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PREFACE

This volume of Reports and Transactions of the NLR contains a selection of reports completed in recent years.

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In addition to the reports which are collected at more or less regular intervals in the volumes of Reports and Transactions, numerous others are published on subjects studied by the NLR.

A complete list of publications issued from 1921 through 1963 is available upon request.

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A. J. Marx

(Director)

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REPORT NLR - TR G.6

The complete solution for the problem of a ring wing around a circular cylindrical fuselage in stationary supersonic flow

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P. J. ZANDBERGEN.

Summary.

The linearized supersonic flow around a ring wing, concentric to a circular cylindrical fuselage, is considered. Use has been made of the Laplace-transform method.

The wing has an arbitrary rotationally symmetric shape and in addition has an axial curvature in the vertical plane; the fuselage and wing can have different angles of attack.

The solution for the general case can be obtained by superposition of the solutions of four more elementary problems. These problems are:

1. '

A case of a rotationally symmetric configuration. The ring wing has a symmetrical thickness distribution with respect to r = B, where B is the mean radius of the ring wing. A case of a rotationally symmetric configuration. The ring wing has a cone angle that is a function of the axial coordinate, but has no thickness. $\mathbf{2}$.

The axis of the ring wing has a certain curvature in the vertical plane. The fusclage and the ring wing have the same angle of attack. З.

First, the pressure distribution on the ring wing is obtained; once this distribution is known, it is possible to obtain the pressure distribution on the fuselage by means of certain influence functions, such as a generalized Wardfunction and some analogous functions.

No numerical values for these functions are given, although their properties have been investigated.

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 - 3.1 Solution for the case of ring wing thickness.
 - 3.2 Solution for a conoidally shaped ring wing.

 - 3.3 Solution for a ring wing with a curved axis. 3.4 Solution for the case of ring wing and
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- t

c(x)shape function of ring wing. pressure coefficient on fuselage. c_{pf} $f(p), f^{*}(p), f(p)$ $g(p), g^{*}(p)$ integration constants. integration constants. g(x)h(x)

List of symbols. $a(p), a^*(p), a(p)$

 $b(p), b^{*}(p)$

 C_p

half the pressure differents on the ring wing. distribution of ring wing thickness. k_n expression given by eq. A (9). l ring wing chord. variable in transformed plane. pcylindrical coordinate. r · integration variable. u velocity in axial direction divided by V. velocity in radial direction di v_r vided by V. velocity perpendicular to \overline{u} and v_t v_r divided by V.

integration constants.

integration constants.

pressure coefficient.

- axial coordinate. \boldsymbol{x} axial coordinate measured from \mathcal{Y}
- ring wing trailing edge. Laplace-transform of $\alpha'(x)$. A'(p)C'(p)Laplace-transform of c'(x)

5 figures.

I_0, I_1	modified Bessel-functions of the
	first kind.
J_{α}, J_{α}	Bessel-functions of the first kind.
K., K.	modified Bessel-functions of the
	second kind
м.	Mach_number
$P(\alpha)$	function defined by $a_1 \wedge (2)$
P(w)	function defined by eq. $A(2)$.
$r_1(x)$	in flags from the defined by eq. A (25).
Q_0	influence function defined by
~	eq. A (20).
Q_1	influence function defined by
	eq. A (38).
R	radius of ring wing.
R'(p)	Laplace-transform of $r'(x)$.
U(x)	Unit step function.
$U_n(z)$	factor in asymptotic series ex-
	pansion of Bessel-functions.
v ·	velocity of undisturbed flow.
V(z)	function defined by $eq B(20)$
W(x) = W(x)	Ward-functions
W(r R)	concratized Ward function
V V	Rescal functions of the second
$1_1, 1_2$	hind
$\mathcal{I}(m, \mathcal{P})$	Alle.
$\mathcal{D}_{0}(u, \mathbf{n})$	D (59)
\mathcal{I} (m \mathcal{D})	eq. D (00)a.
$\Sigma_1(x, \mathbf{n})$	\mathbf{P} (59)
(m)	eq. D (90)a.
$\alpha(x)$	function giving the curvature of
	the ring wing axis.
β	$= V M^2 - 1.$
γ	angle of attack.
$\delta(x)$	Dirac deltafunction.
٩	cylindrical voordinate.
λ	integration variable.
λ_{n}	the n^{th} root of eq. A (7).
μ_n	the n^{th} root of eq. A (27).
ρ	density.
φ	velocity potential.
ō	Laplace-transform of velocity
•	potential.
<i>.</i>	additional velocity potential.
-	Lanlace transform of additional
Ψa	valority notantial (
ሖ	next of velocity notonticl inde
Ψ	part of verocity potential inde-
	pendant or a.

Indices

() ' ,	prime denotes differentiation with respect to independent
	variable.

1 Introduction.

In the last years much work has been done on the problem of a ring wing concentric to a given fuselage.

Many approaches are possible for the determination of the flow variables, as for instance the graphical characteristics method of ERDMANN and OSWATTISCH (ref. 1), an approximate method by VAN DER WALLE (ref. 2) and the analytic approach of EHLERS (ref. 3).

The pressure distribution on the ring wing innerand outer side and on the fuselage can be obtained by either of these methods. On the body there is a region of influence, due to the presence of a ring wing. This region can be split up in two parts. The first part lies between the waves emanating from the wing leading- and trailing edge. The second part lies behind the first.

