789.72 − 1038 .17 i	− 2789.72 + 1038 .17 i	−6280.61 − 2347.80 i	−6280.61 + 2347.80 i	+5321.19	−2503.12	−3666.46	+8936.84	+5648.90
+3154.29 + 5323 .13 i	+ 3154.29 − 5323 .13 i	+687.60 + 5179.61 i	+687.60 − 5179.61 i	+2974.68	+1305.78	−12395.4	−12442 .9	+2632.66
+1677.83 + 3450 .99 i	+ 1677.83 − 3450 .99 i	+1865.18 + 66.24 i	+1865.18 − 66.24 i	+8131.79	−2328.62	−2351 .33	+482.136	+5412.91

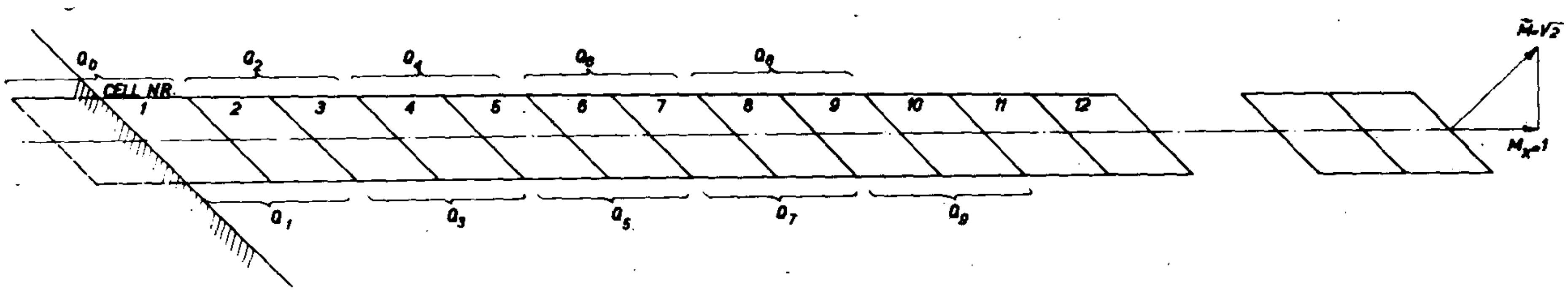
= 10^6 × matrix V_1

Table 9.4 Values of the 10 column matrices $G_2^{\bullet}$ from (9.19) after substitution of the 9 sets λ_i, R_i (i = 1...10) given in table 9.2. Multiplied by 10^6.

λ_1, R_1	λ_2, R_2	λ_3, R_3	λ_4, R_4	λ_5, R_5	λ_6, R_6	λ_7, R_7	λ_8, R_8	λ_9, R_9
−2786.94−2275.35 i	−2786.94 + 2275 .35 i	−1113.16 + 727.946 i	−1113.16 − 727.946 i	−1524.78	+5597.15	+2971.19	+4449 .54	−639.76
−831.508−1725.99 i	−831.508 + 1725 .99 i	−934.112 −43.2113 i	−934.112+43.2113 i	−4071.27	+1182.13	+1091.33	− 859 .20	−4941.66
+120.746−393.417 i	+120.746 + 393.417 i	−213.780 −107.38 i	−213.780+1071.38 i	+ 321.962	+1445.76	−3890.37	−3855 .03	+1511.46
−570.994−1116.09 i	−570.994+1116.09 i	−717.227−31.0948 i	−717.227+31.0948 i	−2527.49	+ 693.99	− 405.21	−7076.11	−3957.40
0	0	0	0	0	0	0	0	0

= 10^6 × matrix V_2

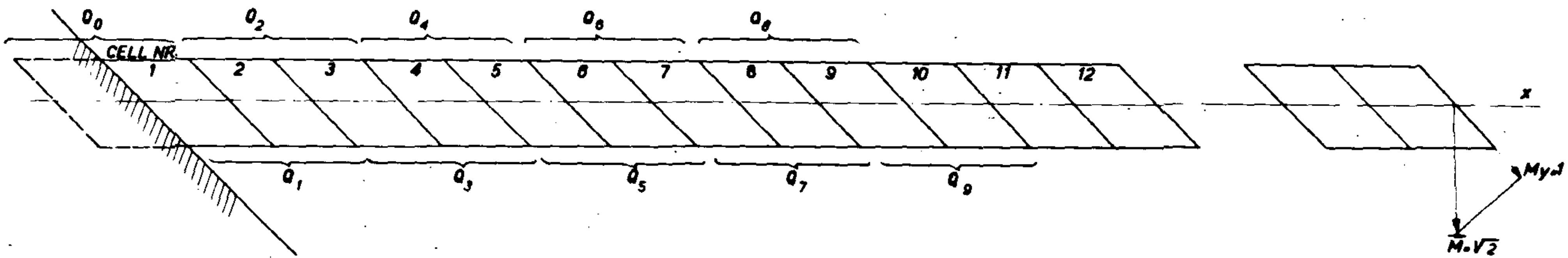
Table 9.5 Participation factors of the supplementary stress systems representing the root perturbation stresses at load (M_x = 1, 0, 0). Multiplied by 10^6.



		Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9		Q_∞	Z_n of (8.74)
participation factors (mult. by 10^6) of the supplementary stress systems in cell nr		1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10			all pairs of cells
	n	0	1	2	3	4	5	6	7	8	9			
Type fig. 7.3	$q_{1.n}$	+2816.42	+ 984.19	+212.11	+13.24	-11.18	-5.28	-1.25	-0.12	+0.04	+0.02		0	0
Type fig. 7.4	$q_{2.n}$	+ 894.19	+ 394.89	+211.77	+80.31	+13.36	-1.33	-1.35	-0.41	-0.07	+0.00		0	-2394.75
Type fig. 7.7	$q_{3.n}$	+ 282.48	+ 398.06	+ 28.28	-49.44	-17.06	-2.16	+0.22	+0.20	+0.07	+0.01		0	0
Type fig. 7.8	$q_{4.n}$	- 43.91	- 34.56	+118.79	+22.78	- 1.79	-1.64	-0.41	-0.07	-0.00	+0.00		0	+ 63.90
Type fig. 8.8	$q_{5.n}$	+ 164.31	- 167.83	+ 7.90	+24.18	+ 7.99	+0.94	-0.15	-0.11	-0.03	-0.01		0	+ 821.84

Solution for the infinite beam for the given load (mult. by 10^6). This column has to be added to all the other columns to obtain total participation factors.
The total state of stress also contains the basic stress system fig.8.5.

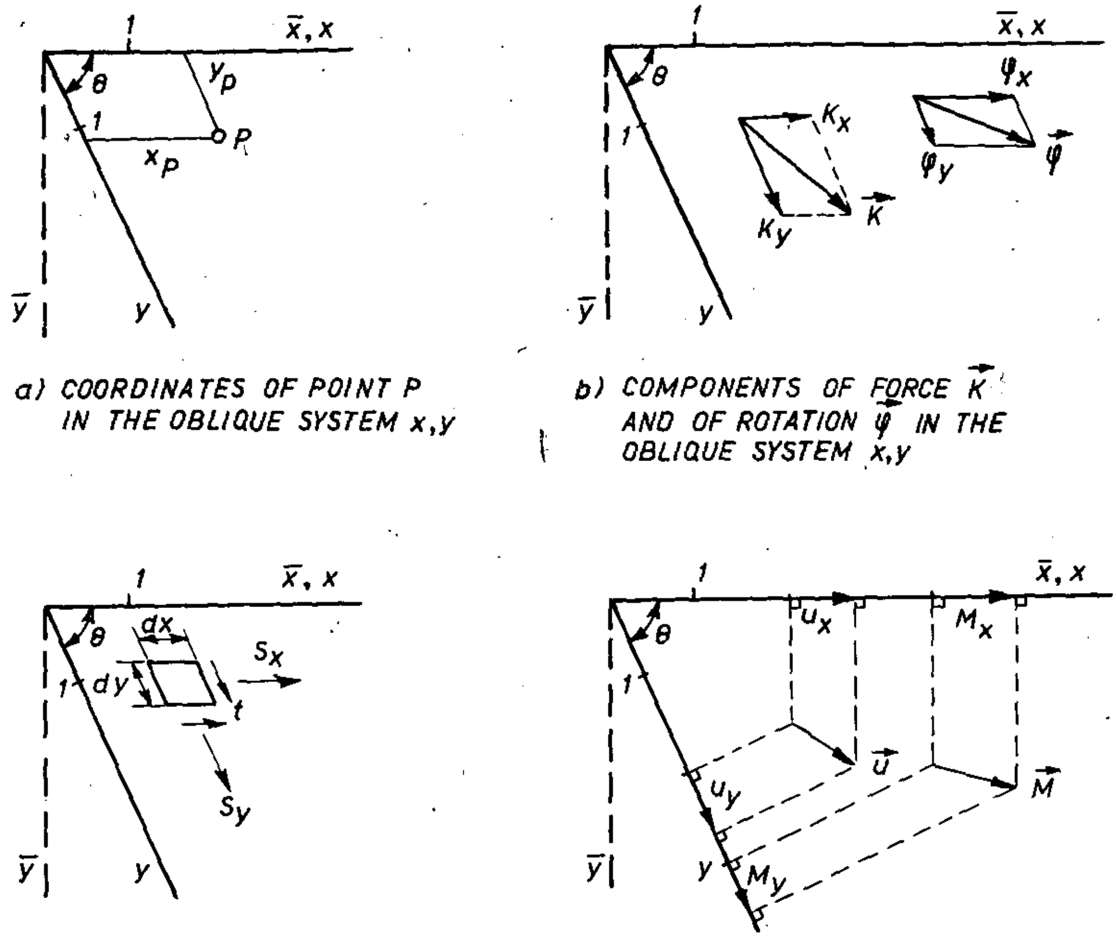
Table 9.6 Participation factors of the supplementary stress systems representing the root perturbation stresses at load (M_y = 1.0). Multiplied by 10^6.



		Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_5	Q_7	Q_8	Q_9		Q_∞	Z_n of (8.75)
participation factors (mult.by 10^6) of the supplementary stress systems in cell nr		1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10			
	n	0	1	2	3	4	5	6	7	8	9			
Type fig.7.3	$q_{1.n}$	−1937.78	−584.74	−103.66	+ 2.08	+ 8.43	+3.13	+0.63	+0.03	−0.03	−0.01		0	0
Type fig.7.4	$q_{2.n}$	− 70.67	−331.67	−156.53	−44.30	− 4.54	+1.49	+0.01	+0.00	+0.00	+0.00		0	−3085.90
Type fig.7.7	$q_{3.n}$	− 806.71	−196.30	+ 35.13	+33.01	+ 8.42	+0.79	−0.21	−0.13	−0.04	−0.01		0	0
Type fig.7.8	$q_{4.n}$	− 6.19	−119.40	− 63.18	−7.13	+ 1.81	+0.93	+0.22	+0.03	−0.00	−0.00		0	−11.4476
Type fig.8.8	$q_{5.n}$	+ 431.19	+ 51.96	− 23.13	−16.25	− 3.91	−0.31	+0.13	+0.07	+0.02	+0.00		0	−147.236

Solution for the infinite beam for the given load (mult. by 10^6). This column has to be added to all the other columns to obtain total participation factors.
The total state of stress also contains the basic stress system fig.8.6.

FIG. 3.1 OBLIQUE COORDINATES.



a) COORDINATES OF POINT P IN THE OBLIQUE SYSTEM x,y

b) COMPONENTS OF FORCE $\vec{K}$ AND OF ROTATION $\vec{\varphi}$ IN THE OBLIQUE SYSTEM x,y

c) STRESS FLOW COMPONENTS IN THE OBLIQUE SYSTEM x,y

d) PROJECTIONS OF DISPLACEMENT $\vec{u}$ AND OF MOMENT $\vec{M}$ IN THE OBLIQUE SYSTEM x, y

FIG. 3.2 THE CONTINUOUSLY STIFFENED PLATE.

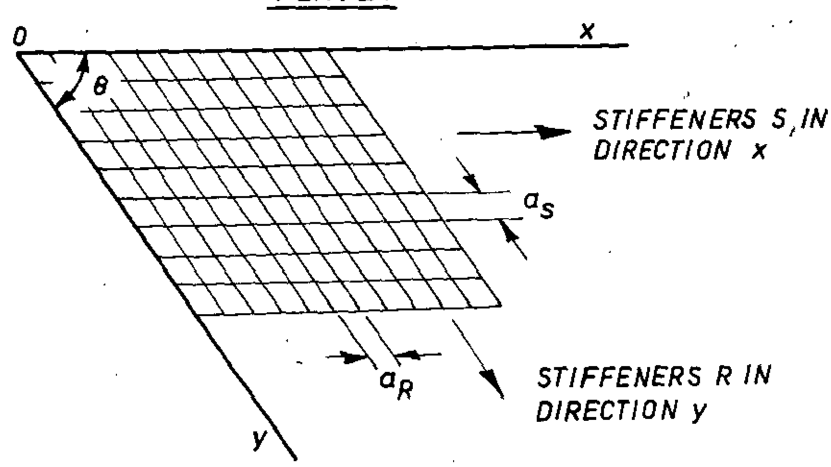


FIG. 3.3 RECTANGULAR PLATE WITH A SYSTEM OF CLOSELY SPACED STIFFENERS.

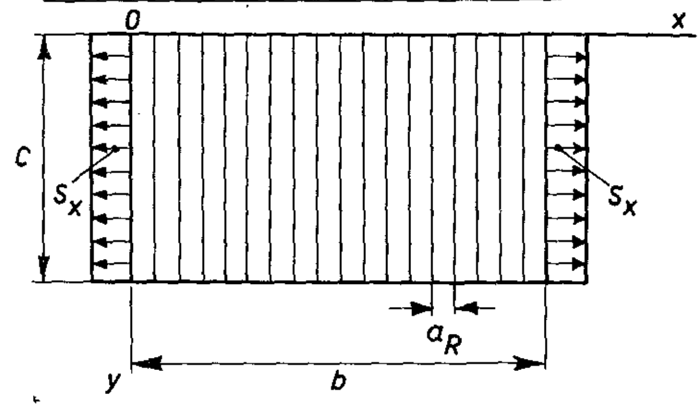


FIG. 4.1 OBLIQUE COORDINATES OF FIGURE 3.1 EXTENDED TO THREE DIMENSIONS. THE z-AXIS IS PERPENDICULAR TO THE x AND y AXIS.

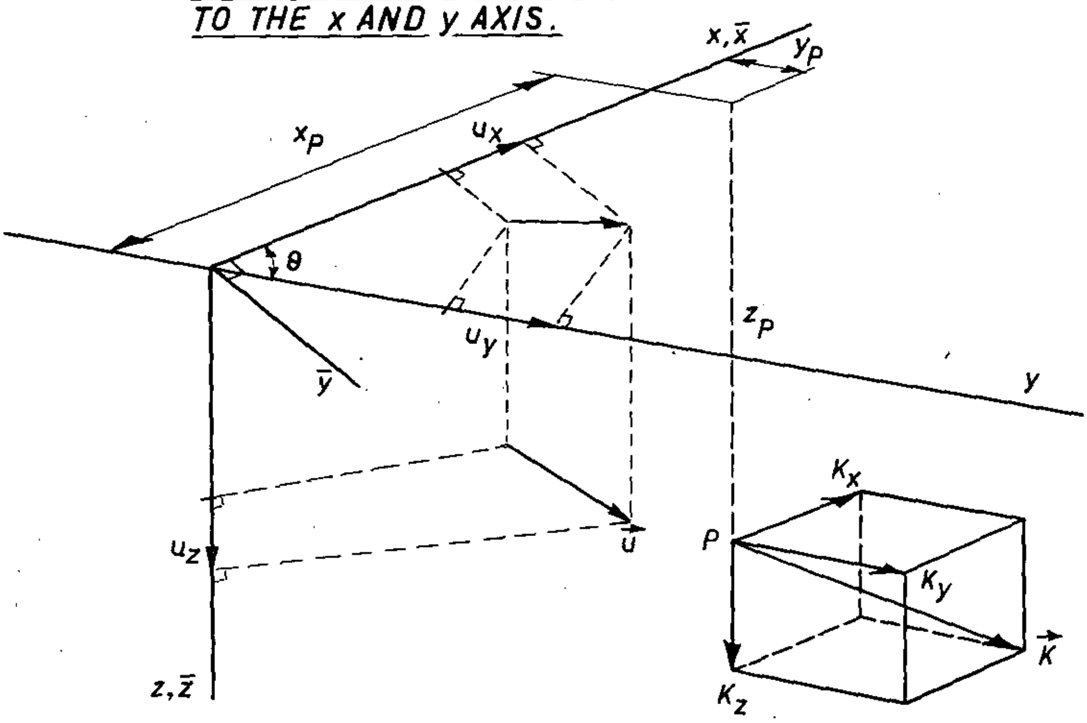


FIG. 4.2 MOMENT $\vec{M}$ PERPENDICULAR TO y AND z AXIS WITH PROJECTIONS $M_x, M_y = 0, M_z = 0$

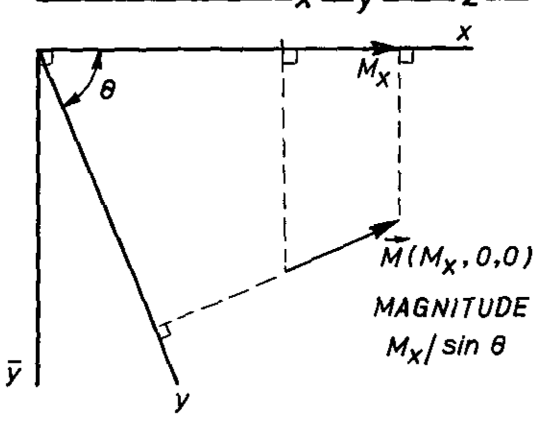


FIG. 4.3 MOMENT $\vec{M}$ PERPENDICULAR TO x AND z AXIS WITH PROJECTIONS $M_x = 0, M_y, M_z = 0$

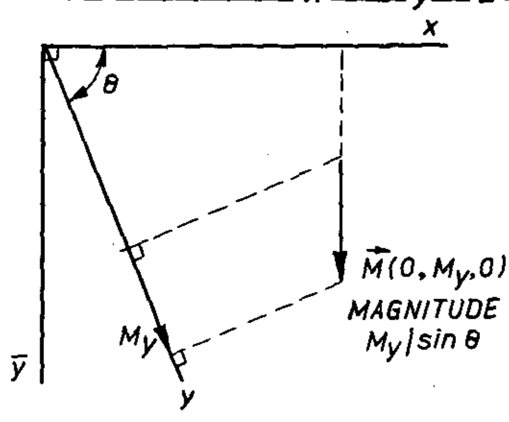


FIG.5.1. RECTANGULAR PLATE WITH STIFFENERS.

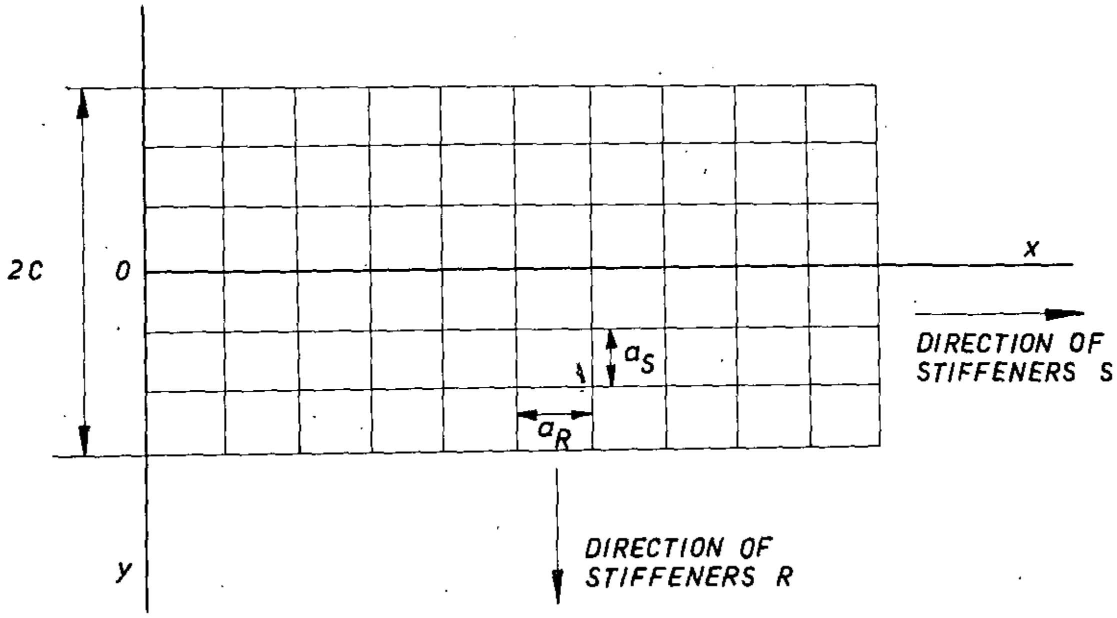


FIG.5.2 PARALLELOGRAM SHAPED PLATE WITH STIFFENERS.

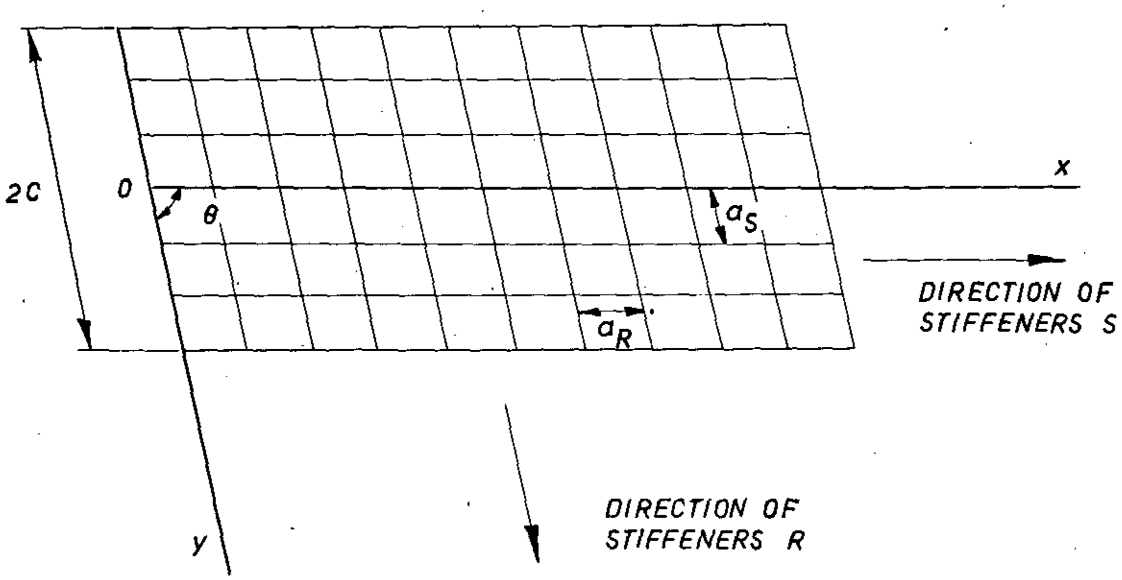


FIG. 7.1 PLANFORM AND CROSS SECTION OF THE 45° SWEPT BOX WITH RIGHTHANDED OBLIQUE COORDINATE SYSTEM.

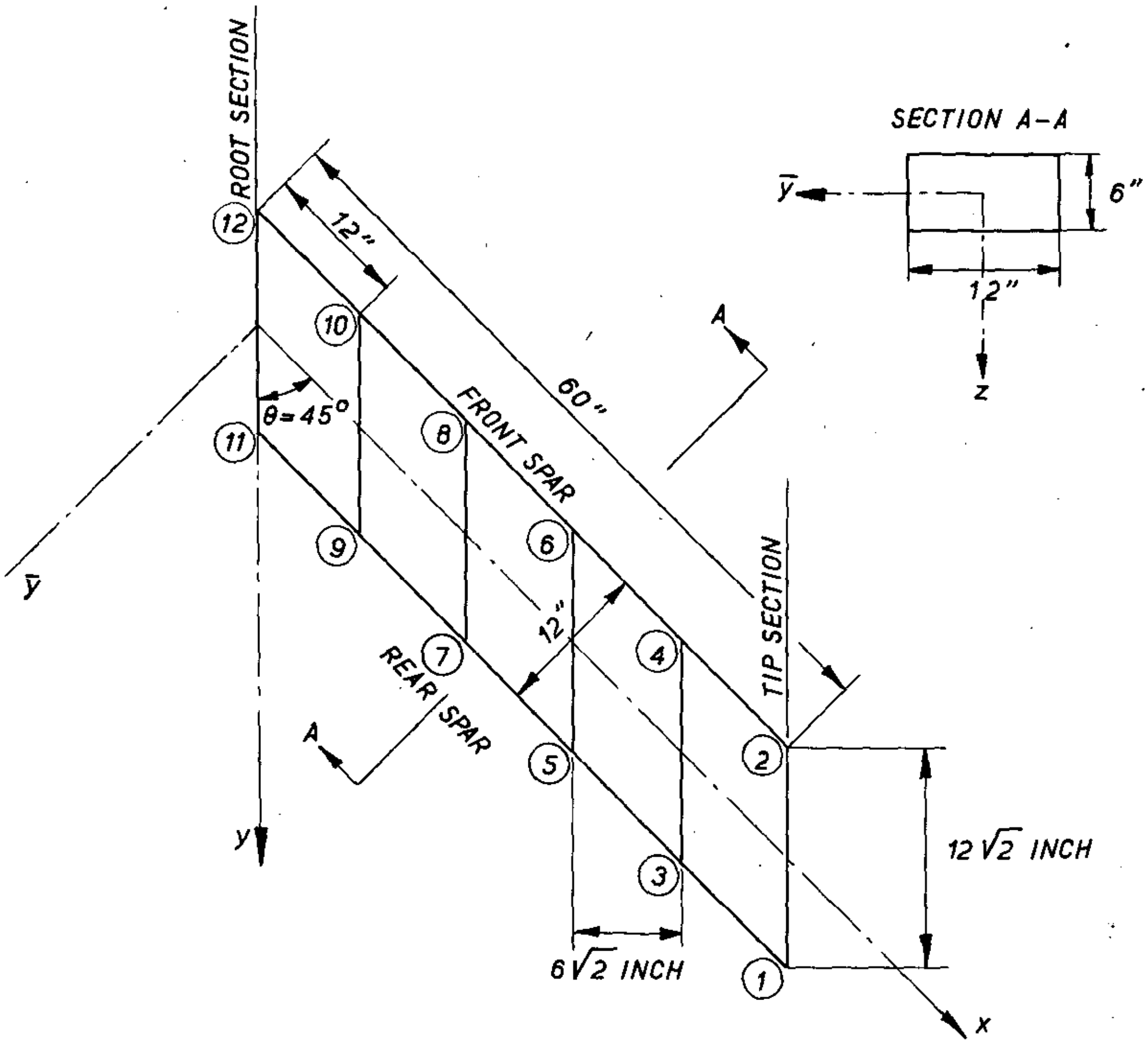


FIG. 7.2 CROSS SECTION A-A OF THE SWEPT BOX OF FIG. 7.1 WITH THE DIMENSIONS OF THE STRUCTURAL ELEMENTS.