Although the graphical characteristics-method is a useful tool in studying the properties of the combinations considered here, it does not lead to analytical expressions which are generally valid. An advantage is the fact that arbitrary forms of fuselages and wings can be considered.

The method of VAN DER WALLE, though giving analytical expressions, has the disadvantage of being only approximate. To be able to use the method, the flow field around the fuselage alone must be known. The effects of the approximation are felt most severely in the second part of the region of influence on the body.

The only approach for an analytic solution of the problem stated here, has been made by EHLERS (ref. 3) who uses the linearized theory of supersonic flow and assumes the fuselage to be a circular cylinder.

However his work contains a number of errors and does not give the complete solution.

The purpose of the present investigation is to derive analytical expressions for the pressure distribution on ring wing and fuselage. The fuselage is supposed to be a circular cylinder, while the ring wing can have an arbitrary shape.

Though some results have been obtained earlier, it was decided to present all results in this report for the sake of completeness.

First, the pressure distribution on the ring wing outer- and inner side will be obtained. Once this distribution is known, it can be used to calculate the pressures on the fuselage by means of certain influence functions, such as a generalized Wardfunction and some new functions.

It is found, in contrast to the results of EHLERS, that there is also a non-zero pressure distribution in the second above-mentioned part of the region of influence on the body.

2 General considerations.

It is assumed that the linearized theory is valid. This means that all disturbations of the free stream velocity are assumed to be small and that the boundary conditions at the ring wing may be applied to a cylindrical surface with radius R.

In this case the problem can be stated in terms of a velocity potential φ_1 , made dimensionless with respect to the free stream velocity V, which is a solution of the following differential equation:

$$\varphi_{rr} + \frac{1}{2} \varphi_r + \frac{1}{r^2} \varphi_{\mathfrak{SS}} - \beta^2 \varphi_{xx} = 0 \quad \text{where} \quad (2.1)$$

x, r and \Im are cylindrical coordinates and $\beta = \sqrt{M^2 - 1}$. The dimensionless perturbation velocities are given by

$$\overline{u} = \frac{\partial \varphi}{\partial x}$$
, $\overline{v}_r = \frac{\partial \varphi}{\partial r}$, $\overline{v}_t = \frac{1}{r}$, $\frac{\partial \varphi}{\partial \theta}$ where (2.2)

The geometry of the ring wing and the fuselage is given in fig. 1. The radius of the cylinder has been used as unity. To present the complete solution the following problems must be solved:

1°. the geometry of the configuration is rotationally symmetric. The fuselage is aligned with the free stream velocity V. The ring wing has a symmetrical thickness distribution with respect to r = R, where R is the mean radius of the ring wing (see fig. 1).

 2° . the geometry of the configuration is rotationally symmetric. The ring wing has a cone-angle that is a function of the axial coordinate, but has no thickness (see fig. 2).

3". the fuselage is aligned with the free stream velocity V. The ring wing axis is a curved line in the vertical phase through the fuselage axis, resulting in a variable angle of incidence of the ring wing elements (see fig. 3).

 4° . the fuselage and the ring wing have the same angle of attack relative to the free stream velocity.

This angle is defined as the angle between the free stream direction and the straight common axis of wing and fuselage (see fig. 4).







Fig. 3. Geometry of ring wing in the case of ring wing axis curvature in the vertical plane.



Fig. 1. Geometry of ring wing and fuselage in the case of a rotationally symmetric thickness distribution.



Fig. 4. Geometry of ring wing and fuselage in the case of common angle of attack,

To calculate the pressure, use will be made of the linearized pressure coefficient. This is given by

$$c_{p} = \frac{\text{difference between local and main stream static pressure}}{\frac{1}{2}\rho V^{2}} = -2 \frac{\partial \varphi}{\partial x}$$
(2.3)

It is then possible to form superpositions of the various cases mentioned. The solution of a general case with arbitrary distributions of thickness, cone angle and incidence can be obtained in this way. To solve the above-mentioned problems, use will be made of the Laplace-transform method. We define,

$$\overline{\varphi} = \int_{0}^{\infty} e^{-px} \varphi \, dx \tag{2.4}$$

Multiplying eq. (2.1) by $e^{-\mu x}$ and integrating from x = 0 to $x \to \infty$, there is obtained, since $\varphi = \frac{\partial \varphi}{\partial x} = 0$ for $x \le 0$

$$\overline{\varphi_{rr}} + \frac{1}{2}\overline{\varphi_r} + \frac{1}{r^2}\overline{\varphi_{\mathfrak{S}}} - \beta^2 p^2 \overline{\varphi} = 0.$$
(2.5)

In the following sections this equation will be solved for the various cases by applying the appropriate boundary conditions.

3 The pressure distribution on the ring wing.

3.1 Solution for the case of ring wing thickness.

The thickness distribution of the ring wing which is symmetrical with respect to r = R is given by

$$h(x) = |r(x) - R|$$
 $r(0) = R.$ (3.1)

The boundary conditions on the ring wing are

$$\vec{v_r} = \frac{dr(x)}{dx} = r'(x) \text{ for the wing outer side}$$

$$\vec{v_r} = -r'(x) \text{ for the wing inner side}$$
(3.2)

The boundary condition on the fuselage is $\overline{v}_r = \frac{\partial \varphi}{\partial r} = 0.$ (3.3)

Since the flow is rotationally symmetric, the solution is independent of 9. Denoting the Laplace-transform of r'(x) by R'(p) we have to solve the following problem

The general solution of the differential equation is

$$\varphi_{01} = a(p)I_0(p\beta r) + b(p)K_0(p\beta r)$$
(3.5)

where the index 01 denotes the thickness case.