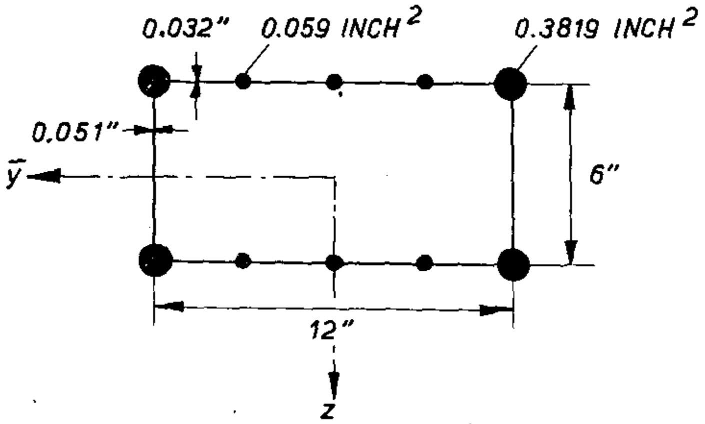


FIG.7.3 SUPPLEMENTARY STRESS SYSTEM. TYPE 1.
FORCES IN lbs. STRESSFLOWS IN lbs/inch.

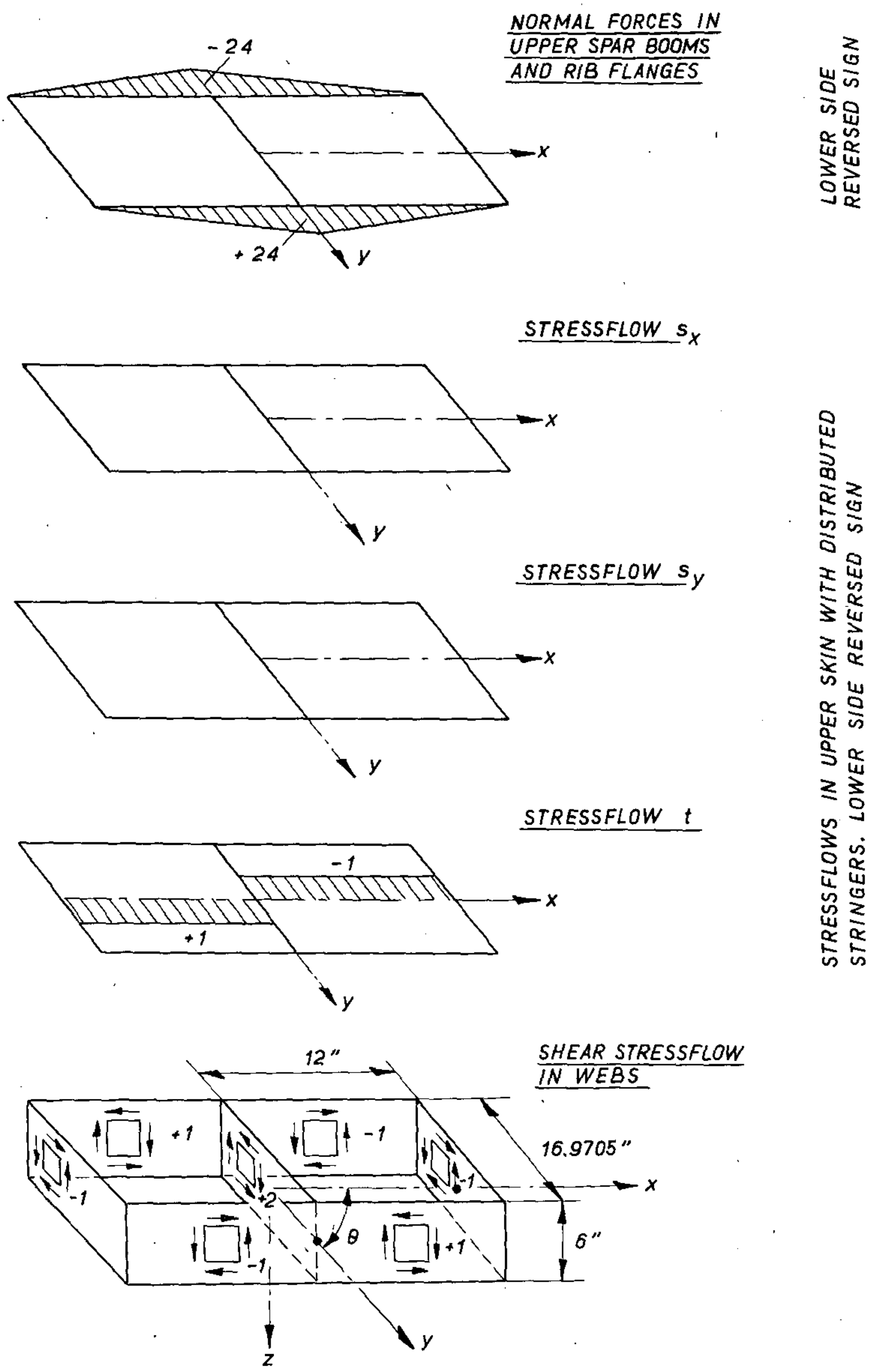


FIG.7.4 SUPPLEMENTARY STRESS SYSTEM. TYPE 2.
FORCES IN lbs. STRESSFLOWS IN lbs/inch.

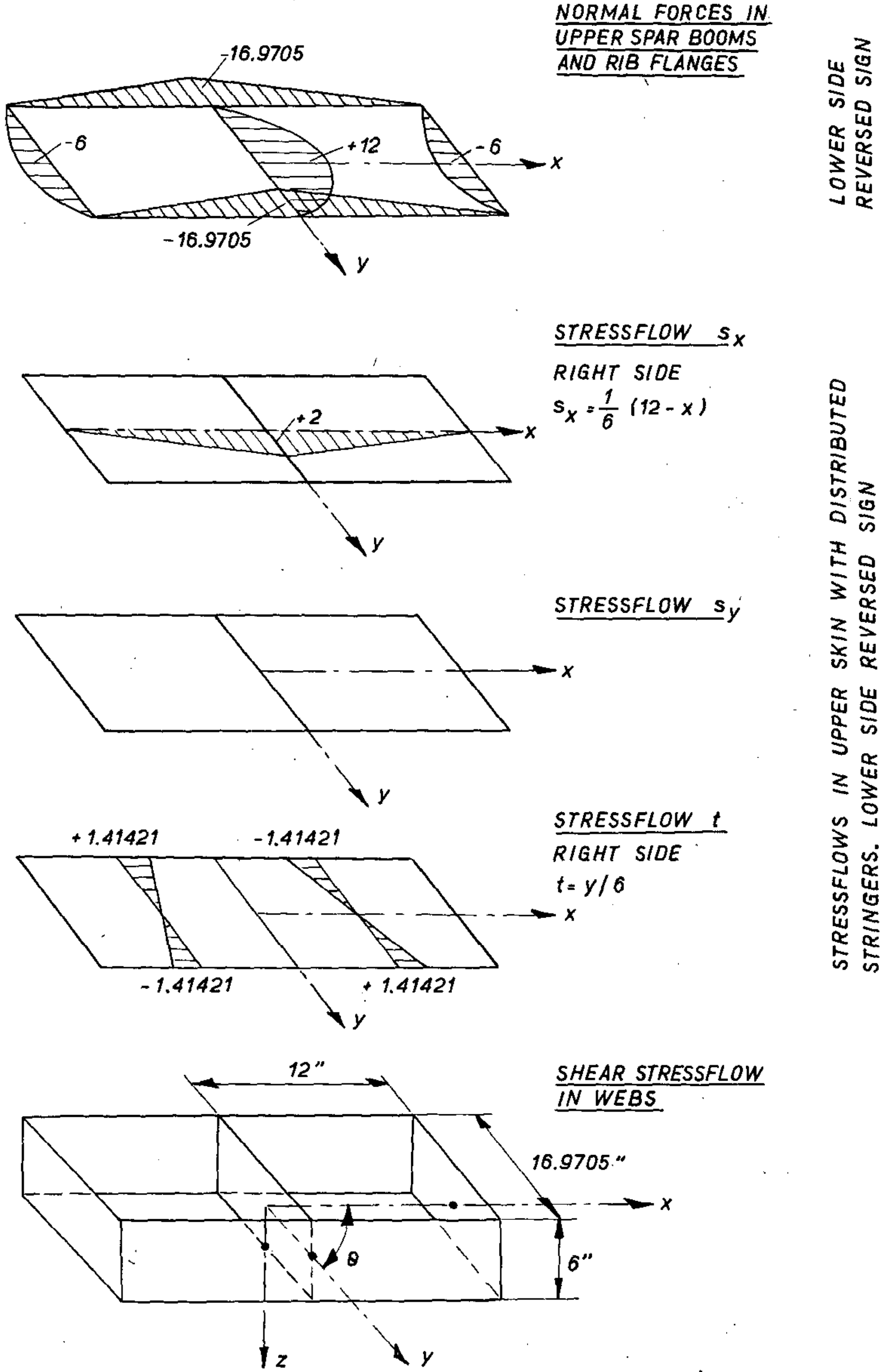
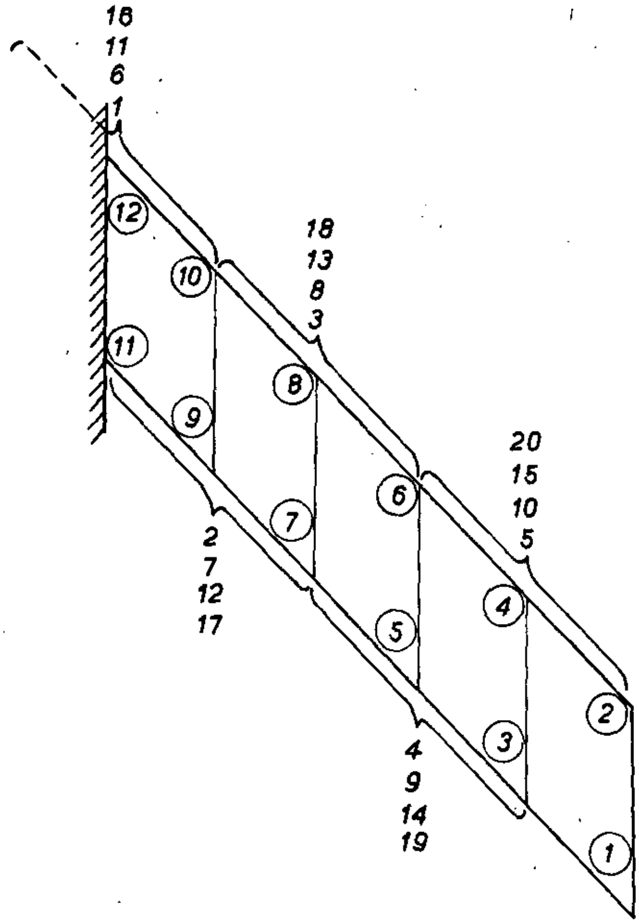


FIG.7.5 POSITION OF SUPPLEMENTARY STRESS SYSTEMS.



Nr 1 , 2 , 3 , 4 , 5	TYPE 1,	FIG. 7.3
6 , 7 , 8 , 9 , 10	TYPE 2,	FIG. 7.4
11 , 12 , 13 , 14 , 15	TYPE 3,	FIG. 7.7
16 , 17 , 18 , 19 , 20	TYPE 4,	FIG. 7.8

FIG. 7.6 BASIC STRESS SYSTEMS FOR THE m LOADING CASES.

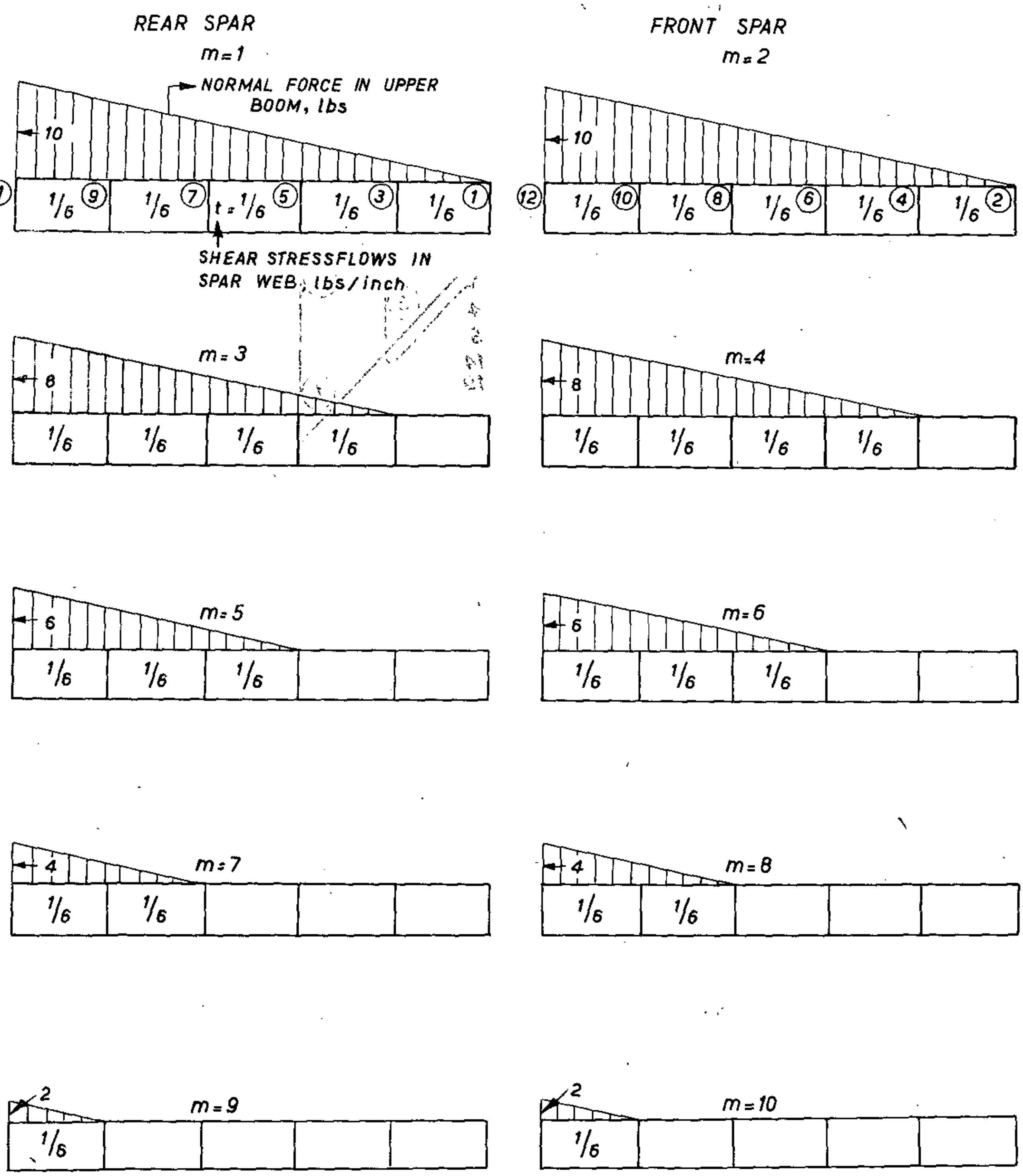


FIG. 7.7 SUPPLEMENTARY STRESS SYSTEM. TYPE 3.
FORCES IN lbs. STRESSFLOWS IN lbs/inch.

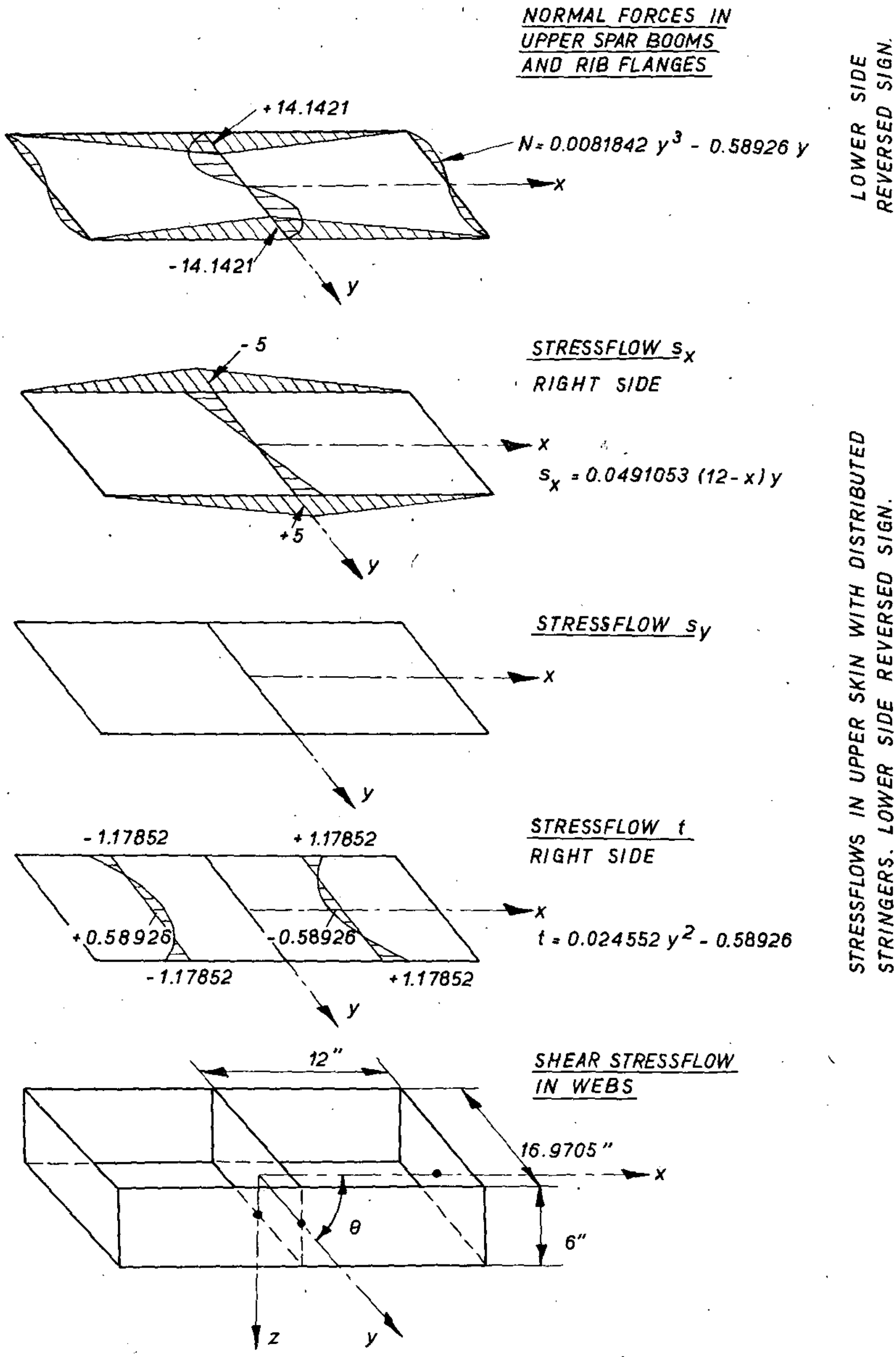


FIG.7.8 SUPPLEMENTARY STRESS SYSTEM. TYPE 4.
FORCES IN lbs. STRESSFLOWS IN lbs/inch.

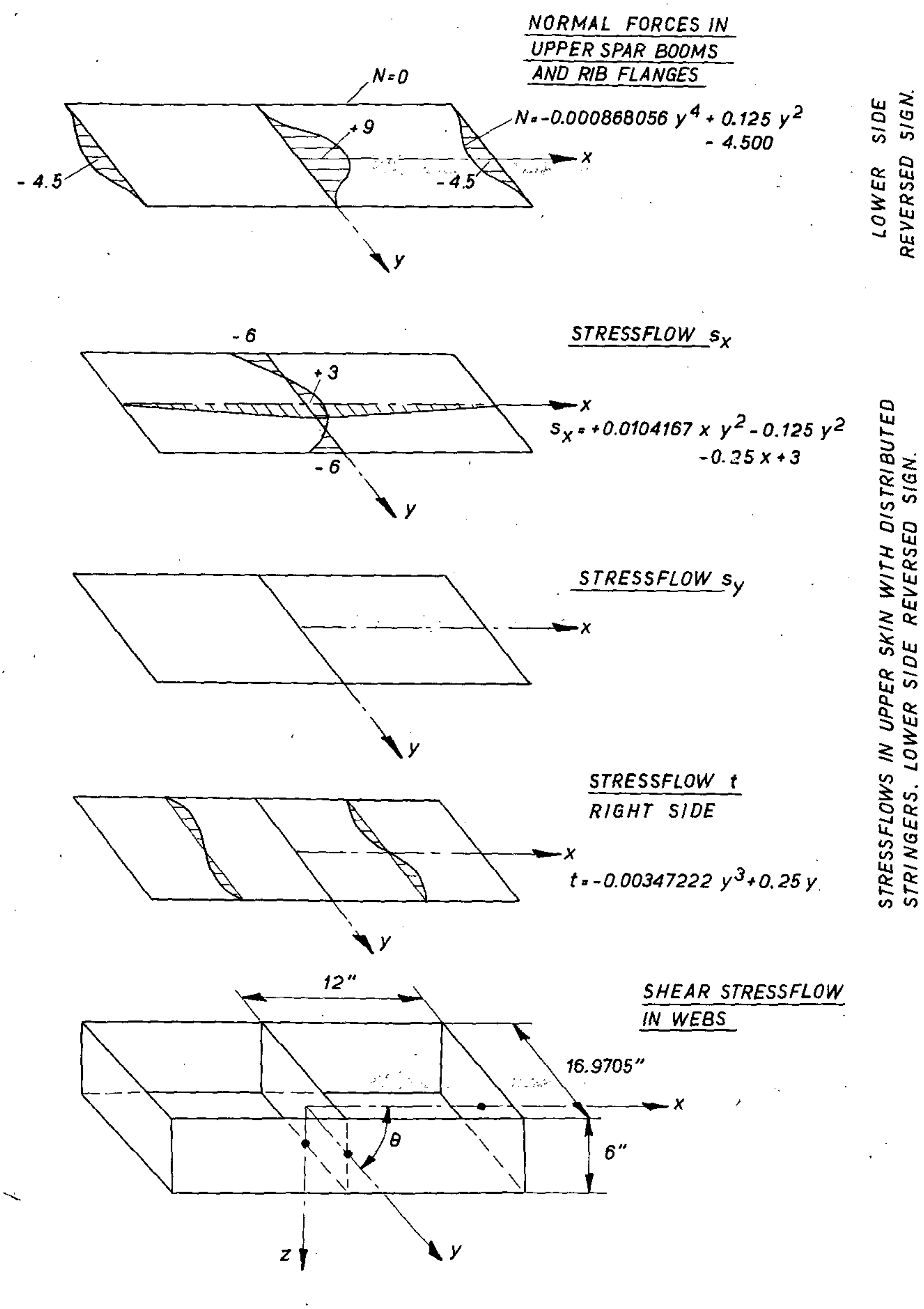
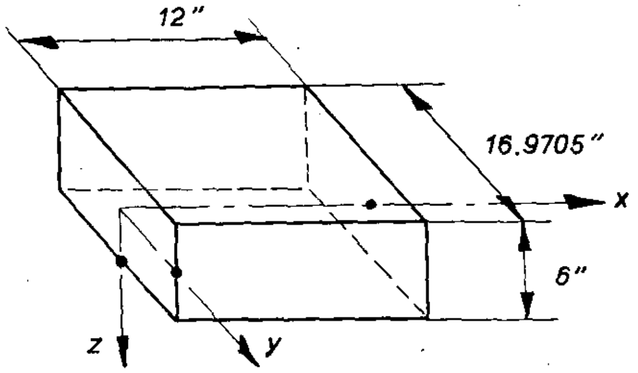


FIG.7.9 SUPPLEMENTARY STRAIN SYSTEM. TYPE 1.
DISPLACEMENTS IN INCHES

VERTICAL DISPLACEMENTS OUT OF CELL : ZERO
DISPLACEMENTS AT THE CORNERS OF THE CELL : ZERO



DISPLACEMENTS WITHIN CELL

UPPER SKIN

$10^3 u_x = \frac{1}{2}x^2 - 6x$

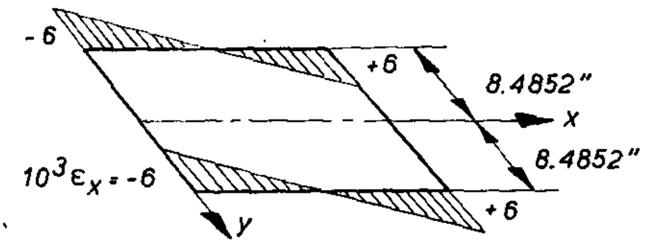
$u_y = 0$

LOWER SKIN REVERSED SIGN

SPAR WEBS

$10^3 u_x = \frac{-1}{6}x^2 z + 2xz$

$10^3 u_z = \frac{1}{18}x^3 - x^2 + 4x$



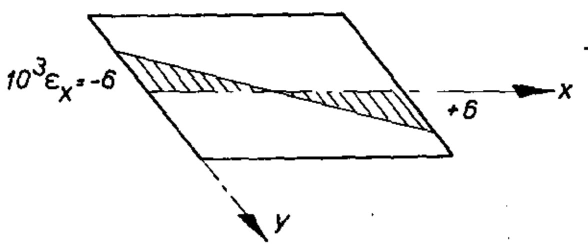
NORMAL STRAINS IN UPPER BOOMS

SPAR BOOMS AND RIB FLANGES

$10^3 \varepsilon_x = x - 6$

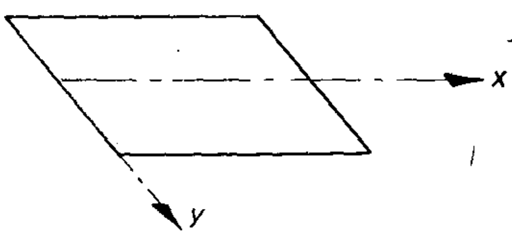
RIB FLANGES $\varepsilon_x = 0$

LOWER SIDE REVERSED SIGN



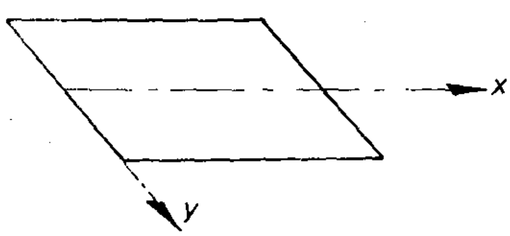
STRAIN ε_x IN UPPER SKIN

$10^3 \varepsilon_x = x - 6$



STRAIN ε_y IN UPPER SKIN

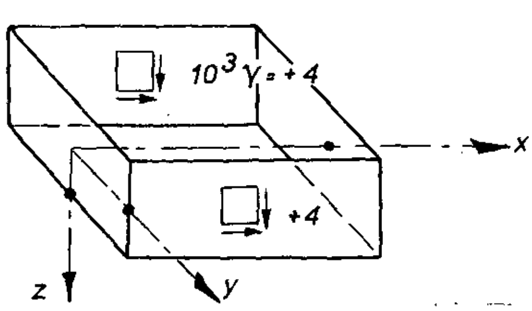
$\varepsilon_y = 0$



STRAIN γ IN UPPER SKIN

$\gamma = 0$

LOWER SKIN REVERSED SIGN



STRAINS IN WEBS

SPAR WEBS

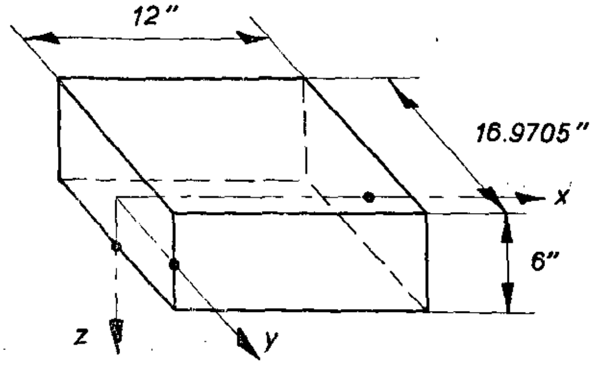
$(10^3 \varepsilon_x = -\frac{1}{3}xz + 2z)$

$\varepsilon_z = 0$

$10^3 \gamma = 4$

FIG. 7.10 SUPPLEMENTARY STRAIN SYSTEM. TYPE 2.
DISPLACEMENTS IN INCHES

VERTICAL DISPLACEMENTS OUT OF CELL: ZERO
DISPLACEMENTS AT THE CORNERS OF THE CELL: ZERO



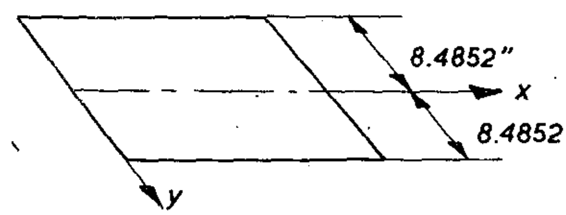
DISPLACEMENTS WITHIN CELL

UPPER SKIN

$u_x = 0$

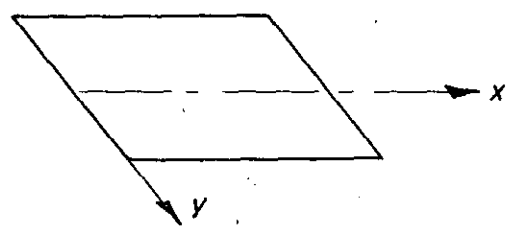
$10^3 u_y = \frac{1}{2}x^2 - 6x$

LOWER SKIN REVERSED SIGN

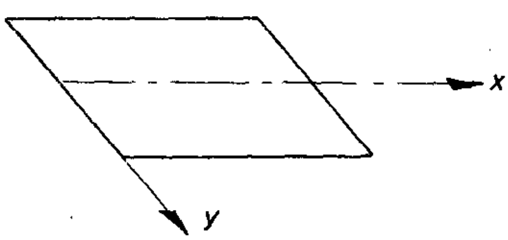


NORMAL STRAINS IN UPPER SPAR BOOMS AND RIB FLANGES

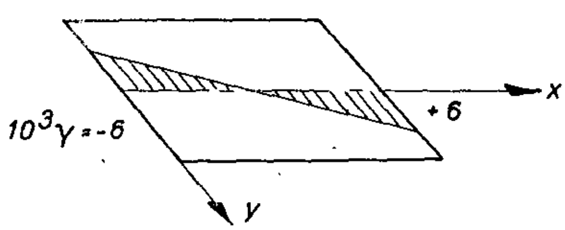
LOWER BOOMS REVERSED SIGN



STRAIN ε_x IN UPPER SKIN



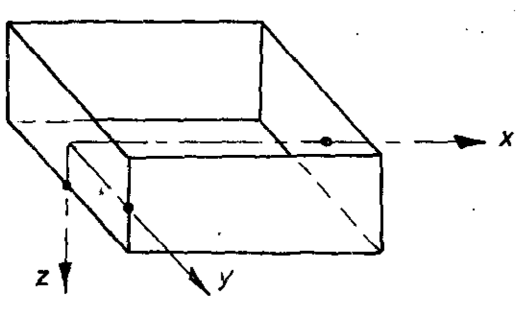
STRAIN ε_y IN UPPER SKIN



STRAIN γ IN UPPER SKIN

$10^3 \gamma = x - 6$

LOWER SKIN REVERSED SIGN

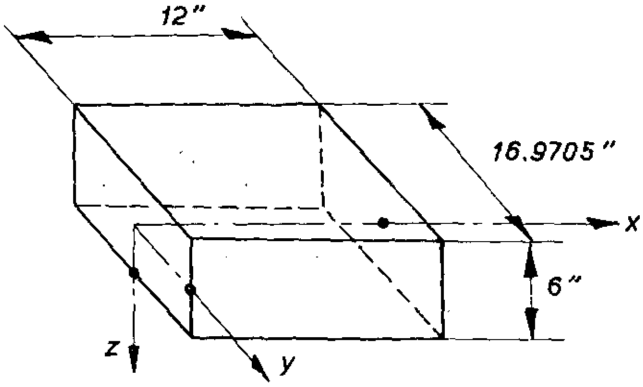


STRAINS IN WEBS

FIG. 7.11 SUPPLEMENTARY STRAIN SYSTEM. TYPE 3. DISPLACEMENTS IN INCHES

VERTICAL DISPLACEMENTS OUT OF CELL: ZERO.

DISPLACEMENTS AT THE CORNERS OF THE CELL: ZERO



DISPLACEMENTS WITHIN CELL

UPPER SKIN

$10^3 u_x = (\frac{1}{2}x^2 y - 6xy) / 8.4852$

$10^3 u_y = (-\frac{1}{6}x^3 + 3x^2 - 12x) / 8.4852$

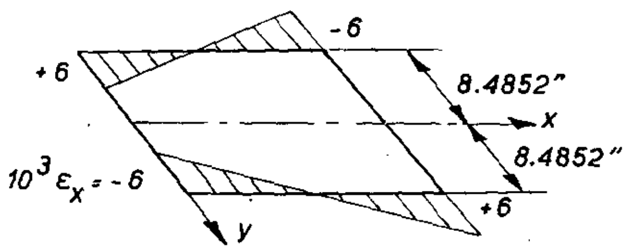
LOWER SKIN REVERSED SIGN

SPAR WEB $y = 8.4852$

$10^3 u_x = -\frac{1}{6}x^2 z + 2xz$

$10^3 u_z = \frac{1}{18}x^3 - x^2 + 4x$

SPAR WEB $y = -8.4852$ REVERSED SIGN



NORMAL STRAINS IN UPPER SPAR BOOMS AND RIB FLANGES

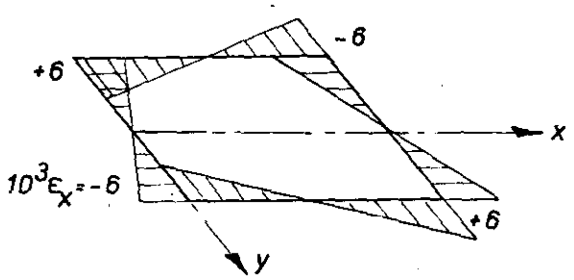
SPAR BOOM $y = 8.4852$

$10^3 \varepsilon_x = x - 6$

SPAR BOOM $y = -8.4852$

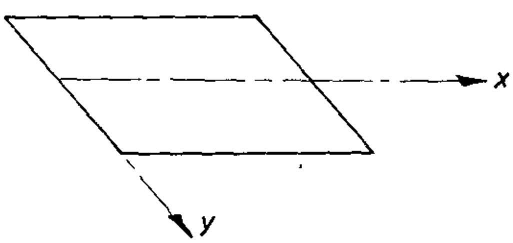
$10^3 \varepsilon_x = -x + 6$

LOWER SIDE REVERSED SIGN



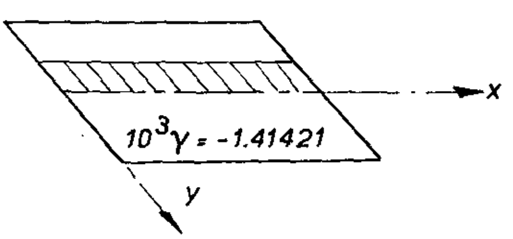
STRAIN ε_x IN UPPER SKIN

$$10^3 \varepsilon_x = \frac{xy - 6y}{8.4852} = \frac{y(x-6)}{8.4852}$$



STRAIN ε_y IN UPPER SKIN

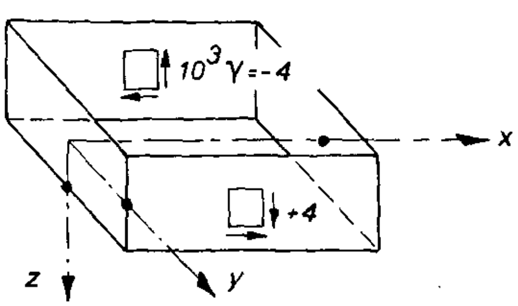
$\varepsilon_y = 0$



STRAIN γ IN UPPER SKIN

$$10^3 \gamma = \frac{-12}{8.4852} = -1.41421$$

LOWER SKIN REVERSED SIGN



STRAINS IN WEBS

SPAR WEB $y = 8.4852$

$(10^3 \varepsilon_x = \frac{1}{3}xz + 2z)$

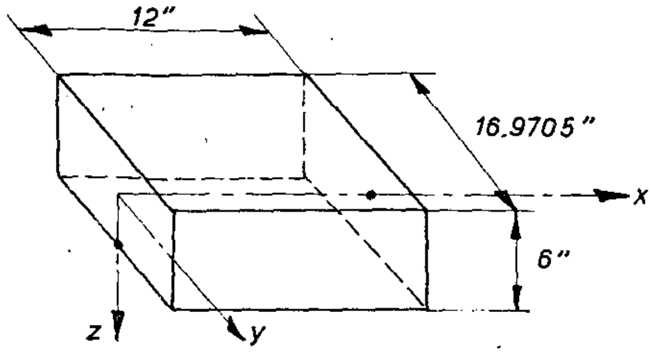
$\varepsilon_z = 0$

$10^3 \gamma = 4$

SPAR WEB $y = -8.4852$ REVERSED SIGN

FIG.7.12 SUPPLEMENTARY STRAIN SYSTEM. TYPE 4.
DISPLACEMENTS IN INCHES.

VERTICAL DISPLACEMENTS OUT OF CELL: ZERO
DISPLACEMENTS AT THE CORNERS OF THE CELL: ZERO



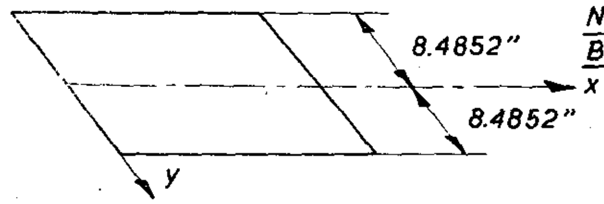
DISPLACEMENTS WITHIN CELL

UPPER SKIN:

$u_x = 0$

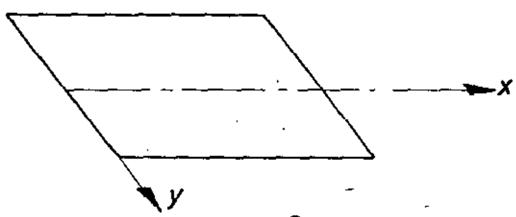
$$10^3 u_y = \frac{x^2 y - 12xy}{8.4852} = \frac{xy(x-12)}{8.4852}$$

LOWER SKIN:
REVERSED SIGN.



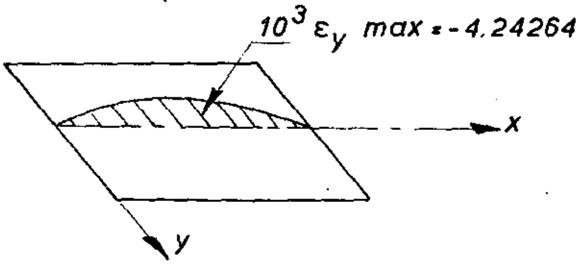
NORMAL STRAINS IN UPPER SPAR BOOMS AND RIB FLANGES

LOWER SIDE REVERSED SIGN



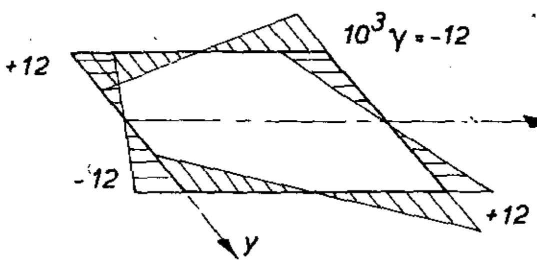
STRAIN ε_x IN UPPER SKIN

$\varepsilon_x = 0$



STRAIN ε_y IN UPPER SKIN

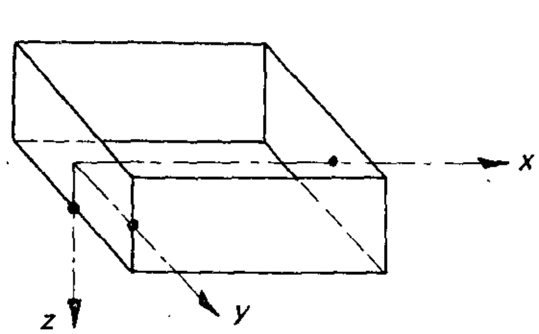
$$10^3 \varepsilon_y = \frac{x^2 - 12x}{8.4852} = 0.117852\, x(x-12)$$



STRAIN γ IN UPPER SKIN

$$10^3 \gamma = \frac{2xy - 12y}{8.4852} = \frac{2y(x-6)}{8.4852}$$

LOWER SKIN REVERSED SIGN



STRAINS IN WEBS

FIG. 7.13 SUPPLEMENTARY STRAIN SYSTEM. TYPE 5.
DISPLACEMENTS IN INCHES

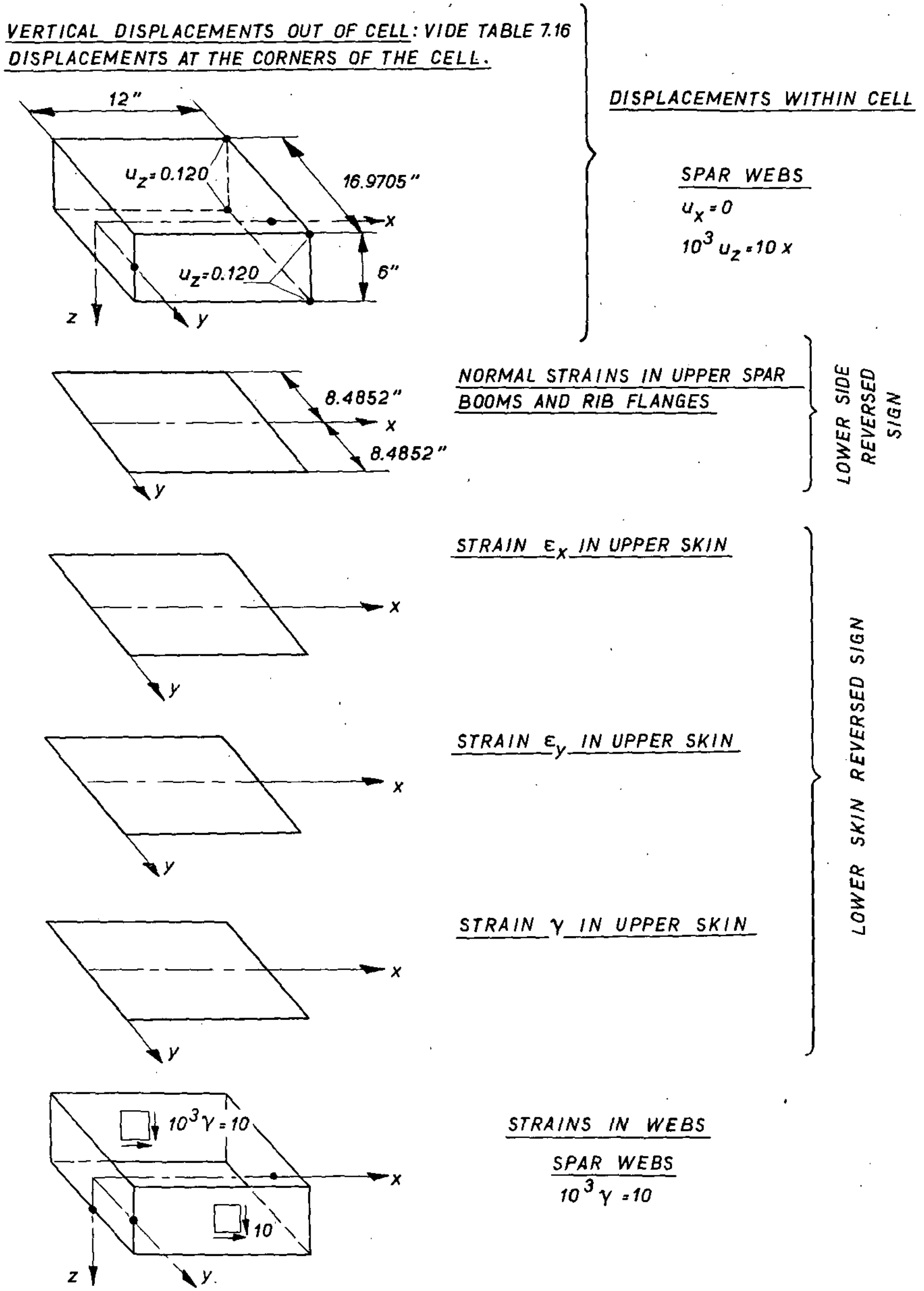
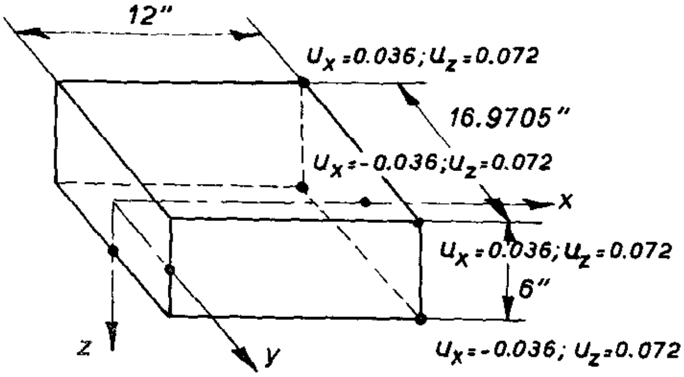


FIG. 7.14 SUPPLEMENTARY STRAIN SYSTEM. TYPE 6.
DISPLACEMENTS IN INCHES

VERTICAL DISPLACEMENTS OUT OF CELL: VIDE TABLE 7.16

DISPLACEMENTS AT THE CORNERS OF THE CELL



DISPLACEMENTS WITHIN CELL

UPPER SKIN

$10^3 u_x = 3x$

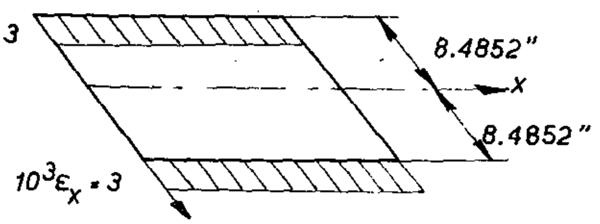
$u_y = 0$

LOWER SKIN REVERSED SIGN

SPAR WEBS

$10^3 u_x = -xz$

$10^3 u_z = \frac{1}{2} x^2$

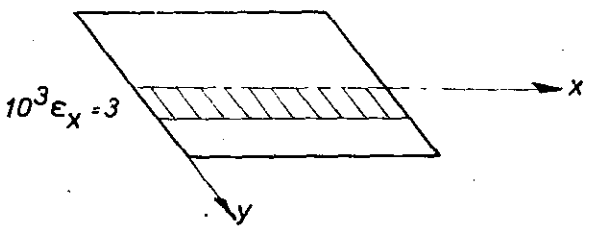


NORMAL STRAINS IN UPPER SPAR BOOMS AND RIB FLANGES

SPAR BOOMS

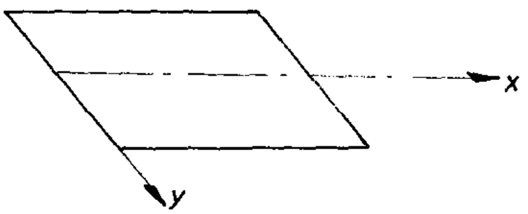
$10^3 \varepsilon_x = 3$

LOWER SIDE REVERSED SIGN



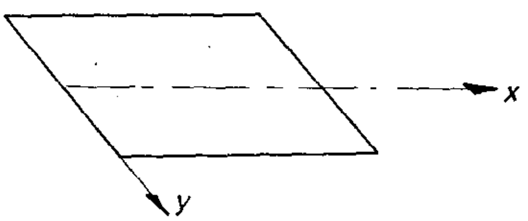
STRAIN ε_x IN UPPER SKIN

$10^3 \varepsilon_x = 3$



STRAIN ε_y IN UPPER SKIN

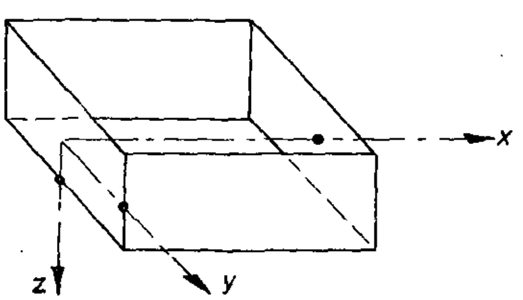
$\varepsilon_y = 0$



STRAIN γ IN UPPER SKIN

$\gamma = 0$

LOWER SKIN REVERSED SIGN



STRAINS IN WEBS

SPAR WEBS

$(10^3 \varepsilon_x = -z)$

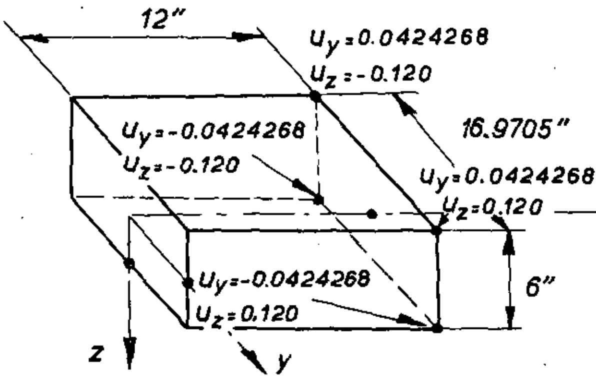
$\varepsilon_z = 0$

$\gamma = 0$

FIG. 7.15 SUPPLEMENTARY STRAIN SYSTEM. TYPE 7.
DISPLACEMENTS IN INCHES

VERTICAL DISPLACEMENTS OUT OF CELL: VIDE TABLE 7.16

DISPLACEMENTS AT THE CORNERS OF THE CELL



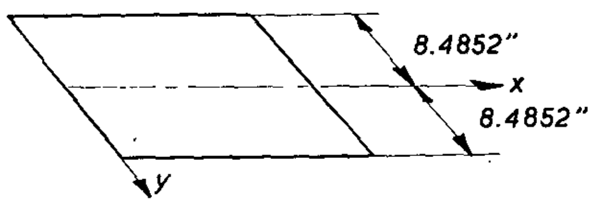
DISPLACEMENTS WITHIN CELL

UPPER SKIN
$u_x = 0$
$10^3 u_y = 3.53557\,x$

LOWER SKIN REVERSED SIGN

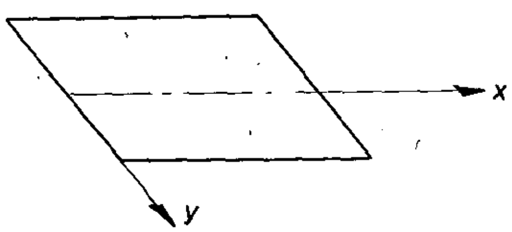
SPAR WEB $y = 8.4852$
$u_x = 0$
$10^3\ u_z = 10\,x$

SPAR WEB $y = -8.4852$ REVERSED SIGN

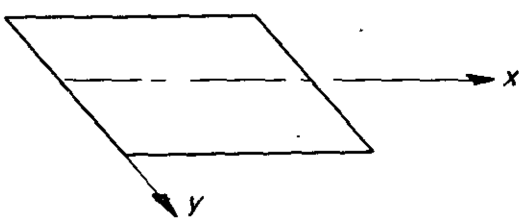


NORMAL STRAINS IN UPPER SPAR BOOMS AND RIB FLANGES

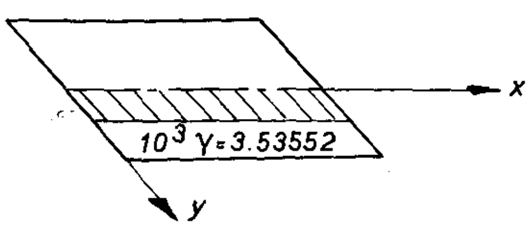
LOWER SIDE REVERSED SIGN



STRAIN ε_x IN UPPER SKIN

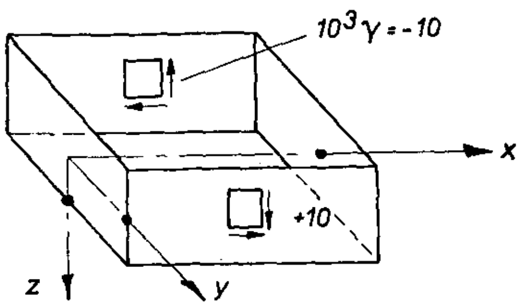


STRAIN ε_y IN UPPER SKIN



STRAIN γ IN UPPER SKIN

LOWER SKIN REVERSED SIGN



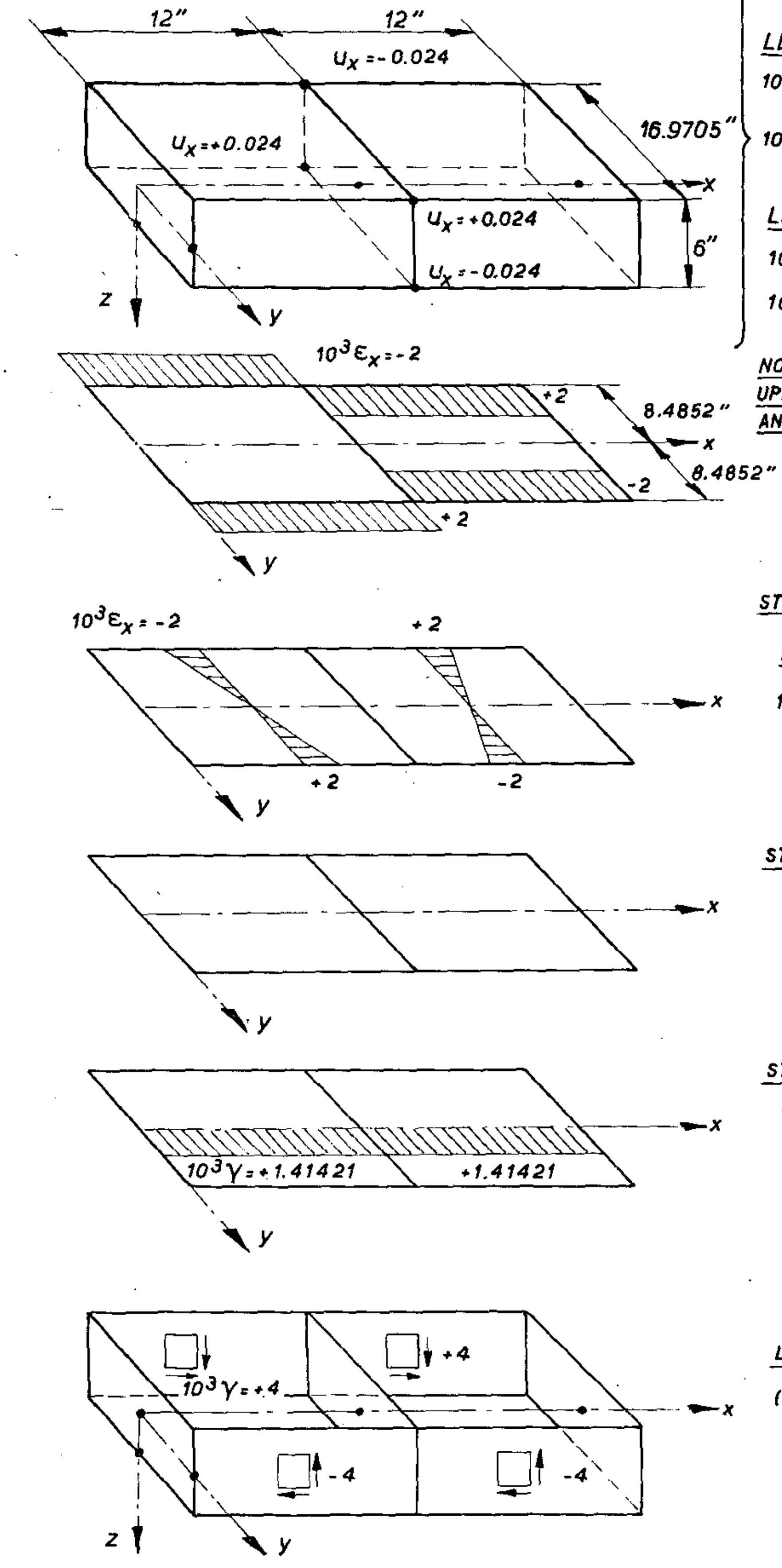
STRAINS IN WEBS

SPAR WEB $y = 8.4852$
$10^3\ \gamma = 10$

SPAR WEB $y = -8.4852$ REVERSED SIGN

FIG. 7.16 SUPPLEMENTARY STRAIN SYSTEM. TYPE 8. DISPLACEMENTS IN INCHES

VERTICAL DISPLACEMENTS OUT OF CELLS : ZERO DISPLACEMENTS AT THE CORNERS OF THE CELLS



DISPLACEMENTS WITHIN LEFT CELL

LEFT UPPER SKIN

$$10^3 u_x = \frac{2xy}{8.4852}$$

$$10^3 u_y = \frac{-2x^2}{16.9705} + 1.41421x$$

(LEFT LOWER SKIN REVERSED SIGN)