Since the potential φ must vanish for $r \to \infty$, the coefficient a(p) must be zero for the region outside of the ring wing. This is equivalent to saying that only waves travelling outwards downstream are present.

Applying the boundary condition given by eq. (3.4, b) we find for this region

$$-\overline{\varphi_{010}} = \frac{R'(p)}{p\beta} - \frac{K_0(p\beta R)}{\overline{K_0'(p\beta R)}} - (3.6)^{-1}$$

when the second index 0 stands for outerside.

The pressure on the outside of the ring wing is given by eq. (2.3)

$$c_{T_{010}} = -2 \frac{\partial \varphi}{\partial x} = -\frac{2}{2 \pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{px}}{\beta} \frac{K_0(p\beta R)}{K_0'(p\beta R)} R'(p) dp.$$

Using the convolution theorem, this van be written as

$$c_{\mathbf{F}_{\mathbf{N}\mathbf{v}}} = \frac{2}{\beta} \left\{ r'(x) - \frac{1}{\beta R} \int_{0}^{x} W_{\mathbf{v}} \left(\frac{x - t}{\beta R} \right) r'(t) dt \right\}.$$
(3.7)

This solution has been obtained by WARD in ref. 4. The function W_0 , known as WARD's function is defined by

$$W_{0}(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \frac{K_{1}(p) - K_{0}(p)}{K_{1}(p)} dp.$$
(3.8)

To obtain the solution for the inner side of the ring wing use has to be made of eqs. (3.4, c) and (3.4, d) together with eq. (3.5). It follows that the transformed potential is given by

$$\overline{\varphi}_{01i} \coloneqq -\frac{R'(p)}{p\beta} \frac{K_0'(p\beta)I_0(p\beta r) - I_0'(p\beta)K_0(p\beta r)}{K_0'(p\beta R) - I_0'(p\beta)K_0'(p\beta R)}$$
(3.9)

where the index i stands for innerside.

The pressure on the ring wing innerside is according to eq. (2.3) -.

$$c_{p_{0!i}} = \frac{2}{2\pi i} \frac{\partial}{\partial x} \int_{a-i\infty}^{a+i\infty} \frac{R'(p)}{p\beta} e^{px} \frac{K_0'(p\beta)I_0(p\beta R) - I_0'(p\beta)K_0(p\beta R)}{K_0'(p\beta R) - I_0'(p\beta)K_0'(p\beta R)} dp.$$
(3.10)

In appendix A the evaluation of this expression has been given. Using eq. A(21) the result can be written as

$$c_{p_{0li}} = \frac{2}{\beta} \left[\frac{r(x) - R}{\beta(R+1)} + r'(x) + 2 \sum_{n=1}^{n=m} r' \left\{ x - 2 n\beta(R-1) \right\} + \int_{0}^{x} r'(t) Q_{0}(x-t) dt \right]. \quad (3.11)$$

This result holds for $2(m+1)\beta(R-1) > x > 2m\beta(R-1)$. The function Q_{q} is defined by eq. A(20).

3.2 Solution for a conoidally shaped ring wing.

In this case the ring wing has no thickness, but only rotationally symmetric camber. The geometry of the wing is given by

$$\mathbf{r} = c(\mathbf{x}) \tag{3.12}$$

where r varies only slightly from r = R.

The only difference with the problem stated in eq. (3.4) is that the boundary conditions on the wing have changed.

They become

$$\overline{\varphi_r} = C'(p) \quad \text{for wing outer side} \\
\overline{\varphi_r} = C'(p) \quad \text{for wing inner side}$$
(3.13)

Herein C'(p) denotes the Laplace-transform of c'(x). Solving the differential equation (3.4, a) with the boundary conditions given by eqs. (3.13), we find by using the eqs. (3.6), (3.7), (3.9) and (3.11)

$$\overline{\varphi_{020}} = \frac{C'(p)}{p\beta} \quad \frac{K_0(p\beta r)}{K_0'(p\beta R)} . \tag{3.14, a}$$

The index 02 applies to the case of cone angle

$$c_{p_{020}} = \frac{2}{\beta} \left\{ c'(x) - \frac{1}{\beta R} \int_{0}^{x} c'(t) W_{o} \frac{x-t}{\beta R} dt \right\}$$
(3.14, b)

$$\overline{\varphi}_{02i} = \frac{C'(p)}{p\beta} \quad \frac{K_0'(p\beta)I_0(p\beta r) - I_0'(p\beta)K_0(p\beta r)}{K_0'(p\beta R) - I_0'(p\beta)K_0'(p\beta R)} \quad (3.14, c)$$

$$\overline{c_{p_{021}}} = -\frac{2}{\beta} \left\{ \frac{c(x) - R}{\beta(R+1)} + c'(x) + 2\sum_{n=1}^{n=m} r'\left(x - n\beta(R-1)\right) + \int_{0}^{x} c'(t)Q_{0}(x-t)dt \right\}$$
(3.14, d)

The last equation holds for $2(m+1)\beta(R-1) > x > 2m\beta(R-1)$.

с'n

3.3 Solution for a ring wing with a curved axis.

In this case the axis of the ring wing is assumed to have a certain curvature in the plane $\varphi = 0$. If the equation of the axis is given by $r = \sigma(x)$ (3.15)

$$(0.10)$$

the boundary conditions on the wing become

$$v_r = \varphi_r = \alpha'(x) \cos \vartheta \text{ for the wing outer side}$$

$$\overline{v_r} = \varphi_r = \alpha'(x) \cos \vartheta \text{ for the wing inner side}$$
(3.16)

Separating variables by putting

$$\varphi = \Phi(xr)\cos\vartheta \tag{3.17}$$

and applying the Laplace-transform method the following equation and conditions, to be fulfilled by Φ , are obtained

$$\overline{\Phi}_{rr} + \frac{1}{2} \overline{\Phi}_r - \left(\frac{1}{r^2} + \beta^2 p^2\right) \overline{\Phi} = 0$$
(3.18, a)

$$\overline{\Phi_r} = A'(p)$$
 for ring wing outer and inner side (3.18, b)

 $\overline{\Phi_r} = 0 \qquad \text{for } r = 1 \text{ (fuselage)} \qquad (3.18, c)$

where A'(p) is the Laplace-transform of $\alpha'(x)$.

The general solution of the differential equation is

$$\overline{\Phi} = f(p)I_1(p\beta r) + g(p)K_1(p\beta r).$$
(3.19)

Using the same arguments as in section 3.1 it can be concluded that the coefficient f(p) has to be zero in the region outside of the ring wing. Applying the boundary condition (3.18, b) one gets for this region:

$$\overline{\Phi}_{110} = \frac{A'(p)}{p\beta} \quad \frac{K_1(p\beta r)}{K_1'(p\beta r)}$$
(3.20)

where the index 11 stands for the case of axis curvature.

The pressure on the outside of the ring wing is given by eq. (2.3)

$$c_{\mathfrak{p}_{110}} = -2 \frac{\partial \varphi}{\partial x} = -\frac{2 \cos \vartheta}{2 \pi i} \int_{a-i\infty}^{a+i\infty} \frac{A'(p)}{\beta} e^{px} \frac{K_1(p\beta R)}{K_1'(p\beta R)} dp.$$

Using the convolution theorem this can be written as

$$c_{p_{110}} = \frac{2\cos\vartheta}{\beta} \left\{ \alpha'(x) - \frac{1}{\beta R} \int_{0}^{x} W_{1}\left(\frac{(x-t)}{\beta R}\right) \alpha'(t) dt \right\}.$$
(3.21)

The function W_1 , which has been discussed in appendix B, is defined by

$$W_{1} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \frac{K_{1}'(p) + K_{1}(p)}{K_{1}'(p)} dp.$$
(3.22)

The solution for the inner side of the ring wing can be obtained by using the conditions (3.18, b, c) to determine the constants f(p) and g(p) of eq. (3.19). One gets:

$$\overline{\Phi_{0i}} = \frac{A'(p)}{p\beta} \frac{K_{1}'(p\beta)I_{1}(p\beta r) - I_{1}'(p\beta)K_{1}(p\beta r)}{K_{1}'(p\beta)I_{1}'(p\beta R) - I_{1}'(p\beta)K_{1}'(p\beta R)}$$
(3.23)

The pressure distribution on the wing inner side is then given by

$$c_{p_{11i}} = -\frac{2\cos\vartheta}{2\pi i} \frac{\partial}{\partial x} \int_{u-i\infty}^{u+i\infty} e^{px} \frac{K_1'(p\beta)I_1(p\beta R) - I_1'(p\beta)K_1(p\beta R)}{K_1'(p\beta R) - I_1'(p\beta)K_1'(p\beta R)} \frac{A'(p)}{p\beta} dp. \qquad (3.24)$$

In appendix A this expression has been investigated. The result is given in eq. A(39). Using this equation the following expression is obtained

$$c_{p_{11i}} = -\frac{2\cos9}{\beta} \left\{ \frac{\alpha(0) - \alpha(x)}{\beta(R-1)} + \alpha'(x) + 2\sum_{n=1}^{n=m} \alpha' \left(x - 2n\beta(R-1) \right) + \int_{0}^{x} \alpha'(t)Q_{1}(x-t)dt \right\}.$$
(3.25)

This result holds for $2(m+1)\beta(R-1) > x > 2m\beta(R-1)$. The function Q_1 is given by eq. A(38) of appendix A.

3.4 Solution for the case of ring wing and fuselage at the same angle of attack.

If the angle of attack is taken to be γ , the potential φ can be written as

្ឋា

$$\varphi \coloneqq \gamma \cos \vartheta \left\{ \left(r + \frac{1}{r} \right) + \Phi \right\}$$
(3.26)

It should be remarked here that the coordinate system is fixed to the body. The first term in this expression is the potential of the fuselage alone at an angle of incidence γ .

The second term represents the additional potential due to the ring wing. The boundary conditions are $\overline{v_r} = 0$, for r = 1 and for r = R. Since $\overline{v_r}$ is given by

$$\overline{v}_r = \frac{\partial \varphi}{\partial r} = \gamma \cos \vartheta \left(1 - \frac{1}{r^2} + \frac{\partial \Phi}{\partial r} \right)$$

we get

$$=0$$
 for $r=1$ (3.27)

$$\frac{\partial \Phi}{\partial r} = -\frac{R^2 - 1}{R^2} \quad \text{for} \quad r = R.$$
(3.28)

As can be seen from eq. (3.26) the Laplace transform Φ of Φ satisfies eq. (3.18, a). Comparing these conditions with the equations (3.18, b, c), equations for the potential and the pressure distribution can be written down by inspection of eqs. (3.20), (3.21), (3.23) and (3.25).