LEFT SPAR WEB $y = 8.4852$

$$10^3 u_x = -\frac{2}{3}xz$$

$$10^3 u_z = \frac{1}{3}x^2 - 4x$$

(LEFT SPAR WEB $y = -8.4852$ REVERSED SIGN)

NORMAL STRAINS IN UPPER SPAR BOOMS AND RIB FLANGES

(LOWER SIDE REVERSED SIGN)

STRAIN ε_x IN UPPER SKIN

LEFT SKIN

$$10^3 \varepsilon_x = \frac{2y}{8.4852}$$

STRAIN ε_y IN UPPER SKIN

$$\varepsilon_y = 0$$

STRAIN γ IN UPPER SKIN

$$10^3 \gamma = 1.41421$$

(LOWER SKIN REVERSED SIGN)

STRAINS IN WEBS

LEFT SPAR WEB $y = 8.4852$

$$(10^3 \varepsilon_x = -\frac{2}{3}z)$$

$$10^3 \gamma = -4$$

(LEFT SPAR WEB $y = -8.4852$ REVERSED SIGN)

FIG. 7.17 SUPPLEMENTARY STRAIN SYSTEM. TYPE 9.
DISPLACEMENTS IN INCHES

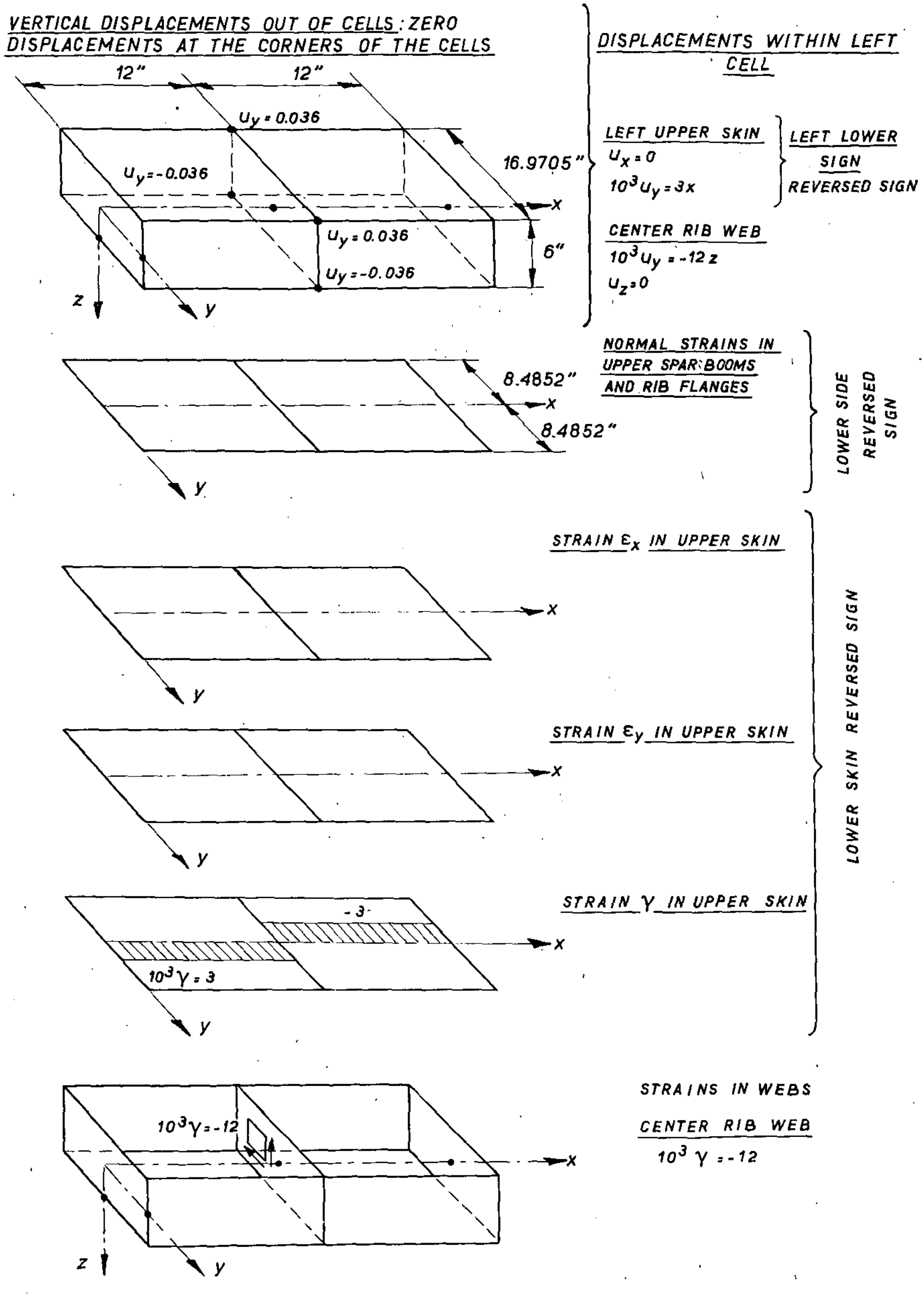


FIG. 7.18 SUPPLEMENTARY STRAIN SYSTEM. TYPE 10.
DISPLACEMENTS IN INCHES

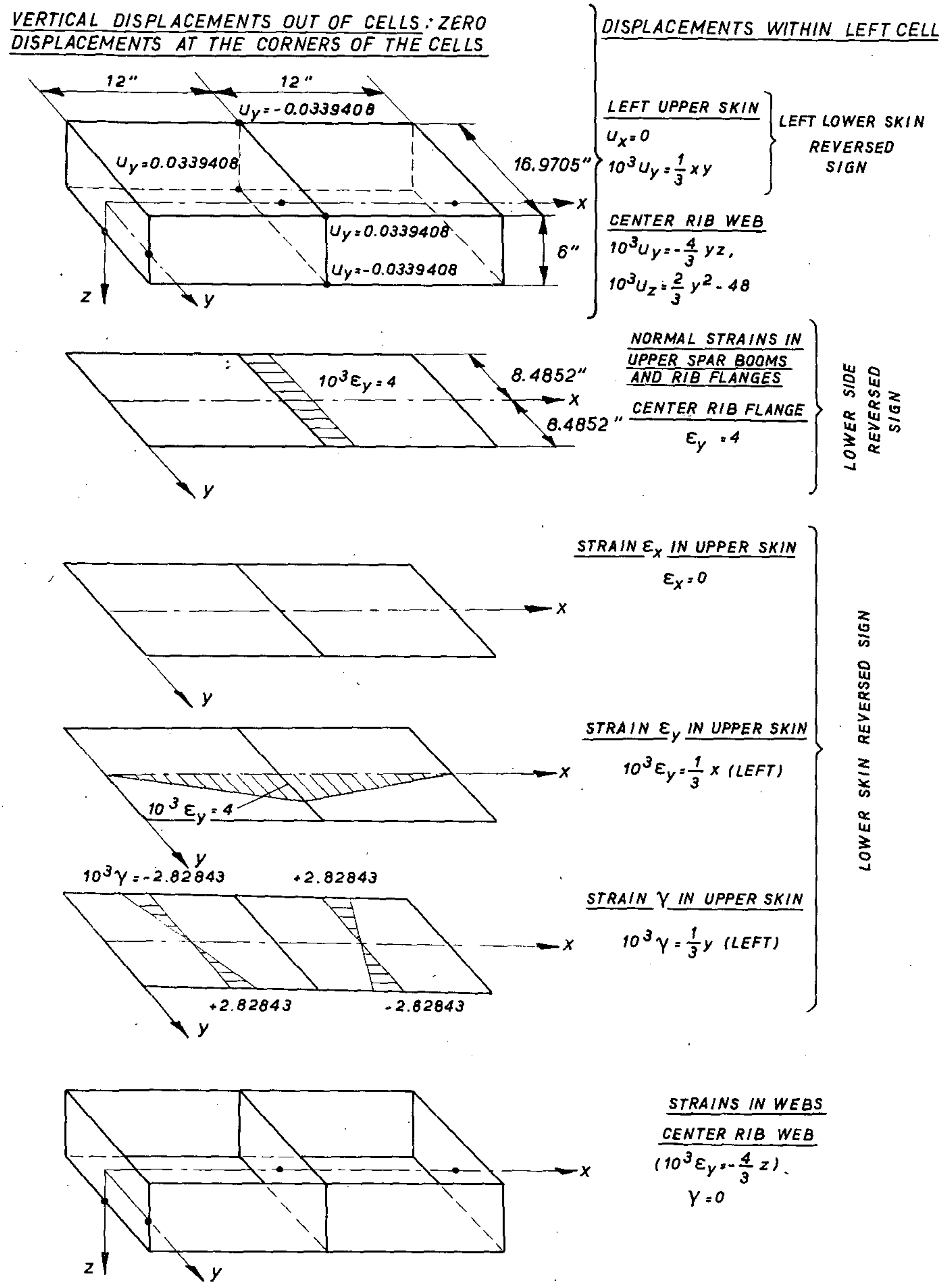
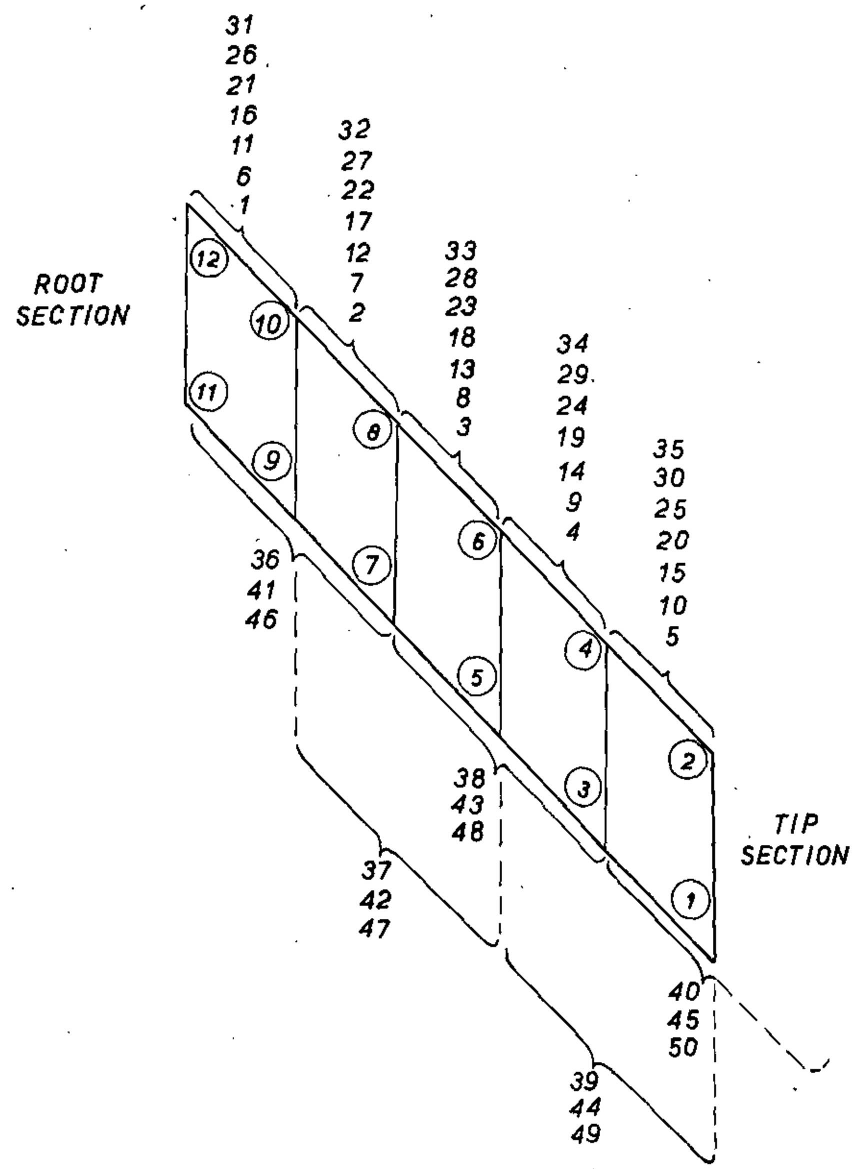


FIG. 7.19. POSITION SUPPLEMENTARY STRAIN SYSTEMS.



NR.	1 --- 5	TYPE 1,	FIG. 7.9
	6 --- 10	TYPE 2,	FIG. 7.10
	11 --- 15	TYPE 3,	FIG. 7.11
	16 --- 20	TYPE 4,	FIG. 7.12
	21 --- 25	TYPE 5,	FIG. 7.13
	26 --- 30	TYPE 6,	FIG. 7.14
	31 --- 35	TYPE 7,	FIG. 7.15
	36 --- 40	TYPE 8,	FIG. 7.16
	41 --- 45	TYPE 9,	FIG. 7.17
	46 --- 50	TYPE 10,	FIG. 7.18

FIG. 7.20. VERTICAL LOAD OF 1 lb IN STATION 3, NORMAL STRESSES IN UPPER SPAR BOOMS.

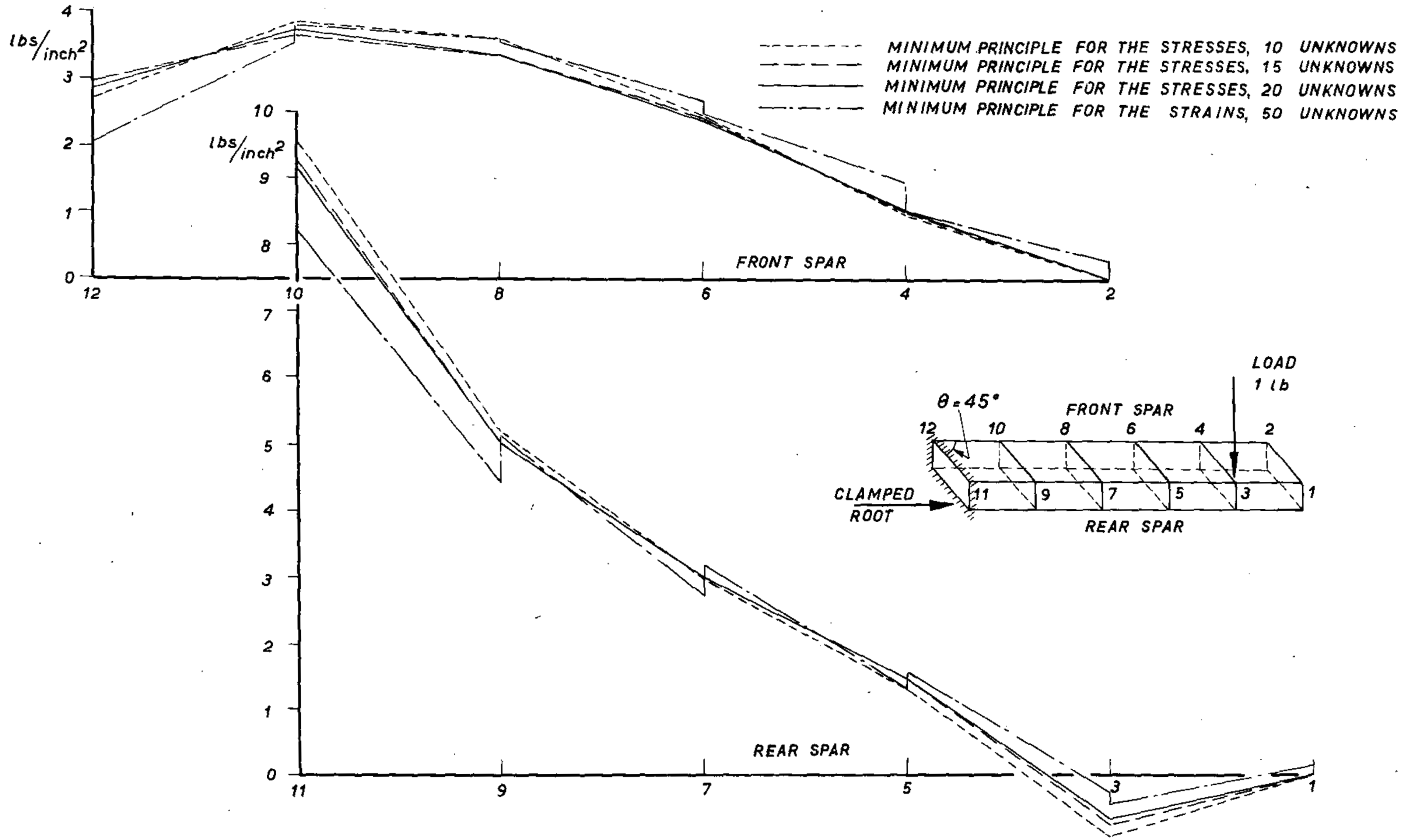


FIG. 7.21. VERTICAL LOAD OF 1 lb IN STATION 3, SHEAR STRESSES IN SPAR AND RIB WEBS.

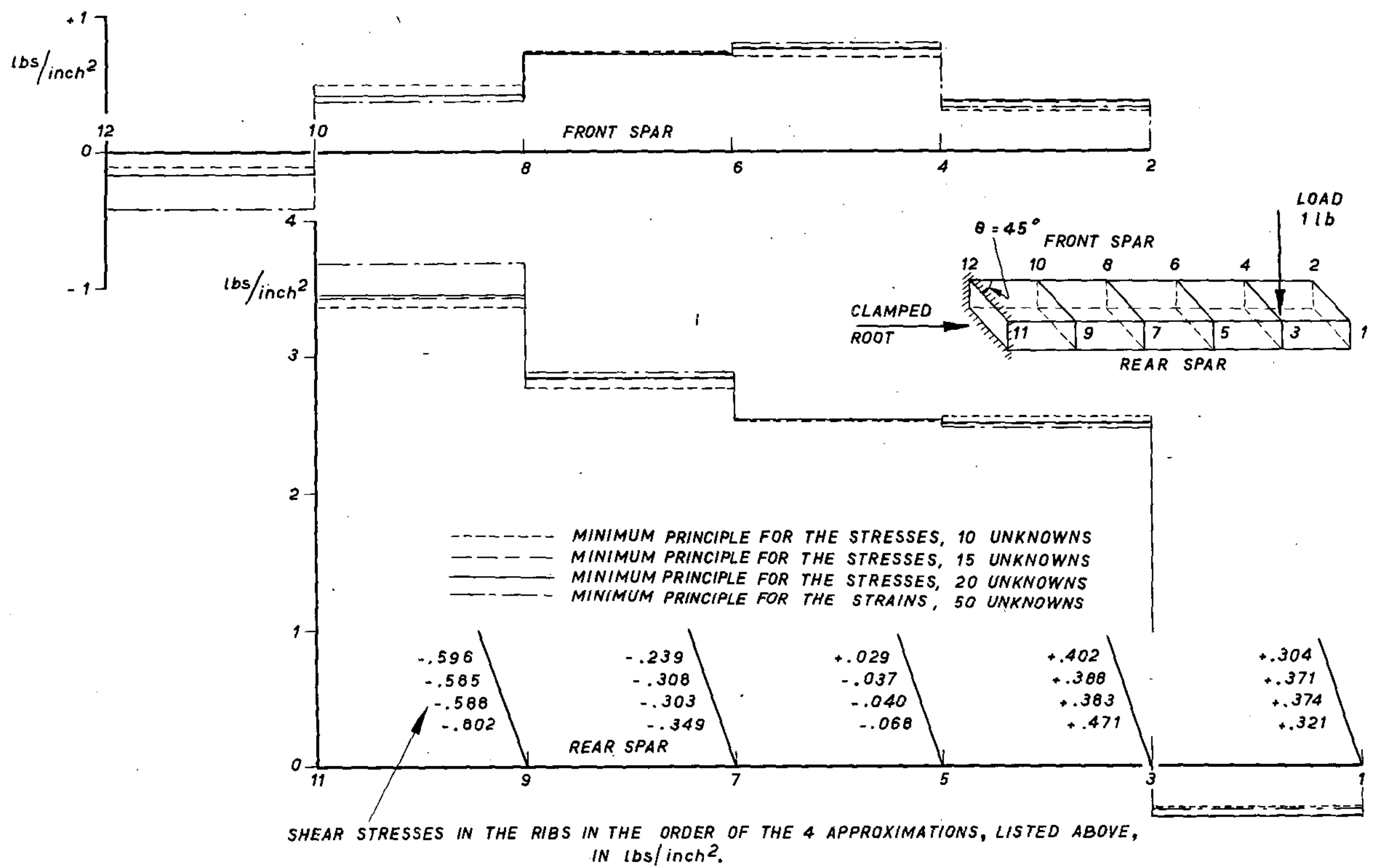


FIG. 7.22. VERTICAL LOAD OF 1 lb IN STATION 3, STRESSFLOWS (lbs/inch) IN UPPER REINFORCED SKIN (COMBINATION OF SKIN AND EQUIVALENT STRINGER PLATE).

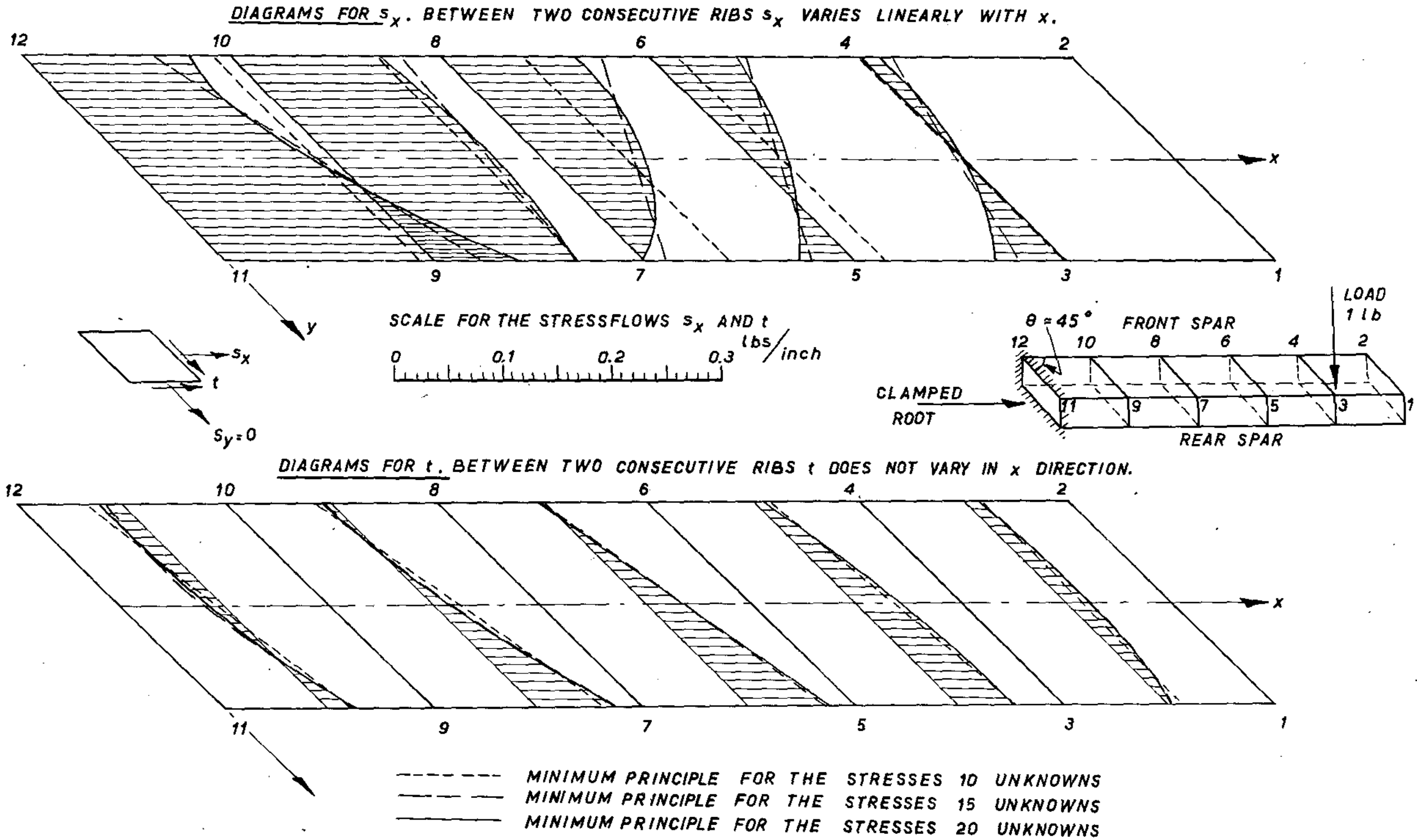


FIG. 7.23. VERTICAL LOAD OF 1 lb IN STATION 3. STRAIN ε_x (MULTIPLIED WITH E) IN UPPER SPAR BOOMS AND IN SKIN IMMEDIATELY ADJACENT TO THE SPAR BOOMS, AS COMPUTED WITH THE MINIMUM PRINCIPLE FOR THE STRESSES AND USING 20 SUPPLEMENTARY STRESS SYSTEMS.