For the outer side of the ring wing the expressions become

 $\partial \Phi$

dr

$$\overline{\Phi}_{120} = -\frac{R^2 - 1}{R} \frac{1}{p^2 \beta} \frac{K_1(p\beta r)}{K_1'(p\beta R)}$$
(3.29)

where the index 12 stands for the case of angle of attack and

$$c_{p_{120}} = - \frac{2\gamma(R^2 - 1)\cos\vartheta}{\beta R^2} \left\{ 1 - \frac{1}{\beta R} \int_0^T W_1\left(\frac{x - t}{\beta R}\right) dt \right\}.$$
(3.30)

And for the inner side of the ring wing:

$$\overline{\Phi}_{12^{j}} = -\frac{R^{2}-1}{p^{2}\beta R^{2}} \frac{K_{1}'(p\beta)I_{1}(p\beta r) - I_{1}'(p\beta)K_{1}(p\beta r)}{K_{1}'(p\beta)I_{1}'(p\beta R) - I_{1}'(p\beta)K_{1}'(p\beta R)}$$
(3.31)

$$c_{p_{12i}} = \frac{2\gamma(R^2 - 1)\cos\vartheta}{\beta R^2} \left\{ -\frac{x}{\beta(R - 1)} + 1 + 2m + \int_{0}^{\infty} Q_1(x - t)dt \right\}.$$
 (3.32)

The last equation holds for $2(m+1)\beta(R-1) > x > 2m\beta(R-1)$.

4 The pressure distribution on the fuselage due to the presence of a ring wing with a finite chord.

4.1 Introduction.

In the preceding sections the pressure on the wing inner and outer side has been obtained. This has been done by assuming that the wing chord extends from x = 0 to $x \to \infty$. In reality however the wing only extends from x = 0 to x = l. The solutions obtained so far are thus valid only in the region between the waves emanating from the wing leading and trailing edge. It can be remarked that these solutions could be used to obtain the pressure on the fuselage in the first part of the region of influence, as here the correct boundary condition on the fuselage has been fulfilled.

However as will be shown in the following sections, the pressure on the fuselage can be written down more easily in terms of the pressure and thickness distribution of the ring wing. The method used to find this solution will be outlined here. It consists of introducing an additional potential φ_a in the region behind the wing trailing edge, which has to be chosen in such a way as to satisfy the conditions posed by the physical picture. These conditions are:

 1° : There can be no jump in the potential across the waves emanating from the trailing edge.

- 2° : There can be no jump in the radial velocity across the wake.
- 3° : There can be no jump in the pressure across the wake.
- 4° : The radial velocity is equal to zero on the fusciage, or written in mathematical terms

1°:
$$\varphi_a = \frac{\partial \varphi_a}{\partial y} = 0$$
 $y \le \beta |r - R|$ where $y = x - l$ (4.1)

where l is the chord of the ring wing.

$$2^{\circ}: \frac{\partial \varphi_{ao}}{\partial r} + \frac{\partial \varphi_{o}}{\partial r} = \frac{\partial \varphi_{ai}}{\partial r} + \frac{\partial \varphi_{i}}{\partial r} \quad y > 0 \quad \text{and} \quad r = R.$$

$$(4.2)$$

The indices 0 and i refer to outer side and inner side of the wake respectively.

$$3^{\circ}: \frac{\partial \varphi_{ao}}{\partial y} + \frac{\partial \varphi_{o}}{\partial y} = \frac{\partial \varphi_{ai}}{\partial y} + \frac{\partial \varphi_{i}}{\partial y} \quad y > 0 \quad \text{and} \quad r = R.$$

$$(4.3)$$

$$4^{\mathfrak{p}} \colon \frac{\partial \varphi_{ao}}{\partial r} = 0 \qquad \qquad r = 1.$$

$$(4.4)$$

Introduce the Laplace-transform

$$\overline{\varphi}_{\mathfrak{a}} = \int_{0}^{\infty} e^{-py} \varphi_{\mathfrak{a}}(y) dy \,. \tag{4.5}$$

Using eq. (2.1) together with eq. (4.1) it is found that $\overline{\varphi_a}$ has to satisfy the differential equation

$$\frac{\partial^2 \overline{\varphi_a}}{\partial r^2} + \frac{1}{r} \quad \frac{\partial \overline{\varphi_a}}{\partial r} + \frac{1}{r} \quad \frac{\partial^2 \overline{\varphi_a}}{\partial \mathfrak{D}^2} - \beta^2 p^2 \overline{\varphi_a} = 0.$$
(4.6)

Multiplying eq. (4.2) by e^{-py} and integrating from y = 0 to $y \to \infty$ gives:

$$\frac{\partial\overline{\varphi}_{ao}}{\partial r} + \int_{0}^{\infty} e^{-py} \frac{\partial\varphi_{o}}{\partial r} dy = \frac{\partial\overline{\varphi}_{ai}}{\partial r} + \int_{0}^{\infty} e^{-py} \frac{\partial\varphi_{i}}{\partial r} dy.$$
(4.7)

The integral may be written as

$$\int_{0}^{\infty} e^{-py} \frac{\partial \varphi_{0}}{\partial r} dy = \int_{l}^{\infty} e^{-p(x-l)} \frac{\partial \varphi_{0}}{\partial r} dx = e^{pl} \int_{0}^{\infty} e^{-px} \frac{\partial \varphi_{0}}{\partial r} dx - e^{nl} \int_{0}^{l} e^{-py} \frac{\partial \varphi_{0}}{\partial r} dx$$