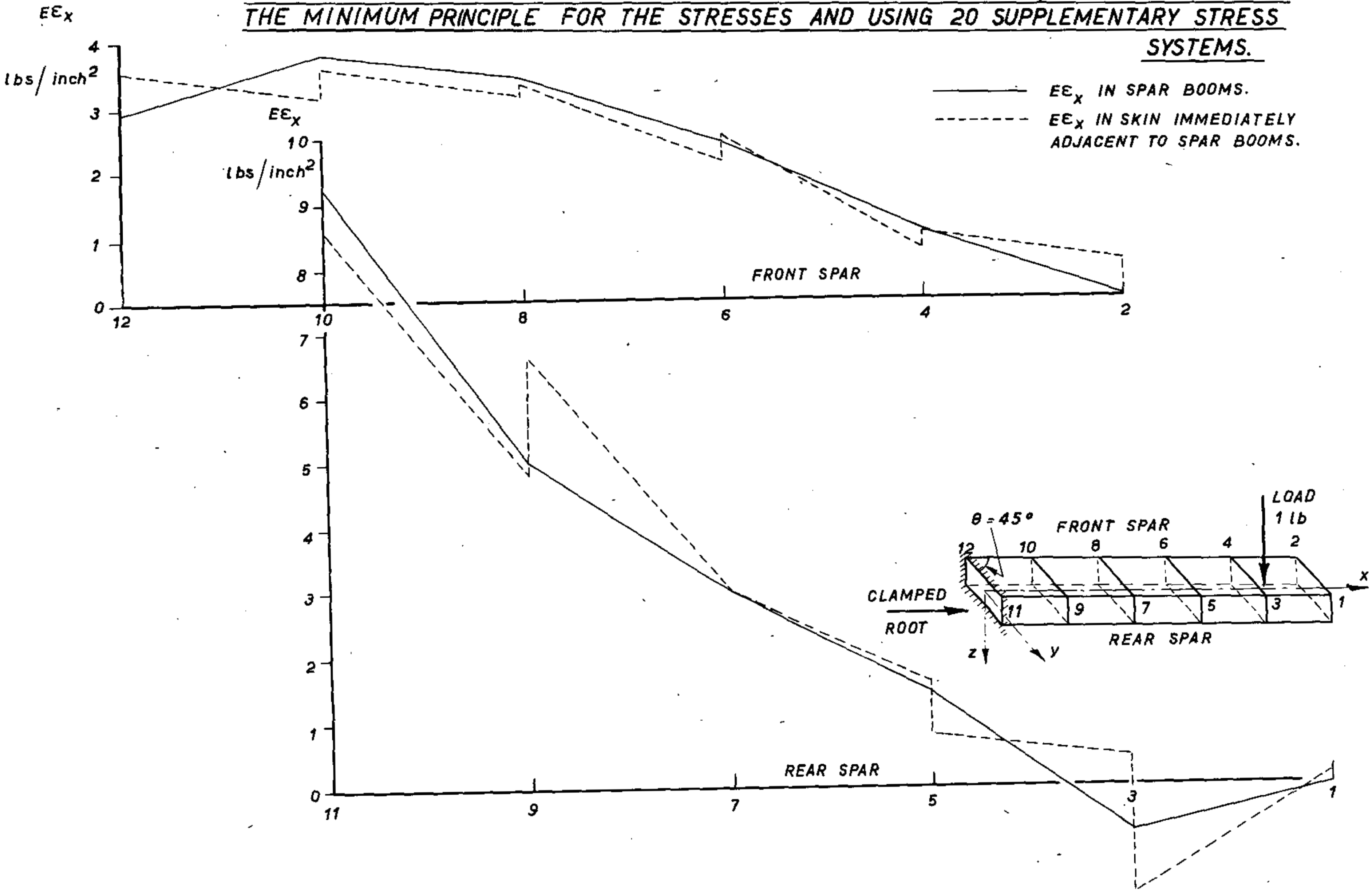


FIG. 7.24. VERTICAL LOAD OF 1 lb IN STATION 4. NORMAL STRESSES IN UPPER SPAR BOOMS

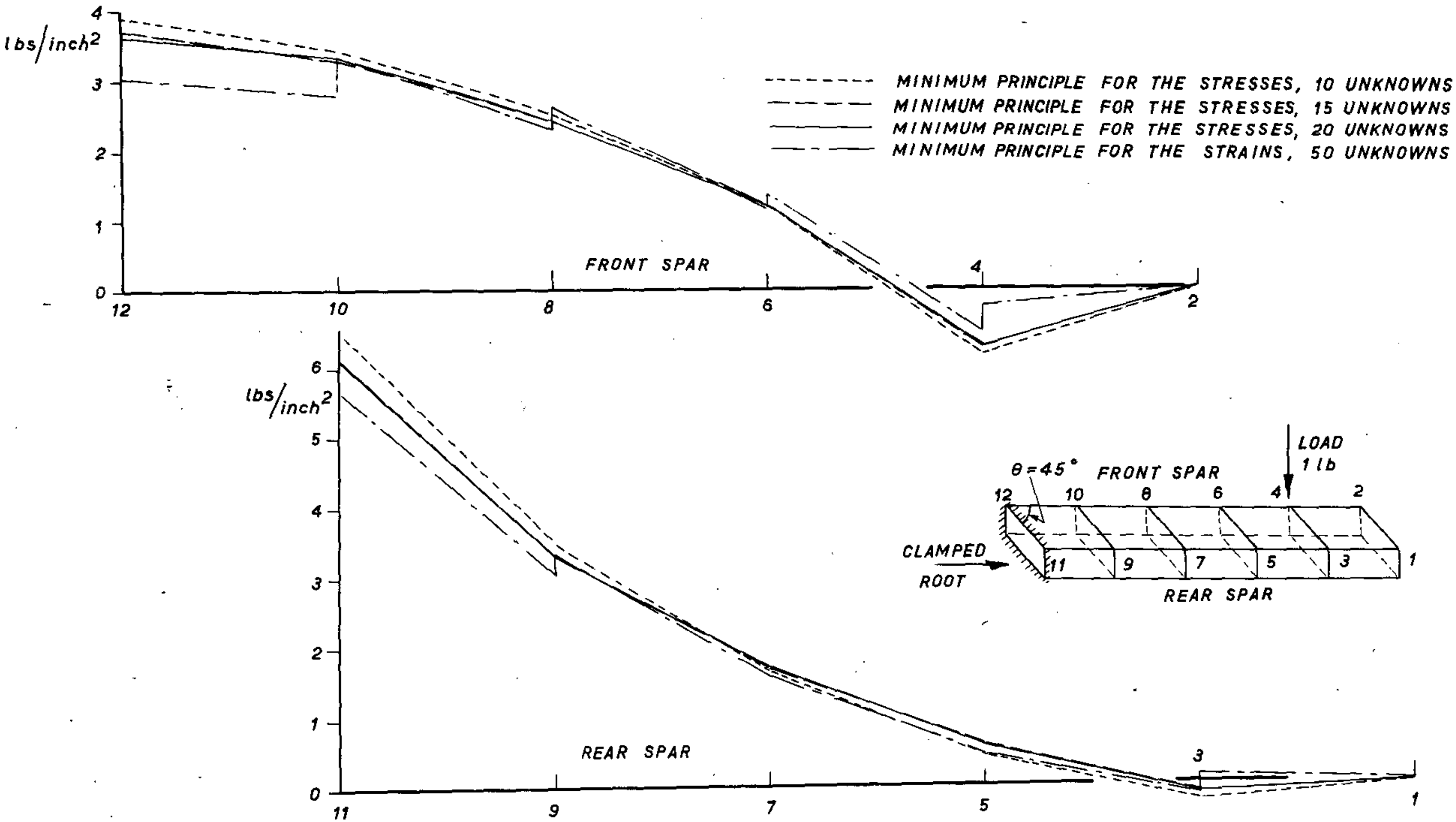
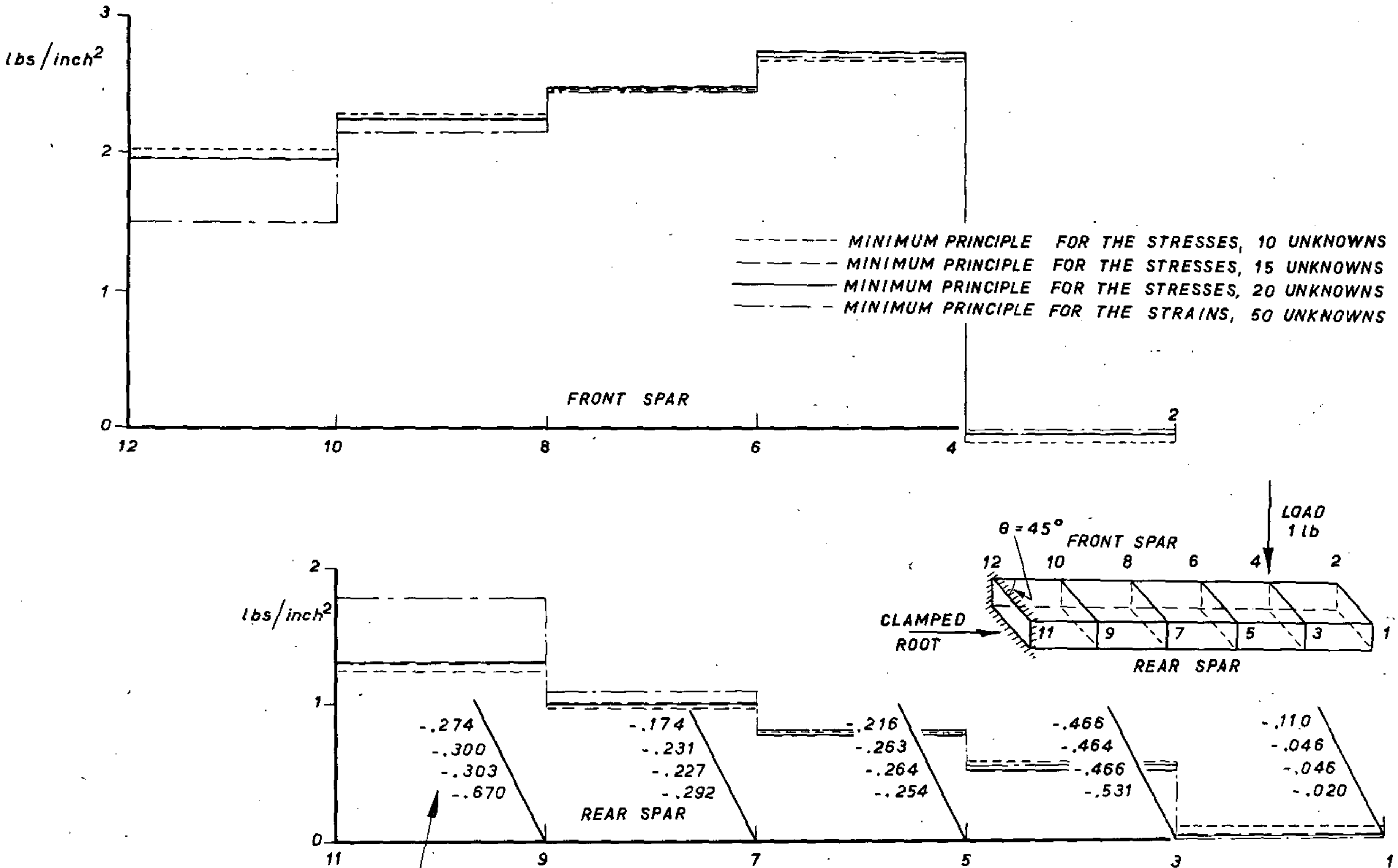


FIG. 7.25. VERTICAL LOAD OF 1 lb IN STATION 4. SHEAR STRESSES IN SPAR AND RIB WEBS.



SHEAR STRESSES IN THE RIBS IN THE ORDER OF THE 4 APPROXIMATIONS, LISTED ABOVE, IN lbs/inch².

FIG. 7.26. VERTICAL LOAD OF 1 lb IN STATION 4. STRESSFLOWS (lbs inch) IN UPPER REINFORCED SKIN. (COMBINATION OF SKIN AND EQUIVALENT STRINGER PLATE)

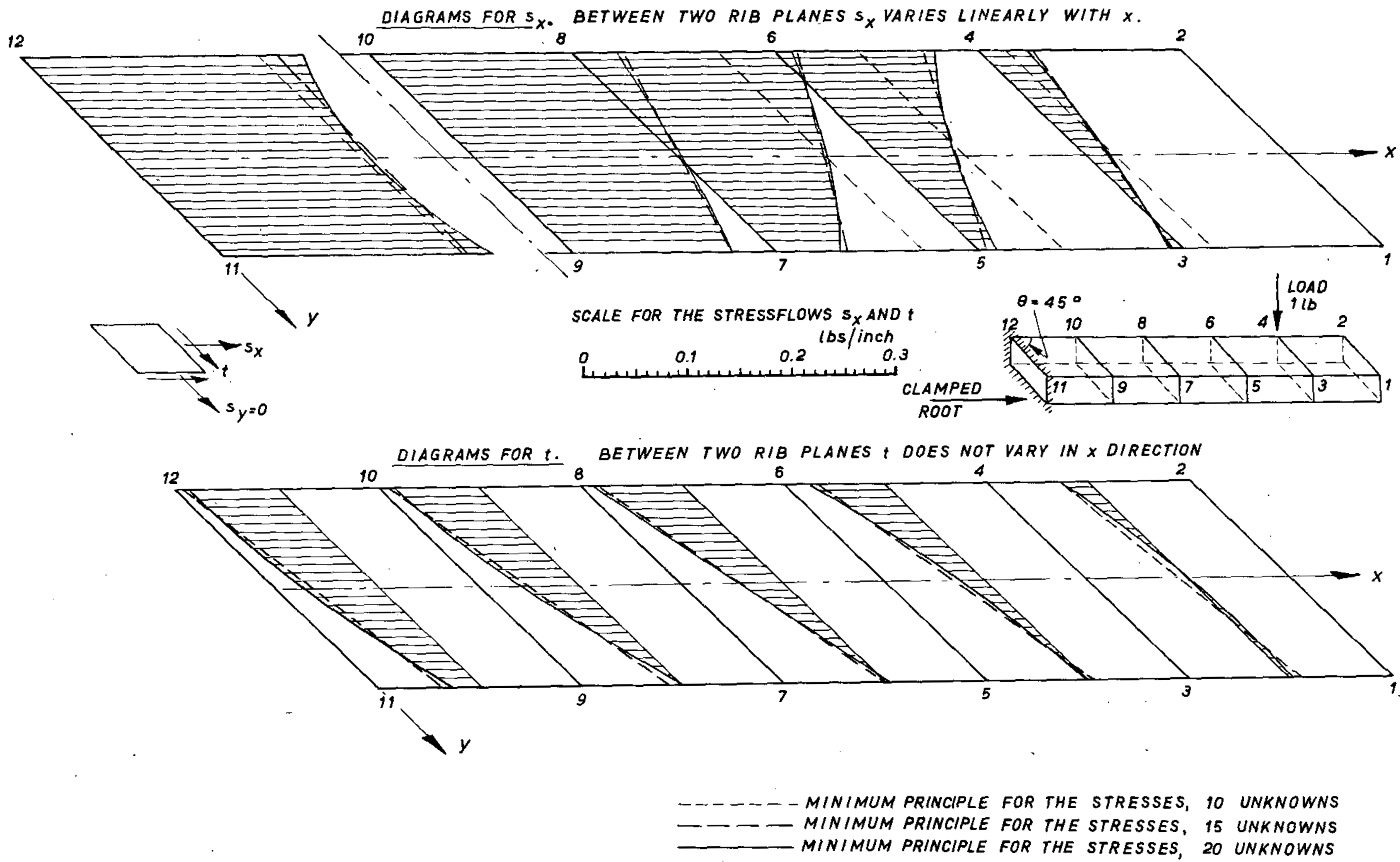


FIG. 7.27. VERTICAL LOAD OF 1 lb IN STATION 4. STRAIN ε_x (MULTIPLIED WITH E) IN UPPER SPAR BOOMS AND IN SKIN IMMEDIATELY ADJACENT TO THE SPAR BOOMS AS COMPUTED WITH THE MINIMUM PRINCIPLE FOR THE STRESSES AND USING 20 SUPPLEMENTARY STRESS SYSTEMS.

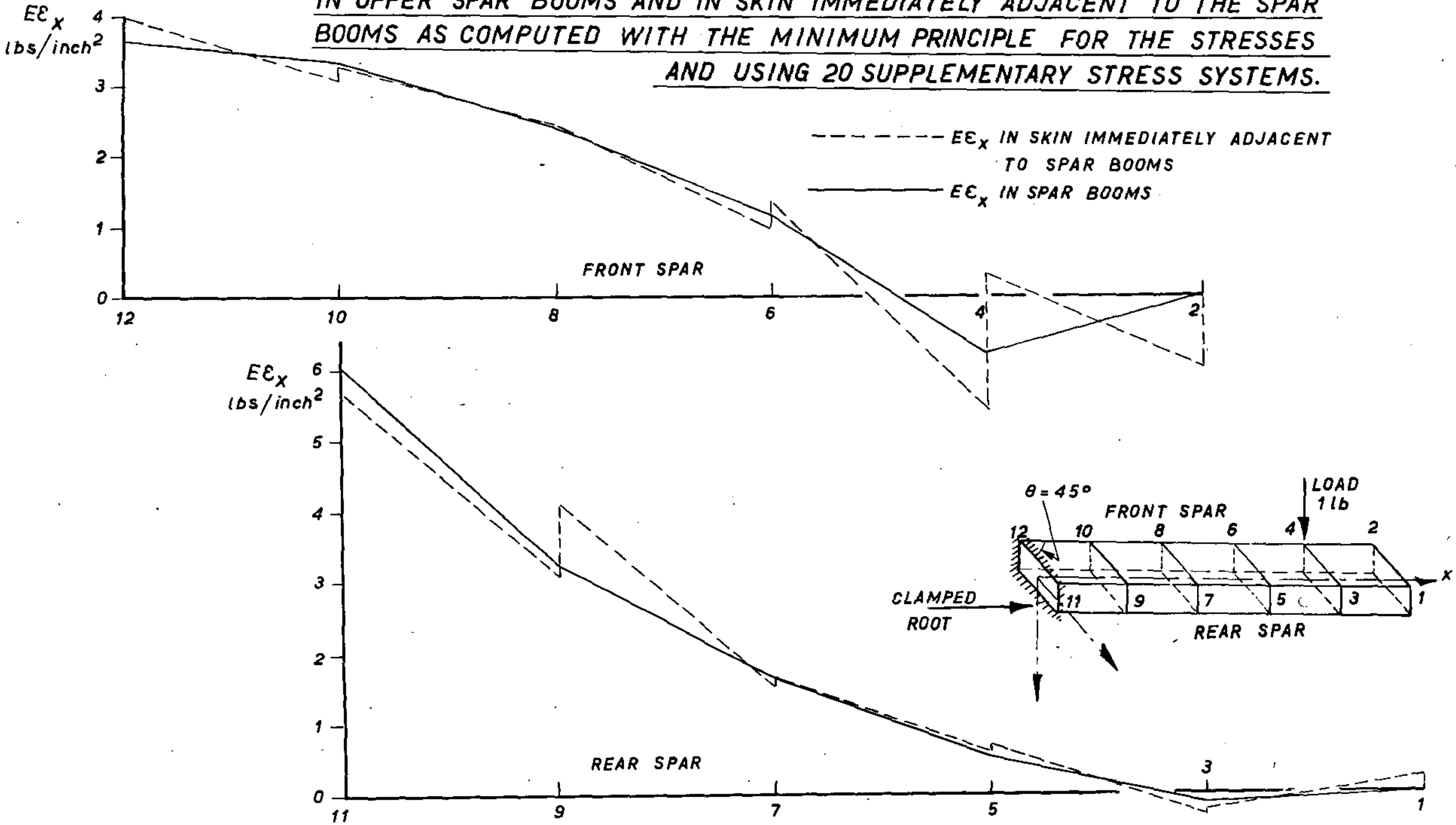


FIG. 7.28 SWEPT BOX ANALYSED BY MORLEY
STRAIN ε_X (MULTIPLIED WITH E) IN UPPER SPAR BOOMS
AND IN SKIN IMMEDIATELY ADJACENT TO THE SPAR BOOMS
FIRST LOADING CASE DERIVED FROM REF. 34

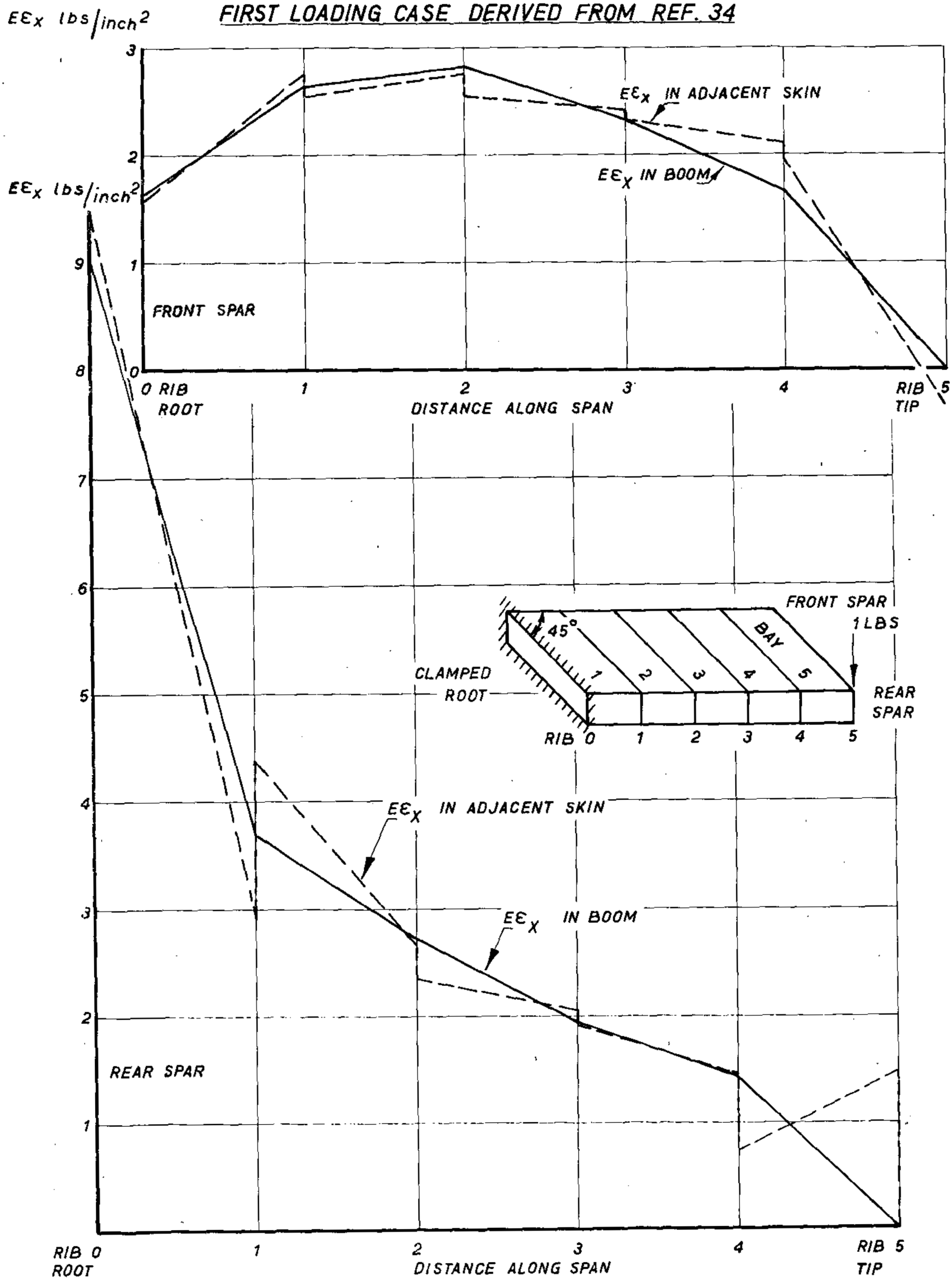


FIG. 7.29 SWEPT BOX ANALYSED BY MORLEY. LOADING CASE OF FIG. 7.28 STRESSFLOW t IN UPPER COMPOSITE SKIN (COMBINATION OF SKIN AND EQUIVALENT STRINGER PLATE). DERIVED FROM REF. 34

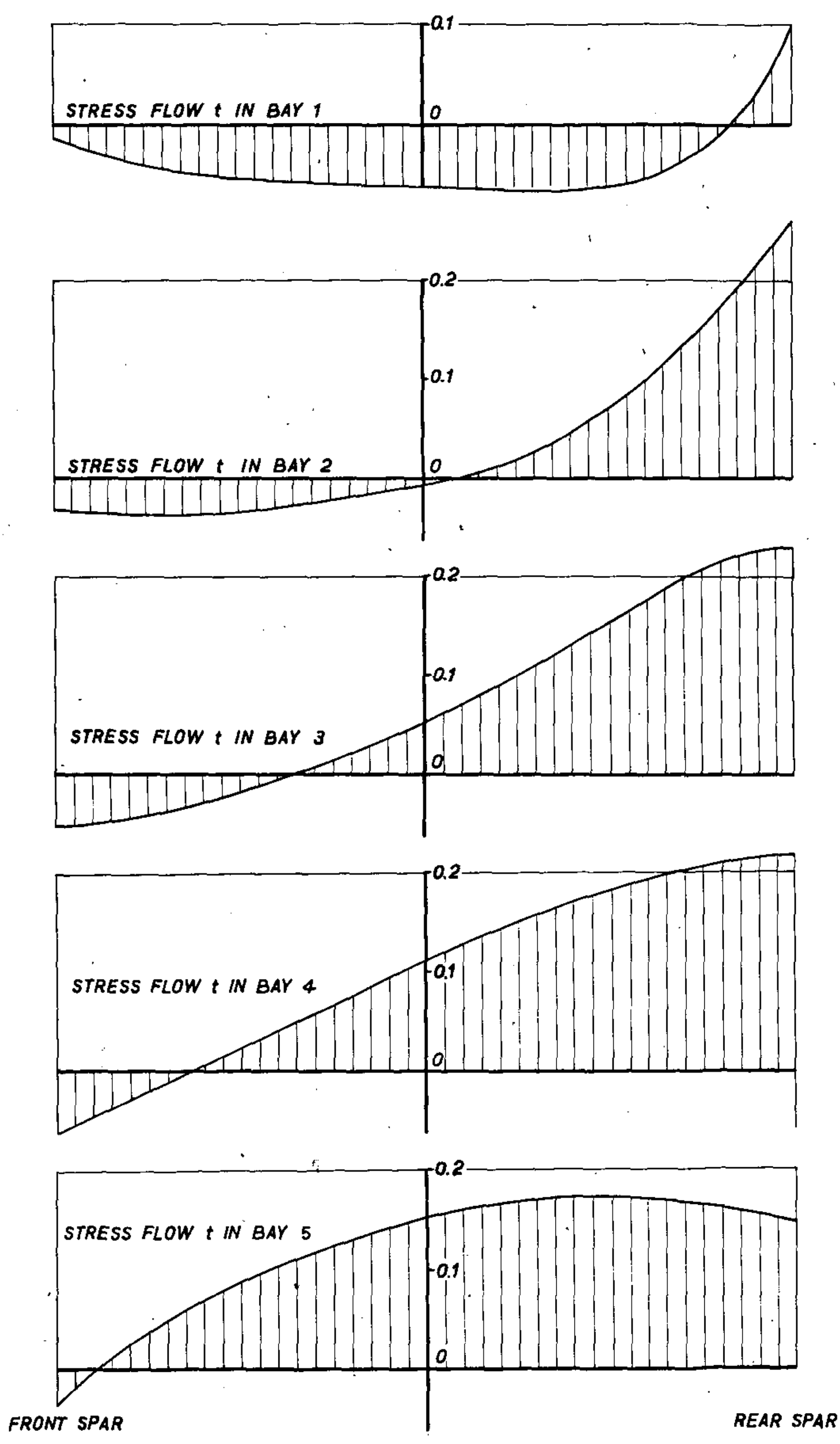


FIG. 7.30 SWEPT BOX ANALYSED BY MORLEY. STRAIN ε_x (MULTIPLIED WITH E) IN UPPER SPAR BOOMS AND IN SKIN IMMEDIATELY ADJACENT TO THE SPAR BOOMS. SECOND LOADING CASE DERIVED FROM REF. 34.