Substituting this in eq. (4.7) and recalling that $\int_{0}^{\infty} e^{-p_{x}} \frac{\partial \varphi_{0}}{\partial r} = \frac{\partial \overline{\varphi}_{0}}{\partial r}$, one obtains:

$$\frac{\partial \overline{\varphi}_{a_0}}{\partial r} + e^{pl} \frac{\partial \overline{\varphi}_{0}}{\partial r} - e^{pl} \int_{0}^{l} e^{-px} \frac{\partial \varphi_{0}}{\partial r} dx = \frac{\partial \overline{\varphi}_{ai}}{\partial r} + e^{pl} \frac{\partial \overline{\varphi}_{i}}{\partial r} - e^{pl} \int_{0}^{l} e^{-px} \frac{\partial \varphi_{i}}{\partial r} dx$$

 $e^{pl}\left[\frac{\partial\overline{\varphi_{o}}}{\partial r} - \frac{\partial\overline{\varphi_{i}}}{\partial r}\right] - e^{pl}\int_{0}^{l} e^{-px}\left\{\frac{\partial\varphi_{o}}{\partial r} - \frac{\partial\varphi_{i}}{\partial r}\right\}dx = \frac{\partial\overline{\varphi_{ai}}}{\partial r} - \frac{\partial\overline{\varphi_{ao}}}{\partial r}r = R.$ (4.8)

Applying the same reasoning to eq. (4.3) results in:

$$pe^{pl} \left[\overline{\varphi_{0}} - \overline{\varphi_{i}}\right] - e^{pl} \int_{0}^{t} e^{-px} \left\{ \frac{\partial \varphi_{0}}{\partial x} - \frac{\partial \varphi_{i}}{\partial x} \right\} dx = p \left\{ \overline{\varphi_{ai}} - \overline{\varphi_{ao}} \right\}.$$

$$(4.9)$$

The fourth condition gives:

$$\frac{\partial \overline{\varphi}_a}{\partial r} = 0. \tag{4.10}$$

These equations will be used in the derivation of the additional potential for the subsequent cases.

4.2 Solution for the case of ring wing thickness.

The flow is rotationally symmetric. Eq. (4.6) reduces thus to

$$\frac{\partial^2 \overline{\varphi_a}}{\partial r^2} + \frac{1}{r} \quad \frac{\partial \overline{\varphi_a}}{\partial r} - \beta^2 p^2 \overline{\varphi_a} = 0.$$
(4.11)

The solution of this eq. is given by different forms for the region inside or outside the surface r = R. Since the potential must vanish if $r \to \infty$, the solution for the outside is given by

$$\overline{\varphi_{oa}} = a(p)K_o(p\beta r) . \tag{4.12}$$

The solution for the inside is

$$\hat{\varphi}_{ai} = a^*(p) I_0(p\beta r) + b^*(p) K_0(p\beta r) .$$
(4.13)

With the aid of eqs. (4.8), (4.9) and (4.10) the solution of the coefficients $\overline{a}(p)$, $a^*(p)$ and $b^*(p)$ can be obtained. The evaluation of these coefficients has been performed in appendix C. The result is given by eqs. C(5), C(6) and C(7).

Substituting these expressions into (4.12) and (4.13) and adding these to eqs. (3.6) and (3.9) a result is obtained which is valid in the whole region aft of the Mach-waves emanating from the wing leading edge. This has been proved in appendix C. The resulting expressions are

$$\overline{\varphi_{0}} = \beta R e^{pt} \frac{I_{0}'(p\beta R) K_{0}'(p\beta) - K_{0}'(p\beta R) I_{0}'(p\beta)}{K_{0}'(p\beta)} K_{0}(p\beta r) \int_{0}^{t} e^{-px} g(x) dx +
+ 2 R e^{pt} \frac{K_{0}(p\beta R) I_{0}'(p\beta) - I_{0}(p\beta R) K_{0}'(p\beta)}{K_{0}'(p\beta)} K_{0}(p\beta r) \int_{0}^{t} e^{-px} r'(x) dx \qquad (4.14)$$

$$\overline{\varphi_{i}} = \beta R e^{pt} \frac{K_{0}'(p\beta) I_{0}(p\beta r) - I_{0}'(p\beta) K_{0}(p\beta r)}{K_{0}'(p\beta)} K_{0}'(p\beta R) \int_{0}^{t} e^{-px} g(x) dx +
- 2 R e^{pt} \frac{K_{0}'(p\beta) I_{0}(p\beta r) - I_{0}'(p\beta) K_{0}(p\beta r)}{K_{0}'(p\beta)} K_{0}(p\beta R) \int_{0}^{t} e^{-px} r'(x) dx \qquad (4.15)$$

when g(x) is half the pressure difference on the ring wing and r'(x) is the local half wedge angle. The pressure on the fuselage can be calculated by using eq. (4.15)

$$\frac{\partial \varphi_i}{\partial x} = -\frac{R}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{p(l+y)} \frac{K_o'(p\beta R)}{K_o'(p\beta)} \int_0^l e^{-px} g(x) dx dp + \frac{2R}{2\pi i\beta} \int_{a-i\infty}^{a+i\infty} e^{\nu(l+y)} \frac{K_o(p\beta R)}{K_o'(p\beta)} \int_0^l e^{-px} r'(x) dx dp.$$
(4.16)