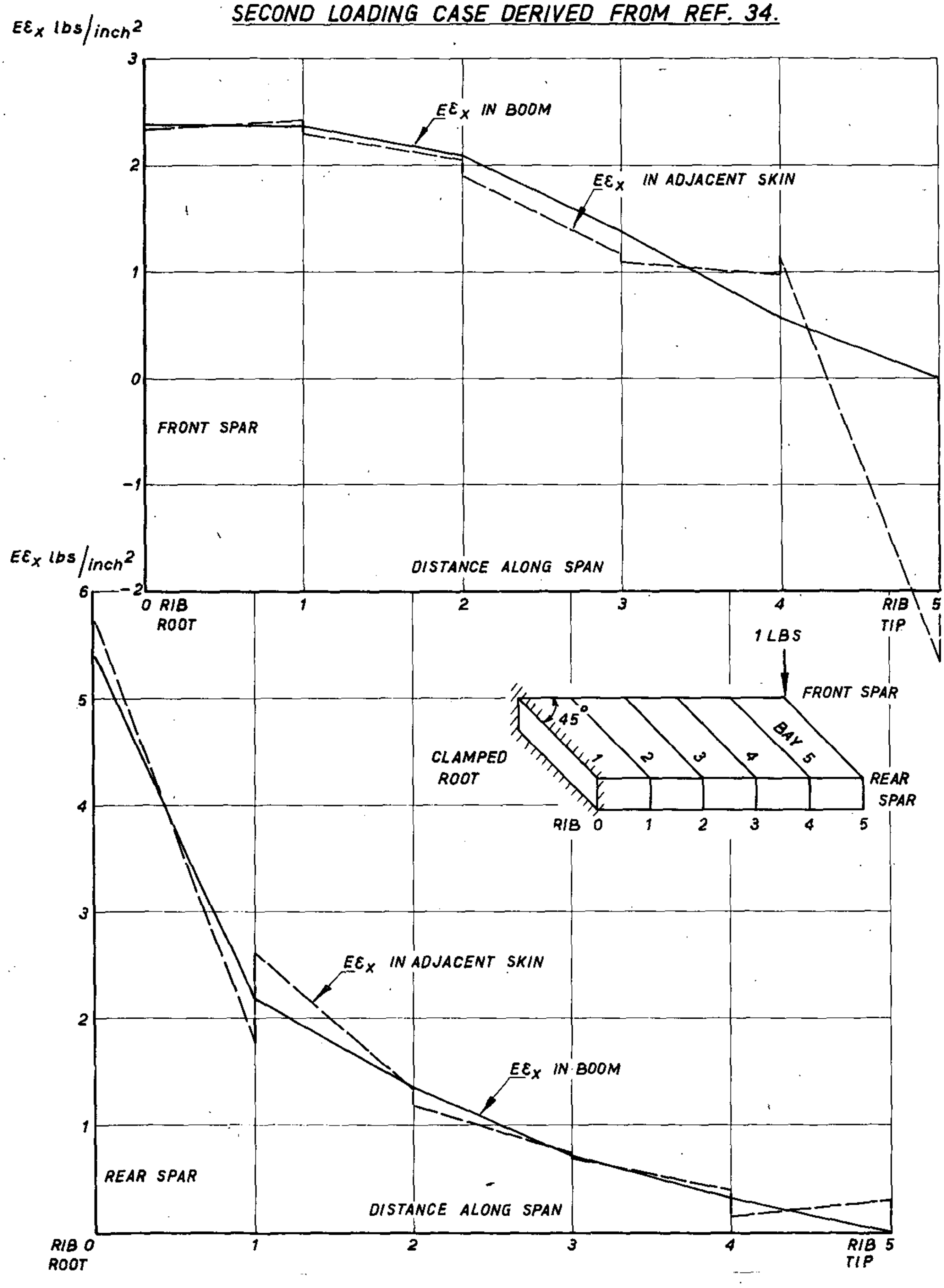


FIG. 7.31 SWEPT BOX ANALYSED BY MORLEY. LOADING CASE OF FIG. 7.30
STRESSFLOW t (lbs/inch) IN UPPER COMPOSITIE SKIN (COMBINATION OF SKIN AND EQUIVALENT STRINGER PLATE). DERIVED FROM REF. 34.

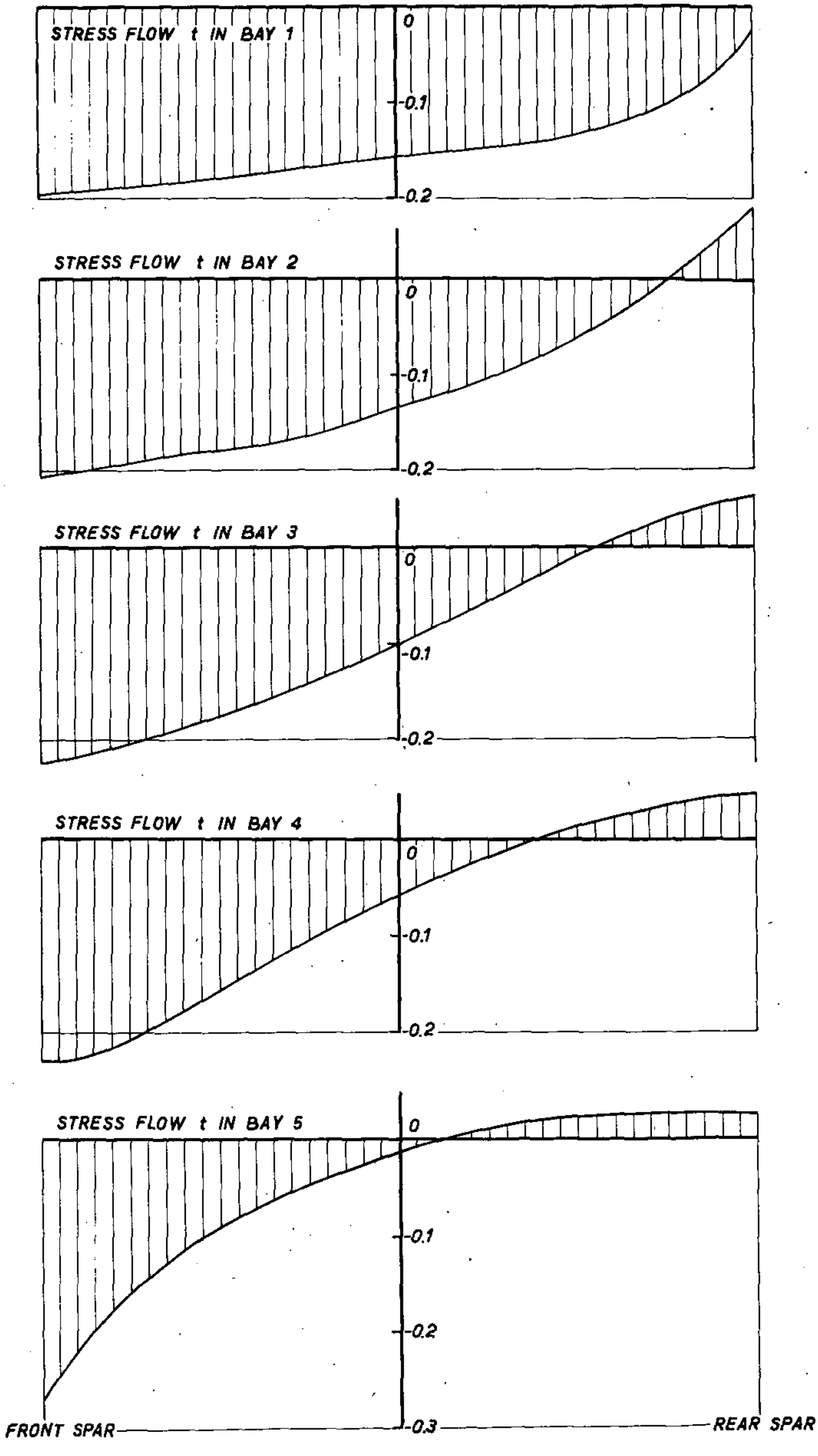


FIG. 8.1 GROUPS OF SUPPLEMENTARY STRESS SYSTEMS IN THE CELLS OF AN INFINITE BOX BEAM INDICATED BY THE COLUMN MATRICES (8.1) OF THEIR PARTICIPATION FACTORS.

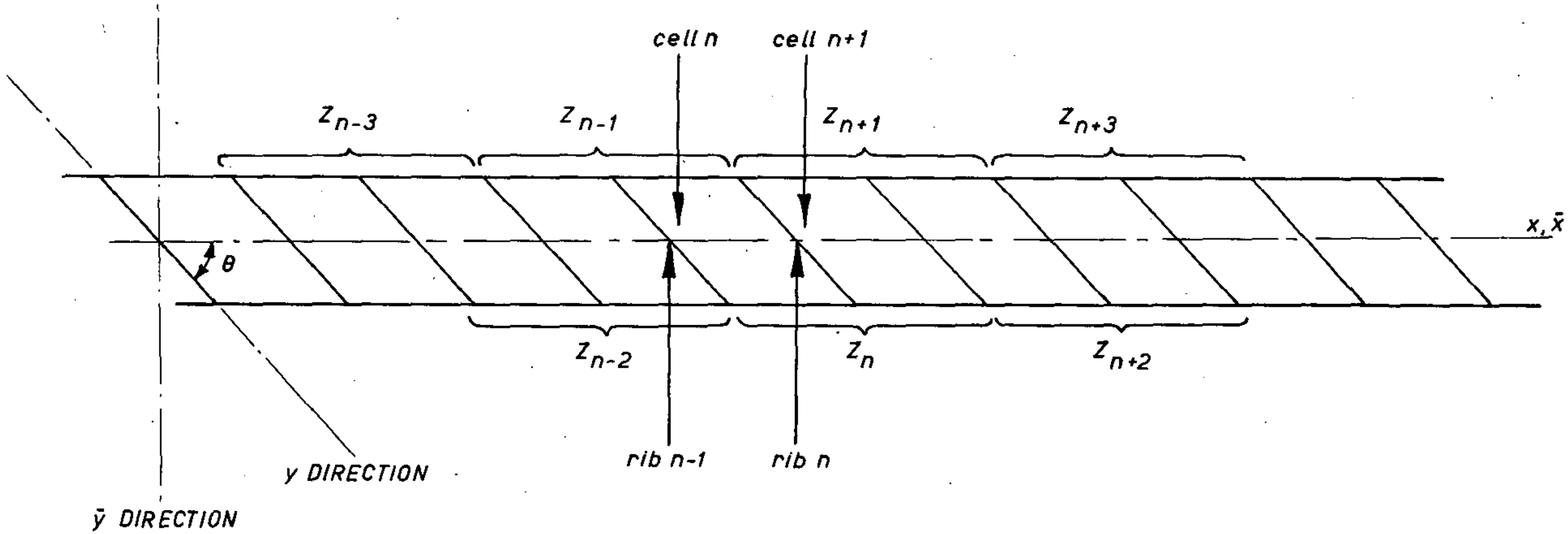


FIG. 8.2. SUPPLEMENTARY STRESS SYSTEM. TYPE 1
IN TERMS OF CELL DIMENSIONS.

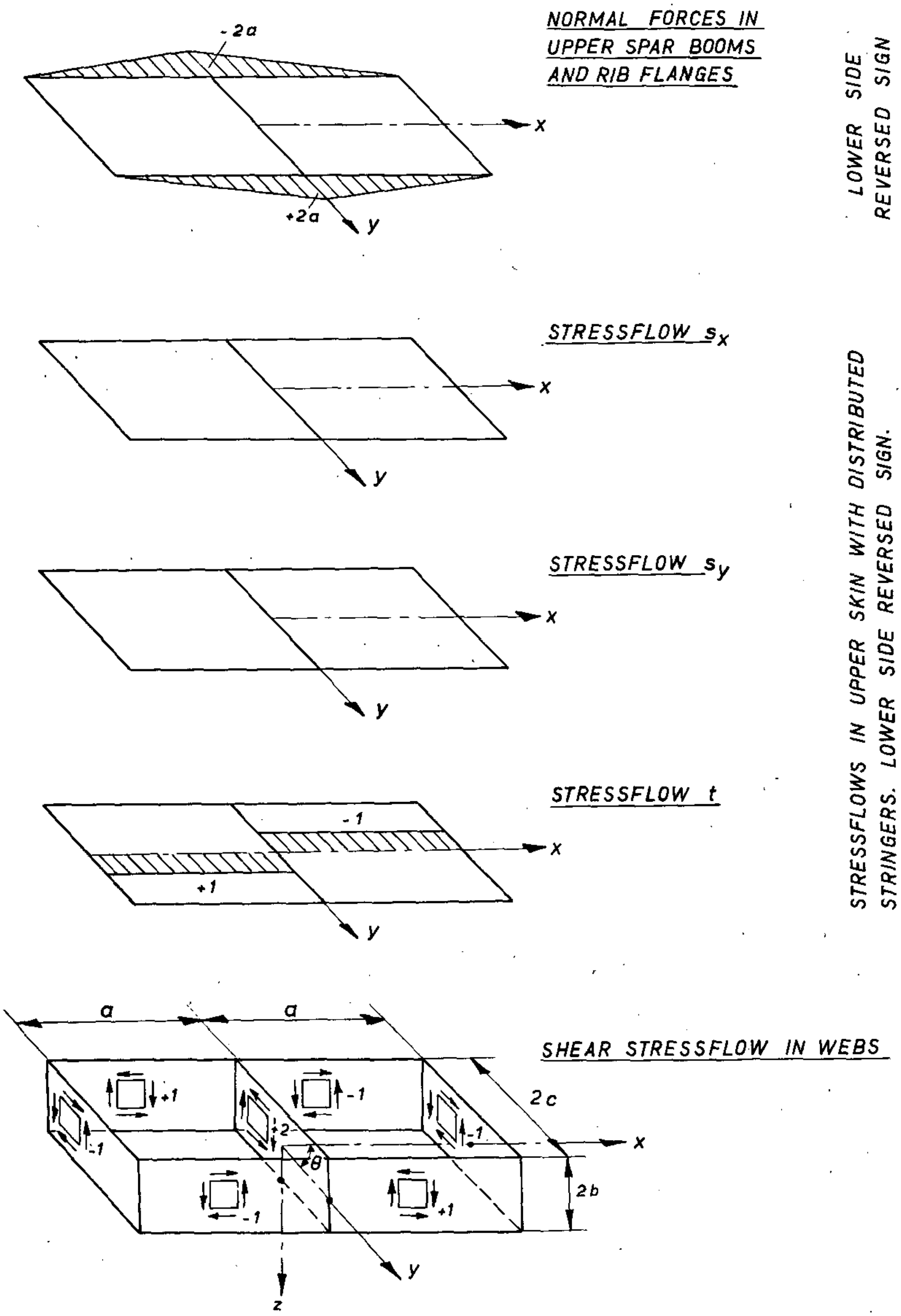


FIG. 8.3 SUPPLEMENTARY STRESS SYSTEM TYPE 2
IN TERMS OF CELL DIMENSIONS.

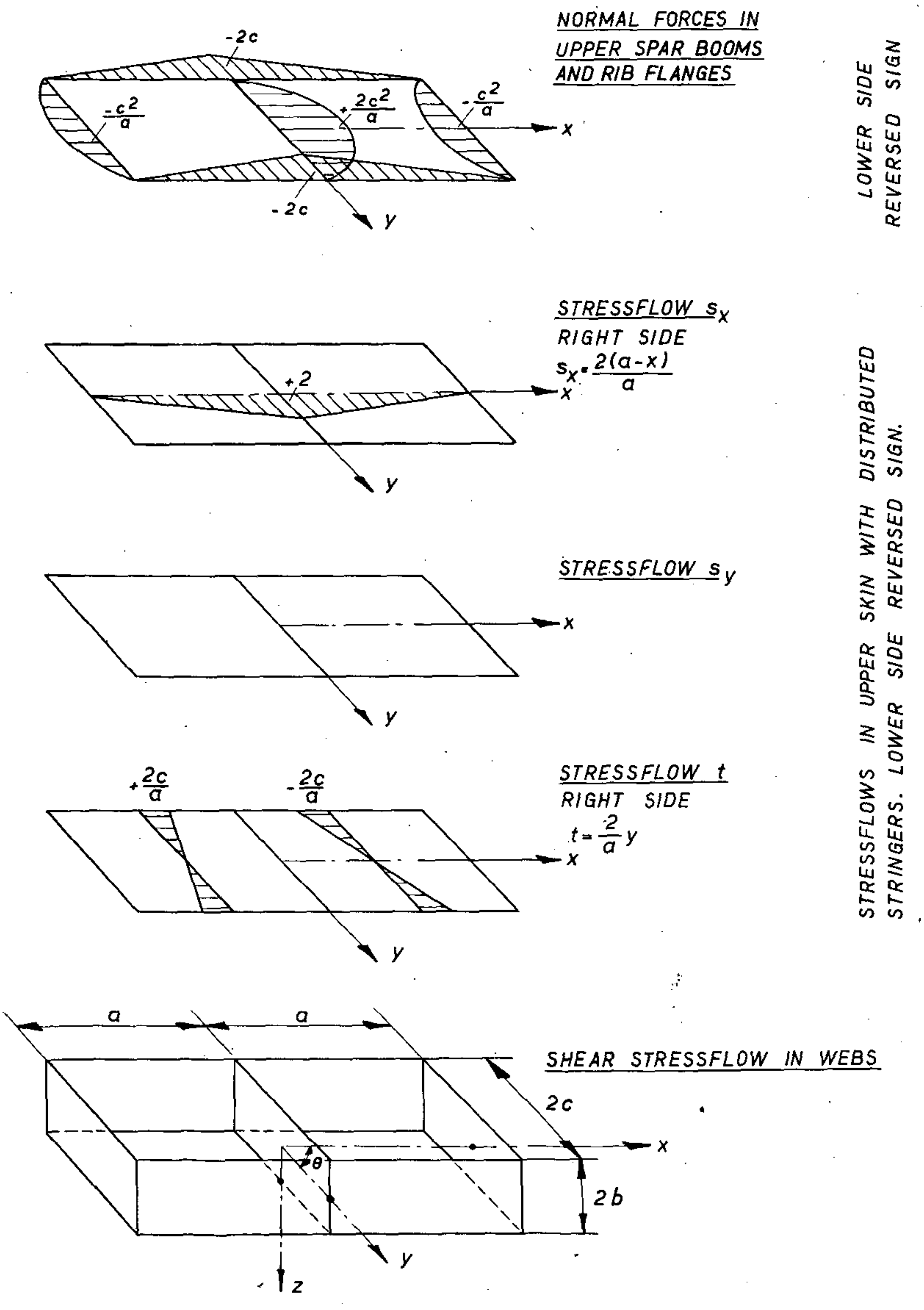


FIG. 8.4 SUPPLEMENTARY STRESS SYSTEM TYPE 3 IN TERMS OF CELL DIMENSIONS.

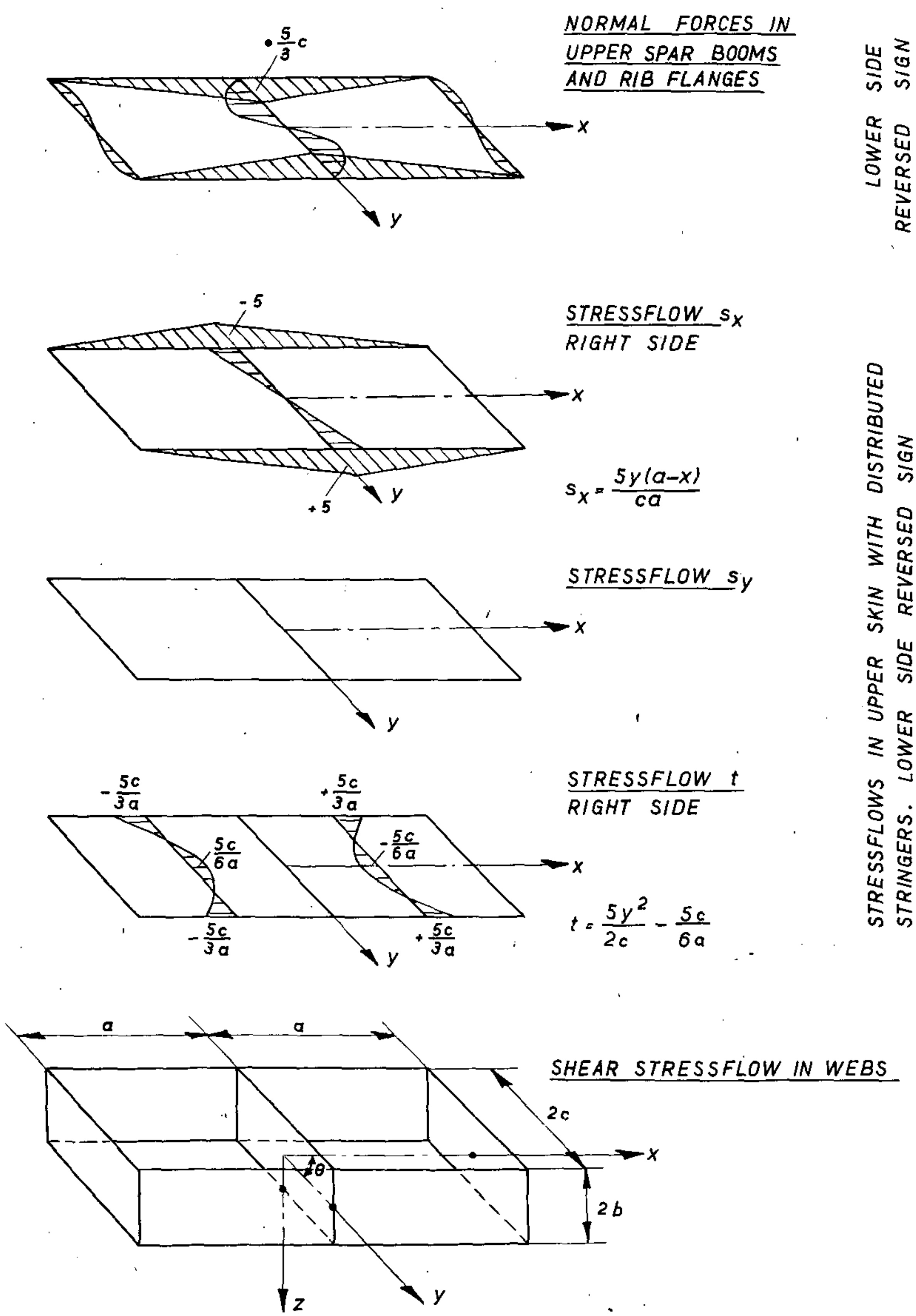
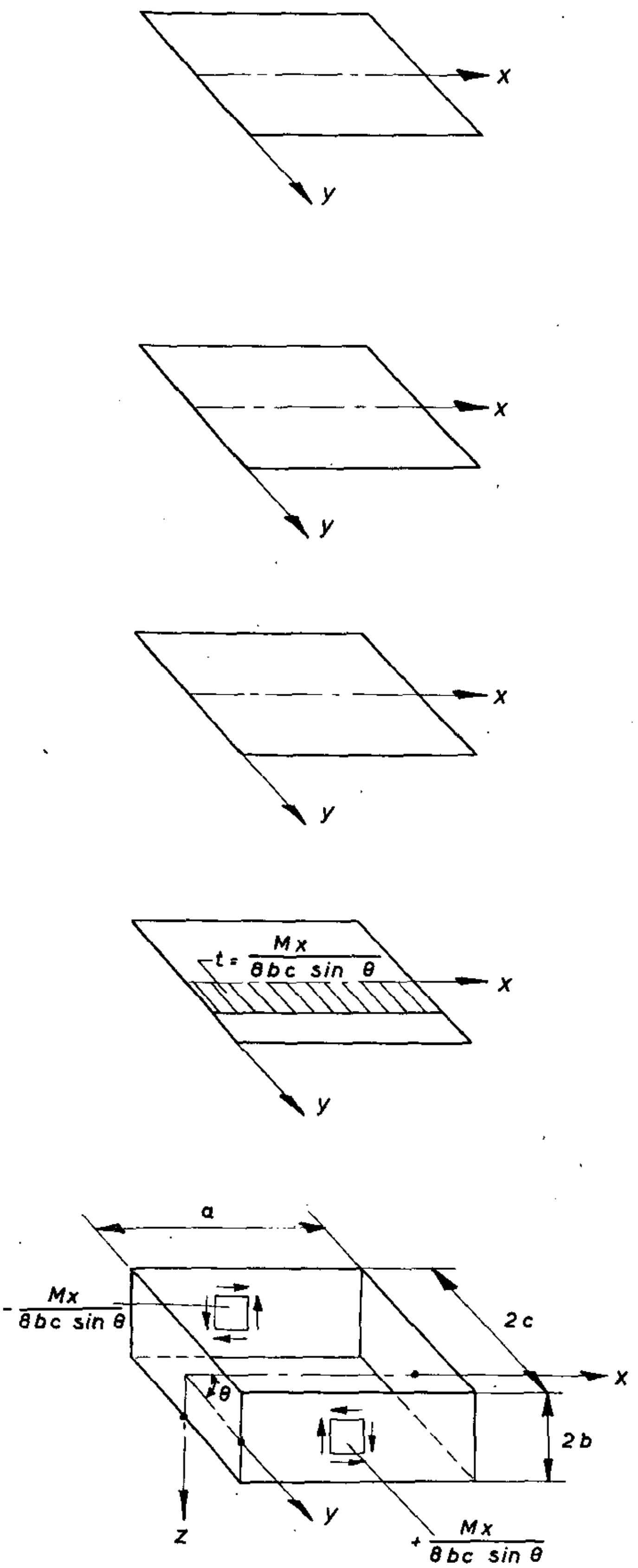


FIG 8.5. BASIC STRESS SYSTEM FOR THE CELLS AT LOAD $\vec{M}(M_x, M_y = 0, M_z = 0)$ IN TERMS OF CELL DIMENSIONS.



NORMAL FORCES IN UPPER SPAR BOOMS AND RIB FLANGES

LOWER SIDE REVERSED SIGN

STRESSFLOW s_x

STRESSFLOW s_y

STRESSFLOW t

STRESSFLOWS IN UPPER SKIN WITH DISTRIBUTED STRINGERS. LOWER SIDE REVERSED SIGN.

SHEAR STRESSFLOW IN WEBS

FIG.8.6. BASIC STRESS SYSTEM FOR THE CELLS AT LOAD $\vec{M}\,(M_x = 0,\ M_y,\ M_z = 0)$ IN TERMS OF CELL DIMENSIONS.

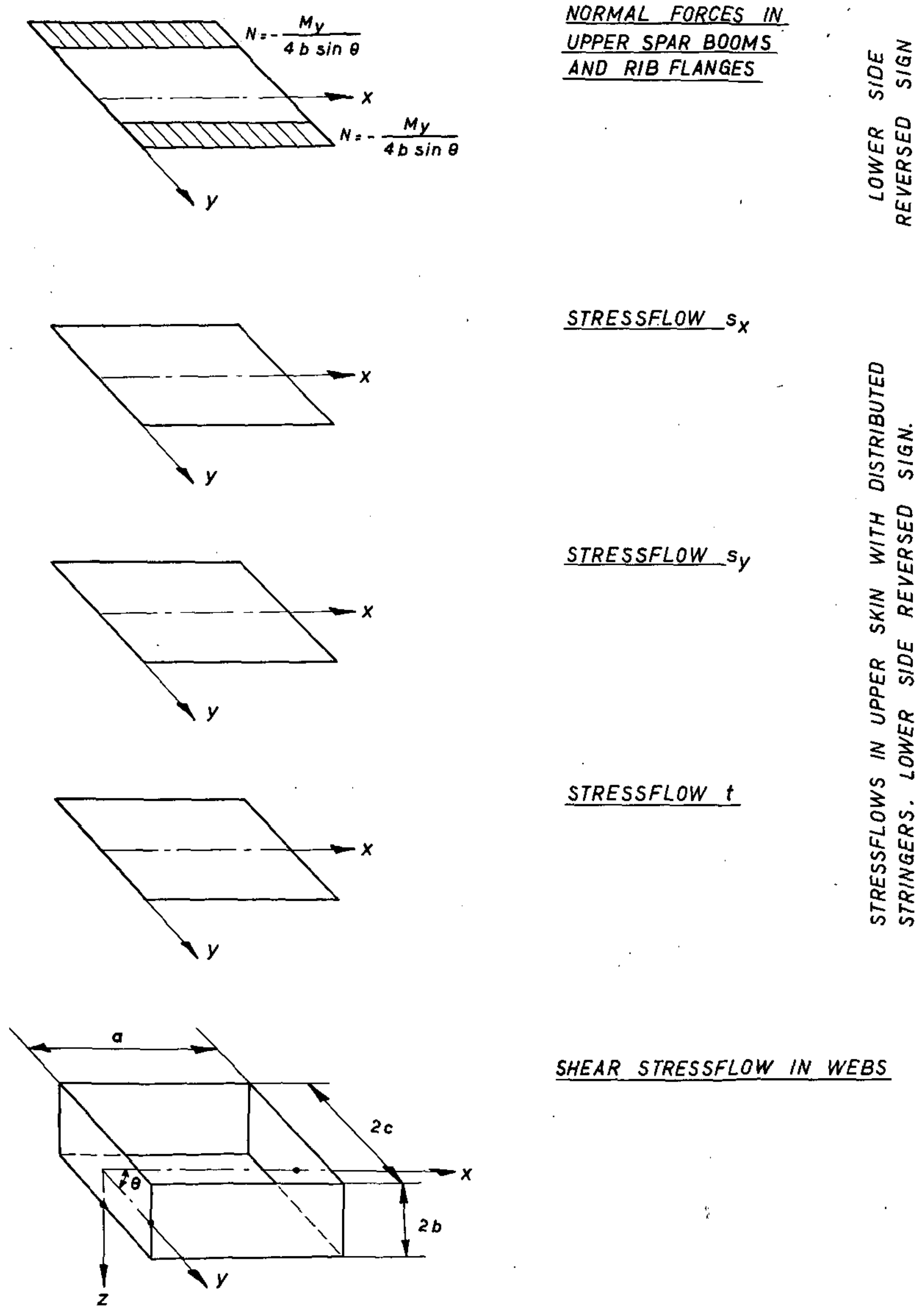


FIG. 8.7. BASIC STRESS SYSTEM FOR THE CELLS AT LOAD $\vec{K}(0,0,K_z)$ IN TERMS OF CELL DIMENSIONS.

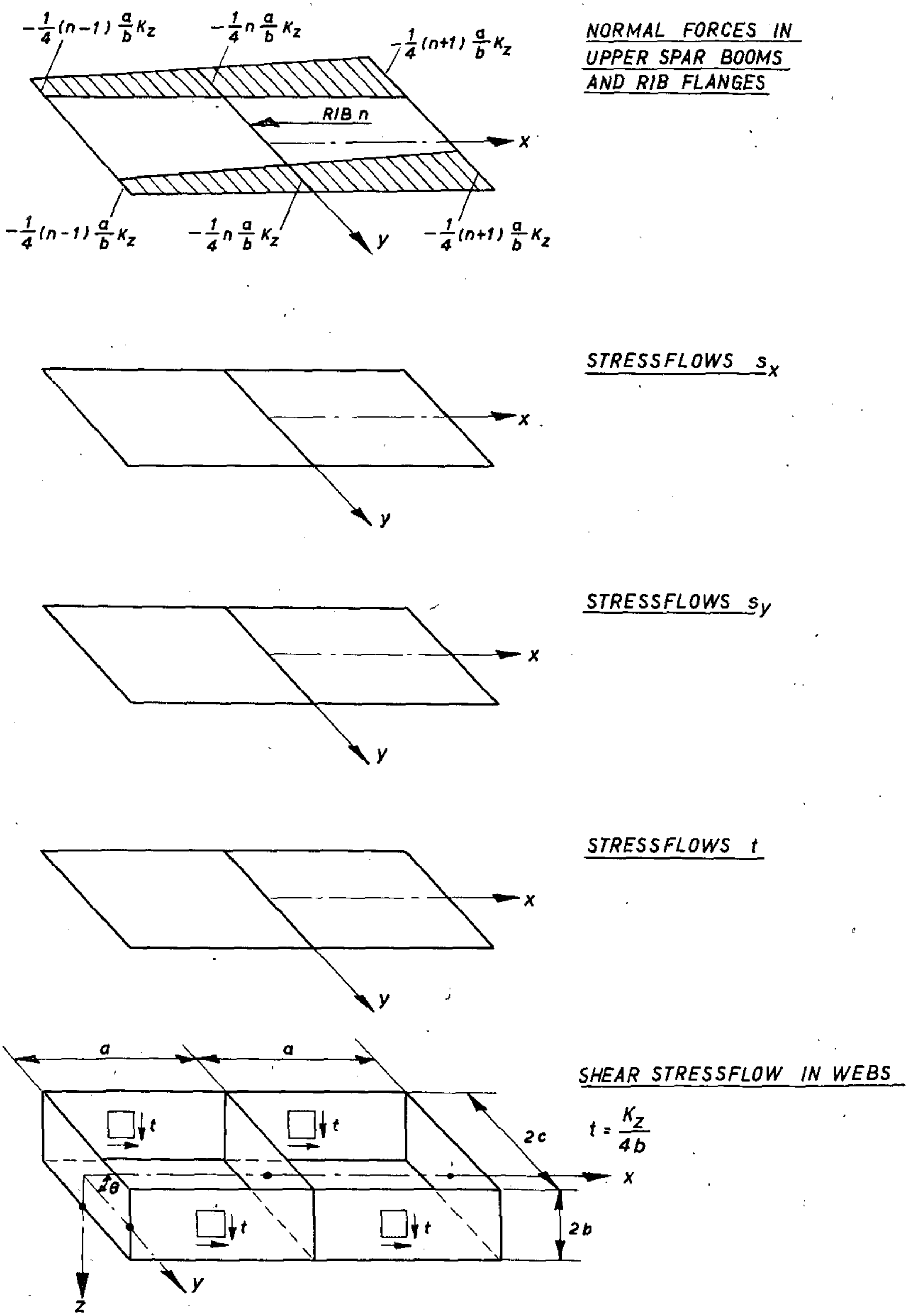


FIG. 8.8. SUPPLEMENTARY STRESS SYSTEM TYPE 5

FORCES IN lbs. STRESSFLOWS IN lbs/inch

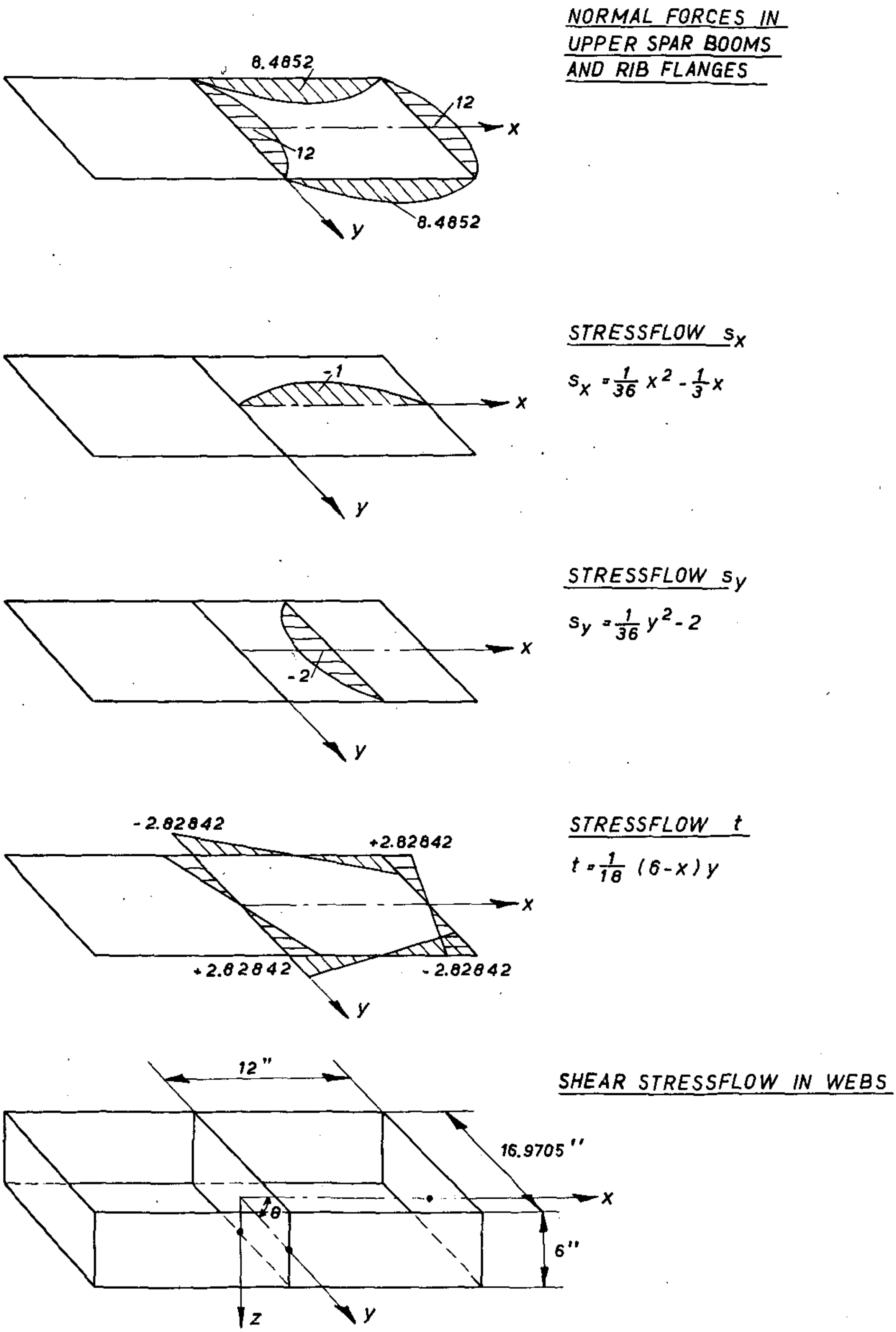
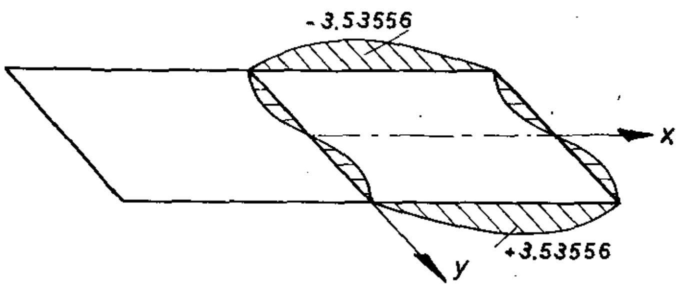


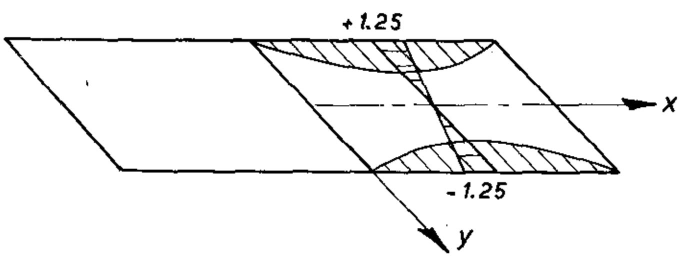
FIG. 8.9. SUPPLEMENTARY STRESS SYSTEM TYPE 6
FORCES IN lbs. STRESSFLOWS IN lbs/inch.



NORMAL FORCES IN UPPER SPAR BOOMS AND RIB FLANGES

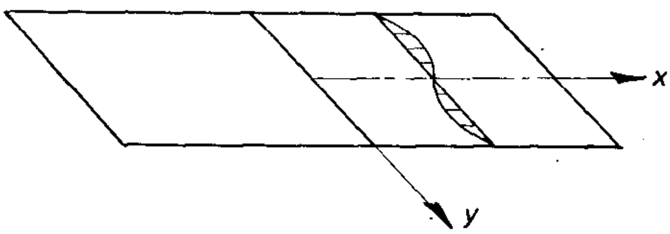
$N = -0.0081842\ y^3 + 0.58926\ y$

LOWER SIDE REVERSED SIGN.



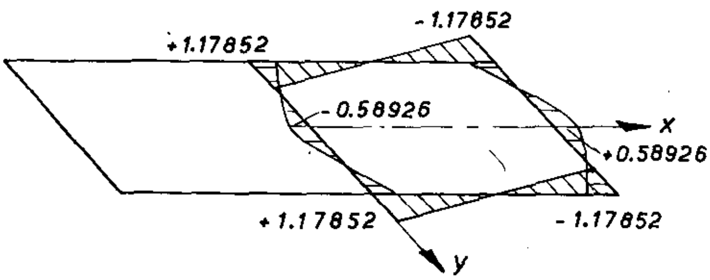
STRESSFLOW s_x

$s_x = 0.00409206\,(x^2 - 12x)\,y$



STRESSFLOW s_y

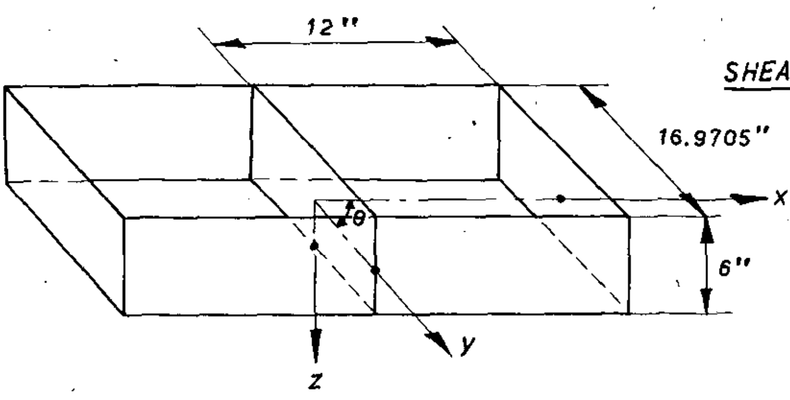
$s_y = 0.00136402\,y^3 - 0.098209\,y$



STRESSFLOW t

$t = (6 - x)\,(0.00409206\,y^2 - 0.098209)$

STRESSFLOWS IN UPPER SKIN WITH DISTRIBUTED STRINGERS. LOWER SIDE REVERSED SIGN.



SHEAR STRESSFLOW IN WEBS

FIG. 8.10 STRAIN ε_x (MULTIPLIED WITH E) IN SPAR BOOM (lbs/inch^2) AND IN ADJACENT SKIN OF INFINITE BOX BEAM AT LOAD $\vec{M}$ (M_x = 1,0,0).

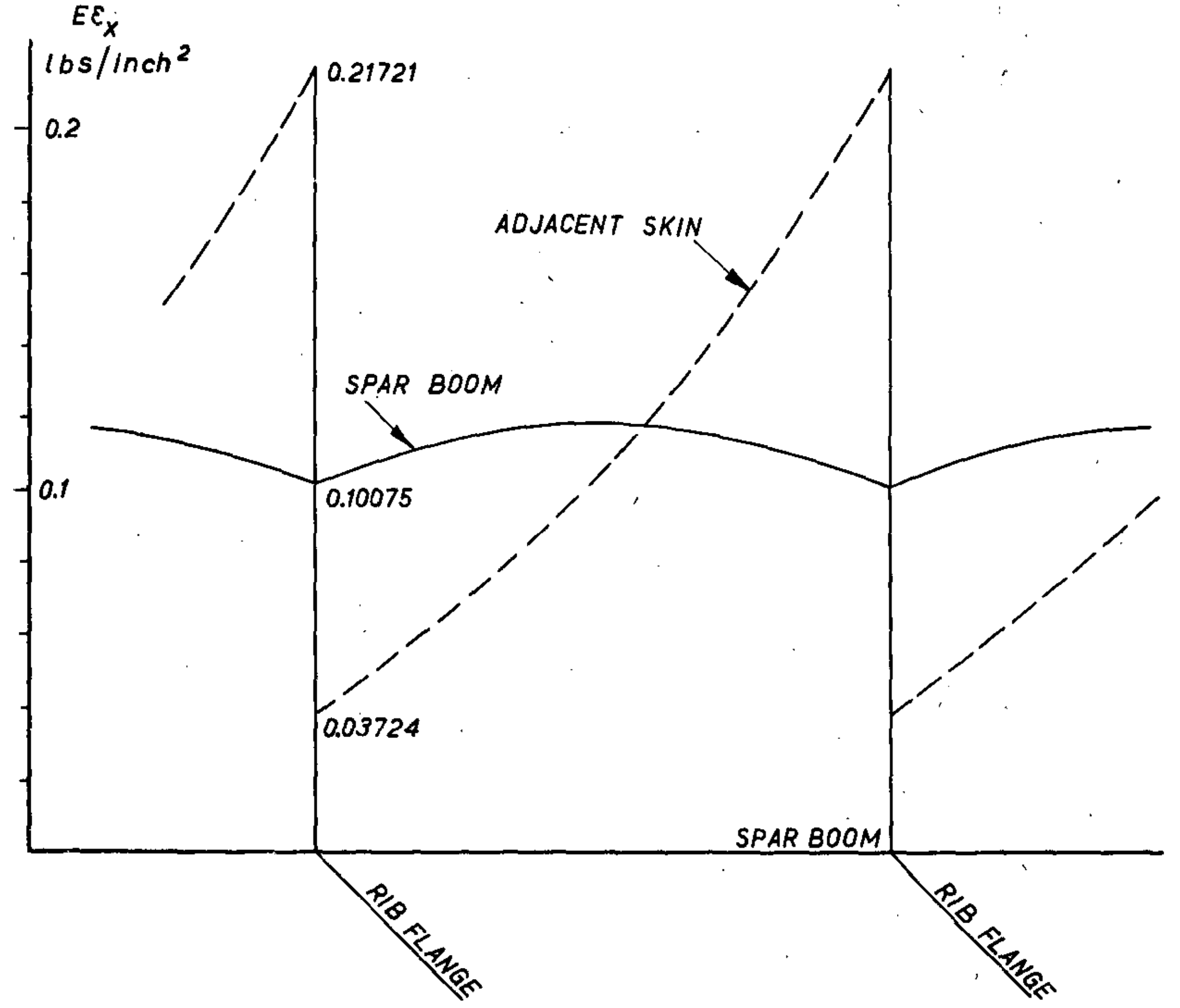


FIG. 8.11 STRAIN ε_y (MULTIPLIED WITH E) IN RIB FLANGE (lbs/inch²) AND IN THE ADJACENT SKIN PANELS OF THE INFINITE BOX BEAM AT LOAD $\vec{M}$ (M_X = 1,0,0)

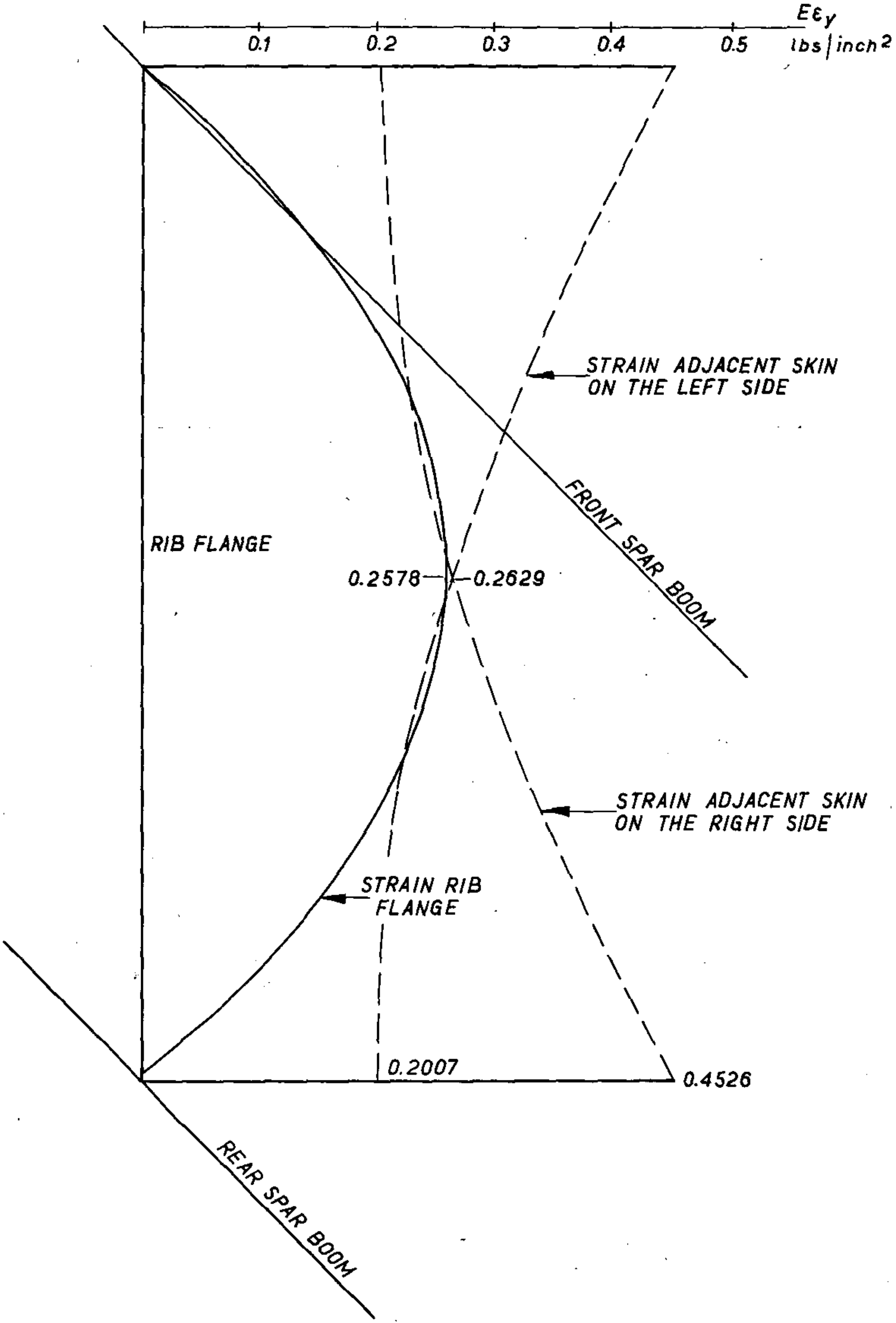


FIG. 8.12 STRAIN ε_x (MULTIPLIED WITH E) IN SPAR BOOM (lbs/inch²) AND IN ADJACENT SKIN OF INFINITE BOX BEAM AT LOAD $\vec{M}(0, M_y = 1, 0)$

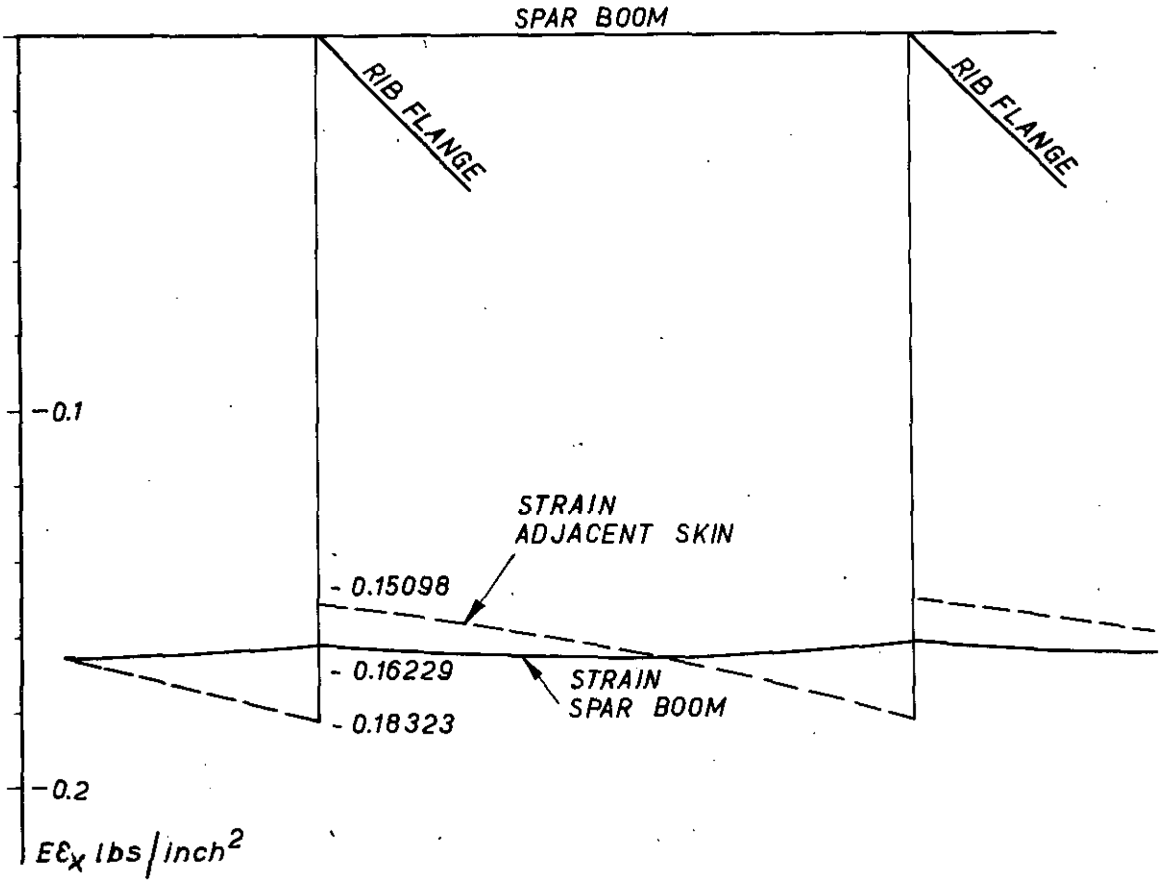


FIG. 8.13 GROUPS OF SUPPLEMENTARY STRAIN SYSTEMS IN THE CELLS OF AN INFINITE BOX BEAM INDICATED BY THEIR PARTICIPATION FACTORS AND COLUMN MATRICES (8.78) FORMED OF THEM.