Writing formally $\int_{0}^{l} e^{-px} g(x) dx = \int_{0}^{\infty} e^{-px} \{g(x)U(x) - g(x)U(x-l)\} dx$ and using the convolution theorem this results in:

$$\frac{\partial \varphi_{i}}{\partial x} = -R \int_{0}^{\infty} g(t) \left\{ U(t) - U(t-l) \right\} \cdot \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{p(l+y-t)} \frac{K_{0}'(p\beta R)}{K_{0}'(p\beta)} dp dt + \frac{2R}{\beta} \int_{0}^{\infty} r'(t) \left\{ U(t) - U(t-l) \right\} \cdot \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{p(l+y-t)} \frac{K_{0}(p\beta R)}{K_{0}'(p\beta)} dp dt .$$
(4.17)

In appendix B the complex integrals have been evaluated. Using the results obtained there, the pressure coefficient on the fuselage becomes

$$c_{pf} = 2R \int_{0}^{\infty} g(t) \left\{ U(t) - U(t-l) \right\} \left\{ \frac{1}{\beta} Z_{0} \left(\frac{y+l-t-\beta(R-1)}{\beta}, R \right) + \frac{\delta_{+} \left\{ y+l-t-\beta(R-1) \right\}}{\sqrt{R}} dt + \frac{4R}{\beta} \int_{0}^{\infty} r^{*}(t) \left\{ U(t) - U(t-l) \right\} \left\{ \frac{1}{\beta} W_{0} \frac{y+l-t-\beta(R-1)}{\beta}, R \right\} - \frac{\delta_{+} \left\{ y+l-t-\beta(R-1) \right\}}{\sqrt{R}} \right\} dt.$$

$$(4.18)$$

Performing the integration gives

$$c_{pl} = 2 \sqrt{R} g \{ y + l - \beta(R-1) \} + \frac{2R}{\beta} \int_{0}^{y+l-\beta(R-1)} g(t) Z_{0} \left(\frac{y+l-t-\beta(R-1)}{\beta}, R \right) dt + \frac{4\sqrt{R}}{\beta} r' \{ y+l-\beta(R-1) \} - \frac{4R}{\beta^{2}} \int_{0}^{y+l-\beta(R-1)} r'(t) W_{0} \left(\frac{y+l-t-\beta(R-1)}{\beta}, R \right) dt.$$
(4.19)

This expression is valid if $\beta(R-1) > y > \beta(R-1) - l$. For $y > \beta(R-1)$ the fuselage pressure coefficient becomes

$$c_{pl} = \frac{2R}{\beta} \int_{0}^{t} g(t) Z_{0} \left(\frac{y+l-t-\beta(R-1)}{\beta}, R \right) dt + \frac{4R}{\beta^{2}} \int_{0}^{t} r'(t) W_{0} \left(\frac{y+l-t-\beta(R-1)}{\beta}, R \right) dt .$$
(4.20)

With the aid of eqs. (4.20) and (4.19) it is possible therefore to determine the pressure coefficient on the fuselage in the whole influence region of the ring wing. The pressure coefficient is directly determined by the pressure differential 2g(x) across the ring wing and by the local half-wedge angle r'(x).

4.3 Solution for a conoidally shaped ring wing.

In this case too the potential will satisfy eq. (4.11). The additional potential can be written as in eqs. (4.12) and (4.13). The derivation of the resulting expressions has been given in appendix C. Expressions valid in the whole region behind the leading edge are presented in eqs. C(20, a) and C(20, b).

$$\int_{0}^{r} = \beta R e^{nt} - \frac{K_{0}(p\beta r)}{K_{0}'(p\beta)} \{ I_{0}'(p\beta R) K_{0}'(p\beta) - K_{0}'(p\beta R) I_{0}'(p\beta) \} \int_{0}^{t} e^{-px} g_{1}(x) dx$$
 (4.21, a)

$$\overline{\varphi_1} = \beta R e^{ii} \frac{K_0'(p\beta R)}{K_0'(p\beta)} \{ K_0'(\overline{p}\beta) I_0(\overline{p}\beta r) - I_0'(p\beta) K_0(\overline{p}\beta r) \} \int_0^1 e^{-px} g_1(x) dx$$
(4.21, b)

where $g_1(x)$ is half the pressure difference on the ring wing.

Using eq. (4.21, b) the pressure on the fuselage can be calculated. As can be seen by comparing eq. (4.21, c) with the first term of eq. (4.14), these expressions are completely identical.

Hence by using the same procedure applied when deriving eqs. (4.19) and (4.20) one gets

$$c_{pf} = 2 \sqrt{R} g_1 \{ y + l - \beta(R-1) \} + \frac{2R}{\beta} \int_{0}^{y+l-\beta(R-1)} g(t) Z_0 \left\{ \frac{y+l-t-\beta(R-1)}{\beta}, R \right\} dt \quad (4.22, a)$$

$$(R-1) \ge y \ge \beta(R-1) - l$$

if
$$\beta(R-1) > y > \beta(R-1)$$

and

$$c_{pl} = \frac{2R}{\beta} \int_{0}^{y+l-\beta(R-1)} g(t)Z_{0} \left\{ \frac{y+l-t-\beta(R-1)}{\beta}, R \right\} dt$$
(4.22, b)

if $y > \beta(R-1)$.

Eqs. (4.19) and (4.22, a) give the pressure on the fuselage in the first part of the region of influence. The main part is given by the first terms which is the two dimensional result multiplied by the factor VR. This multiplication factor indicates the focussing effect of the inward directed Mach-waves.