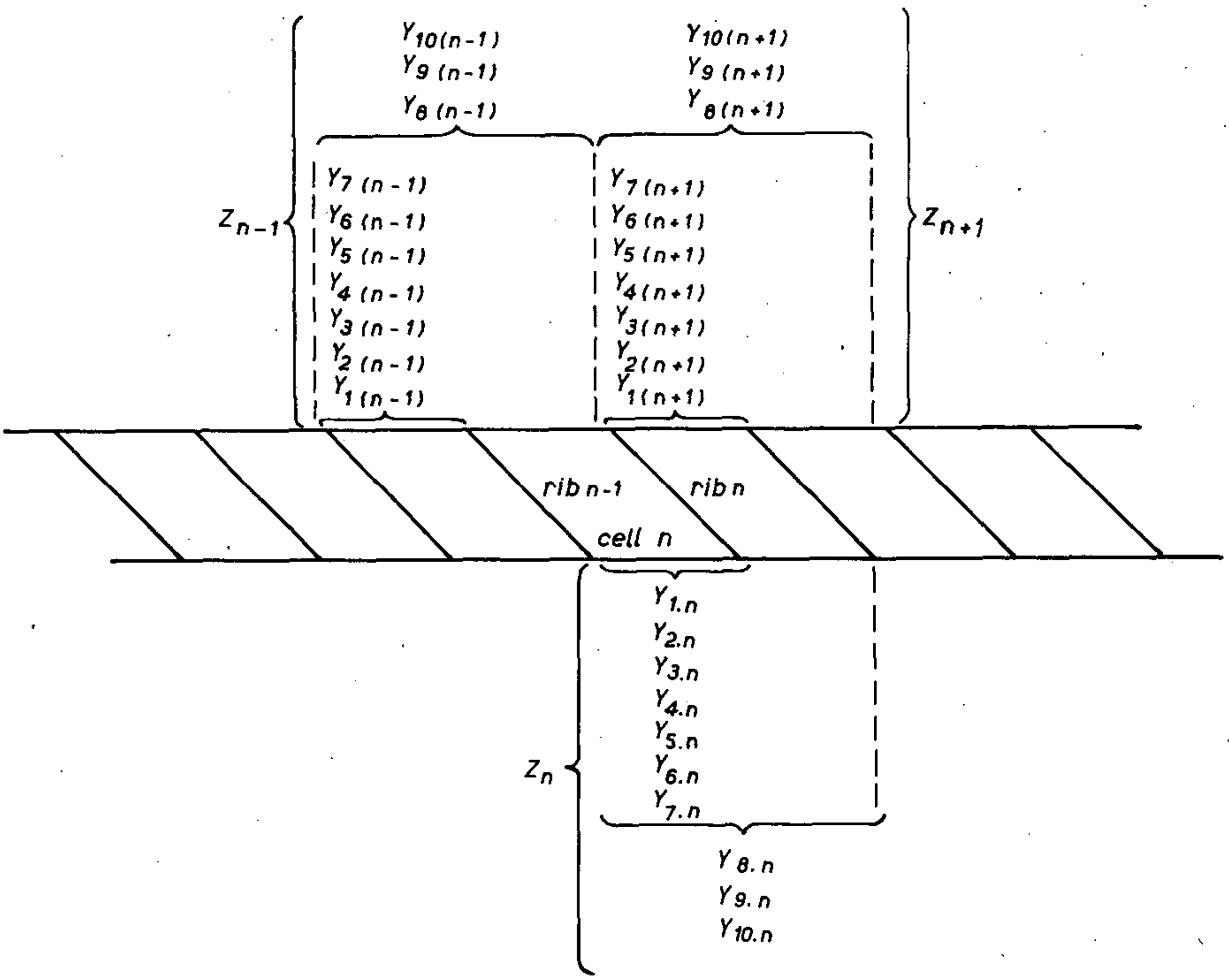


FIG. 8.14. APPLICATION OF A MOMENT $\vec{M}(M_x, 0, 0)$ TO THE RIGHT END BY MEANS OF FORCES P.

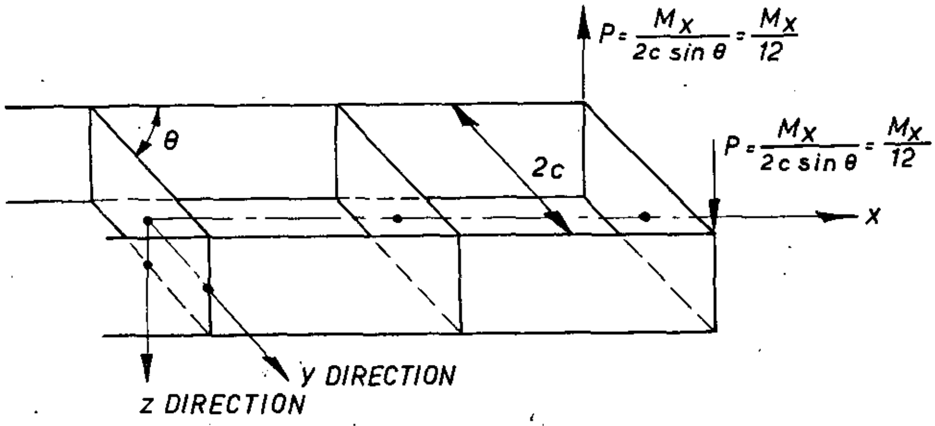


FIG. 8.15. APPLICATION OF A MOMENT $\vec{M}(0, M_y, 0)$ TO THE RIGHT END BY MEANS OF FORCES P.

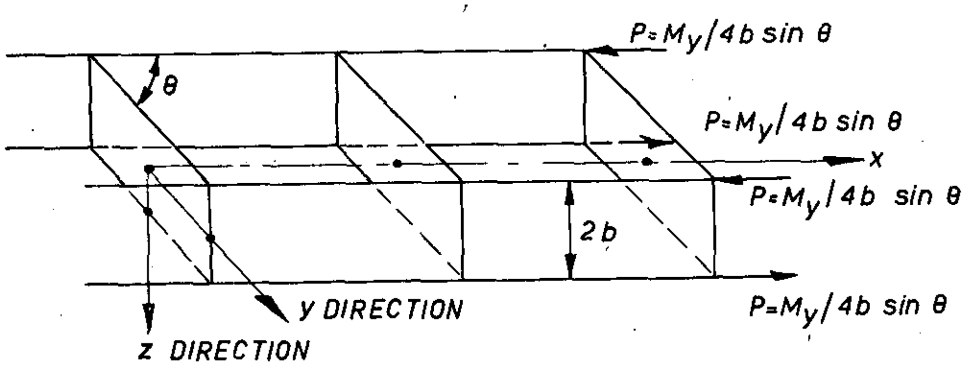


FIG. 8.16. APPLICATION OF A FORCE $\vec{K}(0, 0, K_z)$ BY MEANS OF FORCES P.

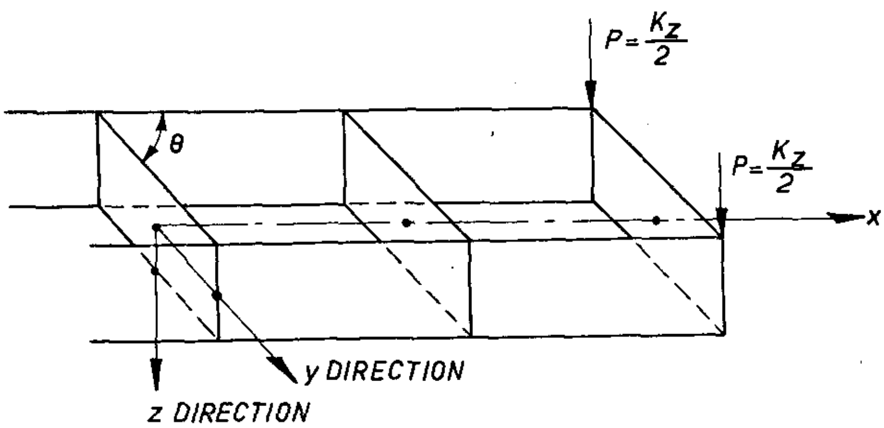
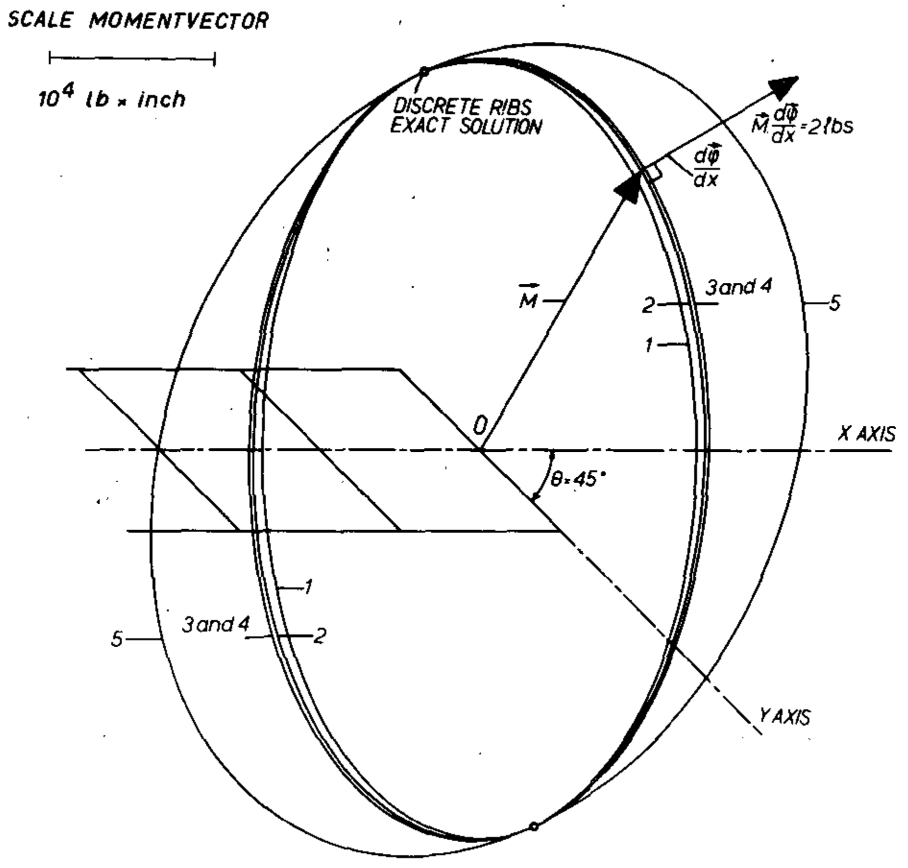


FIG. 8.17. ELLIPSES INDICATING THE ENDPOINTS OF MOMENTVECTORS WHICH CAUSE A STRAIN ENERGY OF 12 lbs x inch PER CELL. OBLIQUE RIBS.



1 DISCRETE RIBS. MIN. PRINCIPLE FOR THE STRESSES WITH 3 TYPES OF SUPPLEMENTARY STRESS SYSTEMS. ALSO EXACT SOLUTION FOR BOX BEAM WITHOUT RIBS.

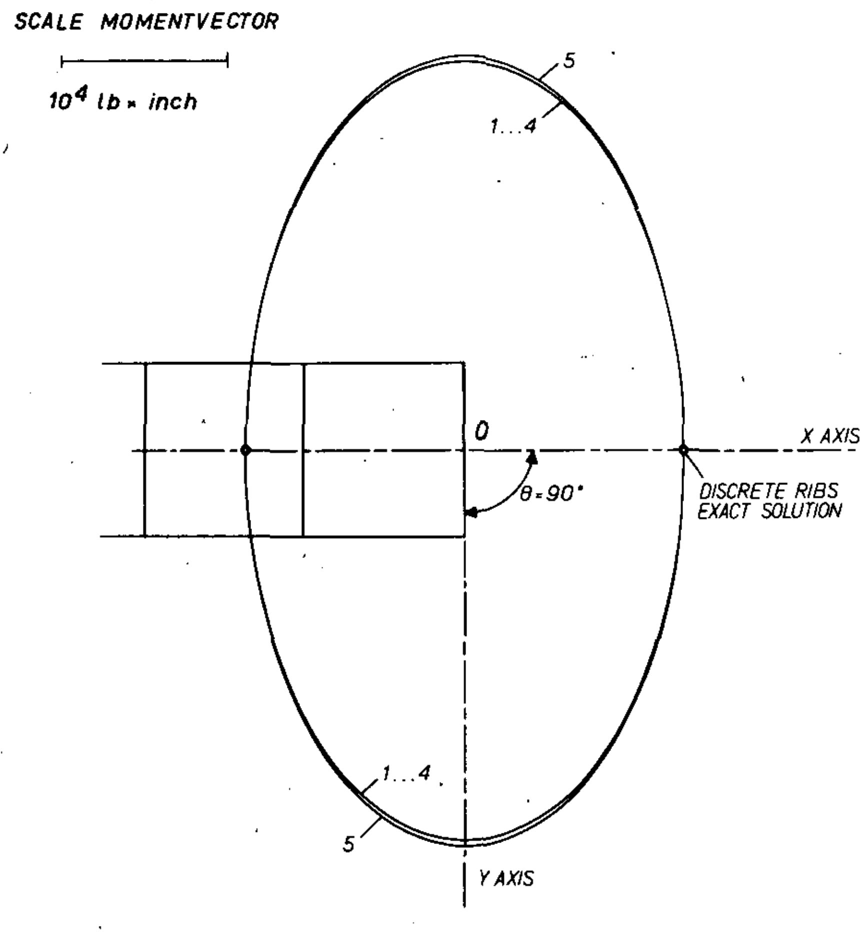
2 DISCRETE RIBS. MIN. PRINCIPLE FOR THE STRESSES WITH 5 TYPES OF SUPPLEMENTARY STRESS SYSTEMS.

3 DISCRETE RIBS. MIN. PRINCIPLE FOR THE STRAINS. ALMOST THE SAME AS 4.

4 CONTINUOUSLY DISTRIBUTED RIBS (HEMP'S SOLUTION). EXACT SOLUTION.

5 CONTINUOUSLY DISTRIBUTED INFINITELY STIFF RIBS. EXACT SOLUTION.

FIG. 8.18. ELLIPSES INDICATING THE ENDPOINTS OF MOMENT-VECTORS WHICH CAUSE A STRAIN ENERGY OF 12 lbs x inch PER CELL. NORMAL RIBS.



THE ELLIPSES 1.... 5, FROM WHICH THE ELLIPSES 1.... 4 ALMOST COINCIDE, CORRESPOND WITH THE 5 ELLIPSES OF FIG. 8.18.

FIG. 8.19. APPLICATION OF AN ELEMENTARY THEORY FOR THE OBLIQUE BEAM.

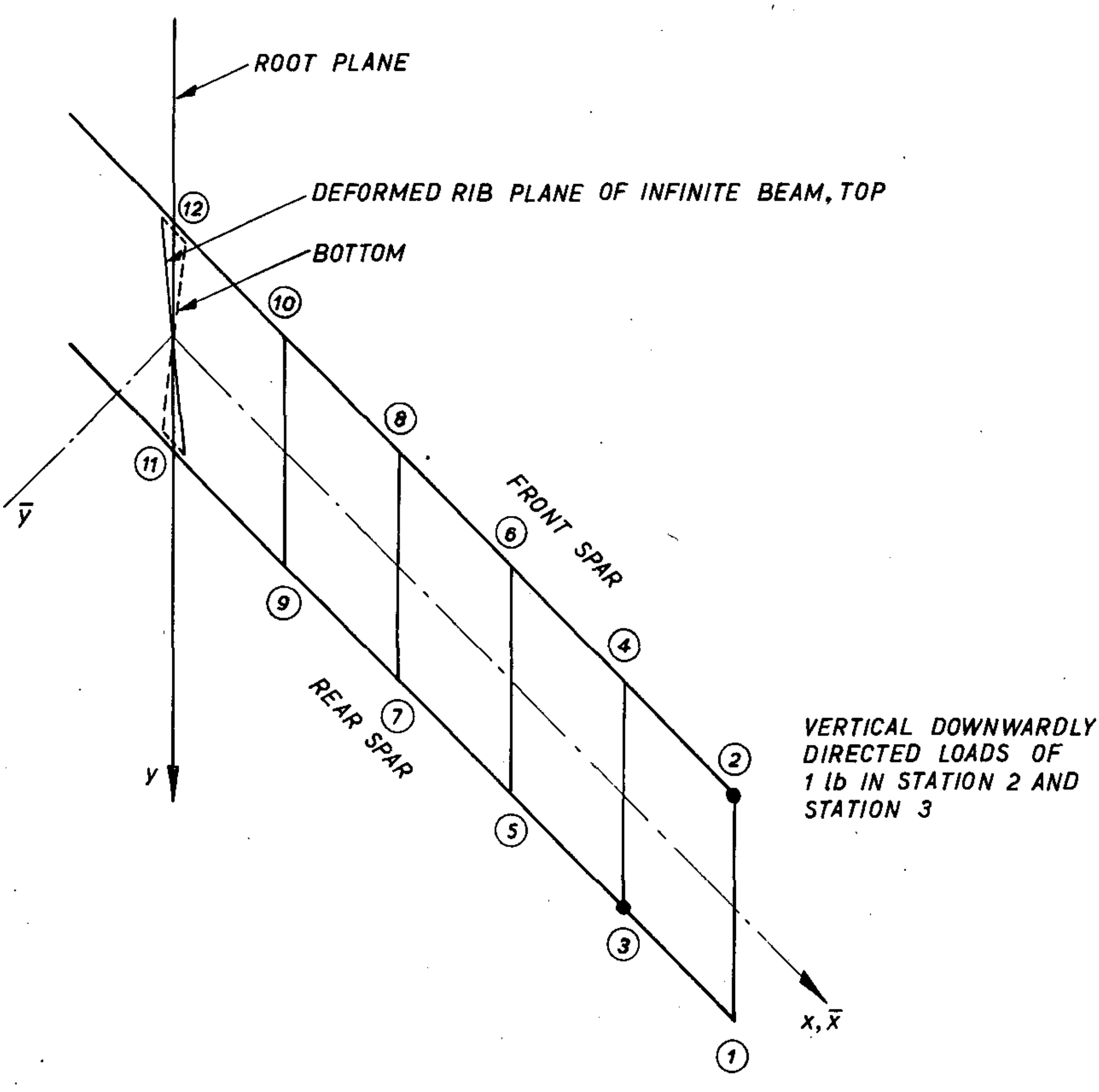


FIG. 9.1. GROUPS OF SUPPLEMENTARY STRESS SYSTEMS IN THE CELLS OF A SEMI INFINITE BOX BEAM WITH ONE CLAMPED END. EACH GROUP Z_n CONSISTS OF THE 5 TYPES FIG. 7.3, 7.4, 7.7, 7.8, AND 8.8.

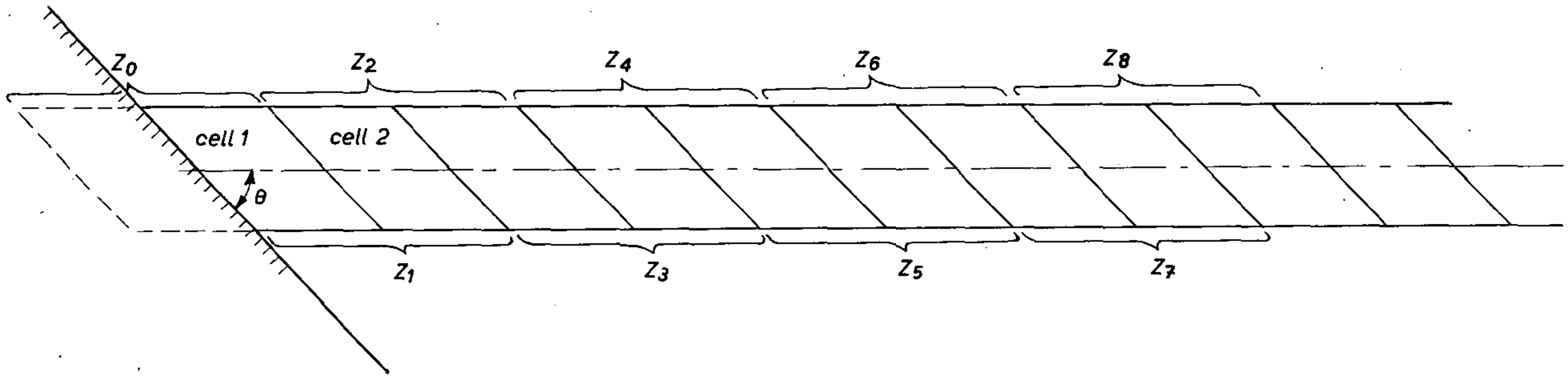


FIG. 9.2. STRESSES IN SPAR BOOMS AND WEBS AT LOAD $\vec{M}(M_x = 1, M_y = 0, M_z = 0)$

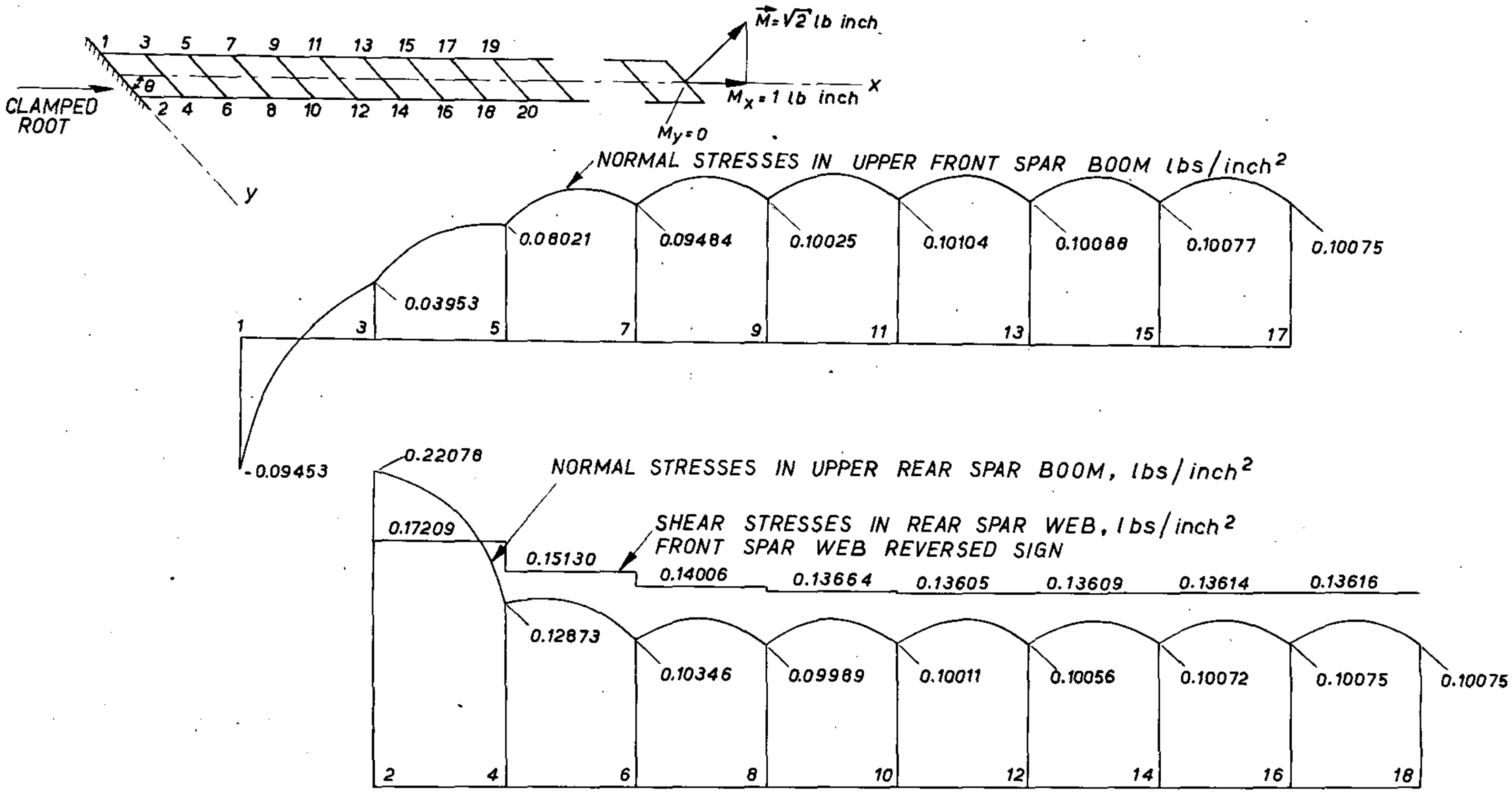
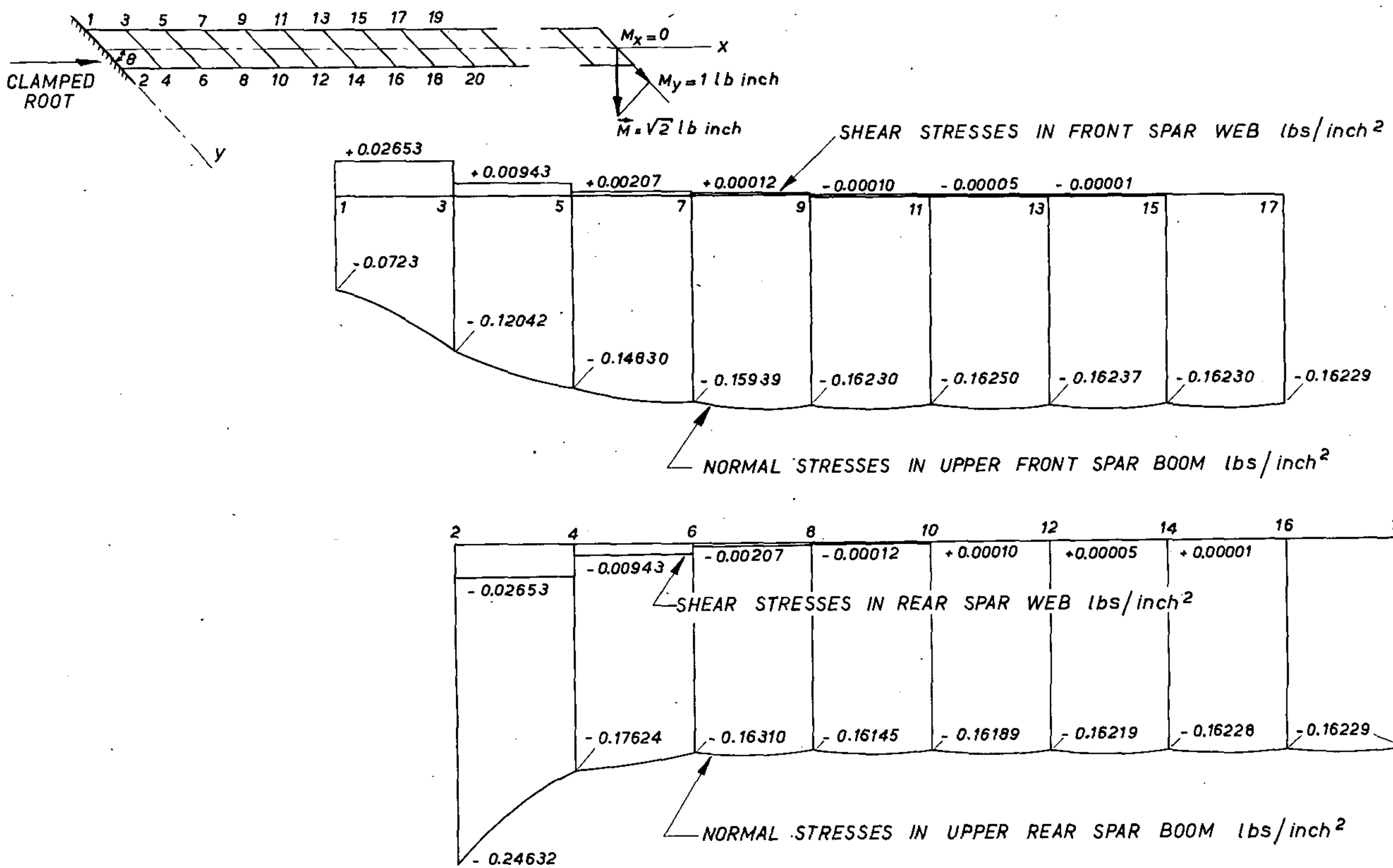


FIG. 9.3. STRESSES IN SPAR BOOMS AND WEBS AT LOAD $\vec{M}(M_x = 0, M_y = 1, M_z = 0)$



S.578 NLL-TR
Nationaal Lucht- en Ruimtevaartlaboratorium (N.L.R.), Amsterdam, the Netherlands. (Nat.Aero- and Astron.Inst.)
ON THE ANALYSIS OF SWEPT WING STRUCTURES.
J.P. Benthem, text, tables, figures.

The ribs are placed parallel to the longitudinal direction of the aircraft. There are oblique panels between the ribs andtthe spars. A structure with such oblique panels asks methods of stress analyses quite different from those used for non-swept wings. Methods based on the principles of variation calculus, the use of oblique coordinates for stiffened skin panels and matrix notation are developed. Application to a structure with 5 ribs, to the infinitely long beam and to the root restraint of the semi-infinitely lnng beam.

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