The second terms can be considered as corrections to these terms which have the effect of advancing in x-direction and thereby distort the purely two-dimensional result. Eqs. (4.20) and (4.22, b) give the pressure in the second part of the region of influence.

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4.4 Solution for a ring wing with a curved axis.

In this case the potential can be written as

$$\Phi_a := \Phi_a \cos \varphi. \tag{4.23}$$

Eq. (4.6) reduces therefore to .

$$\frac{\partial^2 \overline{\Phi}_a}{\partial r^2} + \frac{1}{r} \quad \frac{\partial \overline{\Phi}_a}{\partial r} - \frac{1}{r^2} \ \overline{\Phi}_a - \beta^2 p^2 \overline{\Phi}_a = 0.$$
(4.24)

The additional potential for the outside can be written as

$$\Phi_{ao} = f(p) K_1(p\beta r) \tag{4.25, a}$$

and for the innerside

$$\overline{\Phi}_{ai} = f^{*}(p)K_{1}(p\beta r) + g^{*}(p)I_{1}(p\beta r).$$
(4.25, b)

Application of the eqs. (4.8), (4.9) and (4.10) gives the three equations which determine the unknown quantities $\overline{f}(p)$, $f^*(p)$ and $g^*(p)$. The derivation of these quantities has been given in appendix C. Just as in the former two cases, expression for the potential result which are valid in the whole influence region.

These expressions are, according to eqs. (C.28, a) and (C.28, b),

$$\overline{\Phi}_{0} = \beta R e^{pt} \left\{ I_{1'}(p\beta R) K_{1'}(p\beta) - K_{1'}(p\beta r) I_{1'}(p\beta) \right\} \frac{K_{1}(p\beta r)}{K_{1'}(p\beta)} \int_{0}^{t} e^{-px} g_{2}(x) dx \qquad (4.26, a)$$

and

$$\overline{\Phi}_{i} = \beta R e^{pl} \{ I_{1}(p\beta r) K_{1}'(p\beta) - K_{1}(p\beta R) I_{1}'(p\beta) \} \frac{K_{1}'(p\beta R)}{K_{1}'(p\beta)} \int_{0}^{t} e^{-px} g_{2}(x) dx, \qquad (4.26, b)$$

where $g_2(x)$ is half the pressure difference between wing outer and inner surface.

Using eq. (4.26, b) an expression for the pressure on the fuselage can be derived

$$\left(\frac{\partial\varphi_i}{\partial x}\right)_{r=1} = -\frac{R}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{r(t'+y)} \frac{K_1'(p\beta R)}{K_1'(p\beta)} \int_0^t e^{-px} g_2(x) \, dx \, dp \,. \tag{4.27}$$

Using the convolution theorem this gives

$$\left(\frac{\partial\varphi_i}{\partial x}\right)_{r=1} = -R \int_0^\infty g_2(t) \left\{ U(t) - U(t-l) \right\} \cdot \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{p(t+y-t)} \frac{K_1'(p\beta R)}{K_1'(p\beta)} dp dt .$$
(4.28)

The complex integral has been studied in appendix B. It is analogous to the first complex integral in eq. (4.17).

Applying the result obtained in the appendix we have .

$$c_{pl} = 2R \int_{0}^{\infty} g_{2}(t) \left\{ U(t) - U(t-l) \right\} \left\{ \frac{1}{\beta} Z_{1} \left(\frac{y+l-t-\beta(R-1)}{\beta}, R \right) + \frac{\delta + \left\{ y+l-t-\beta(R-1) \right\}}{\sqrt{R}} \right\} dt.$$

Performing the integration:

$$c_{pl} = 2 \, \sqrt{R} \, g_2 \left\{ y + l - \beta(R-1) \right\} + \frac{2R}{\beta} \, \int_0^{y+l-\beta(R-1)} g_2(t) Z_1 \left(\frac{(y+l-t-\beta(R-1))}{\beta}, R \right) dt \,. \tag{4.29}$$

This expression is valid if:

$$\beta(R-1) > y > \beta(R-1) - l.$$

For $y > \beta(R-1)$ the pressure coefficient c_{pf} is given by

$$p_{pl} = \frac{2R}{\beta} \int_{0}^{t} g_{2}(t) Z_{1} \left(\frac{y+l-t-\beta(R-1)}{\beta}, R \right) dt.$$
(4.30)

The pressure on the fusciage in the case of ring wing axis curvature is given therefore directly by the pressure on the ring wing and a characteristic function Z_1 , which has to be calculated once.

4.5 Solution for the case of ring wing and fuselage at the same angle of attack.

The solution for this case is completely identical with the solution for the case of ring wing axis curvature.

The pressure is thus given by eqs. (4.29) and (4.30), if the function $g_2(t)$ is replaced by the pressure difference on the ring wing calculated with the aid of section 3.4. **Conclusions.**

The supersonic flow around a ring wing concentric to a circular cylindrical fuselage is considered. Use has been made of the linearized theory. With the aid of the Laplae-transform method, the pressure distribution on the ring wing and the fuselage has been determined for four elementary cases.

By superposition of these cases a more general problem can be solved.

The pressure distribution on the ring wing has been given in terms of integrals containing the product of a geometrical parameter and an influence function. These influence functions are dependent on only one variable (the axial coordinates).

To obtain the pressure distribution on the fuselage use has to be made of the pressure distribution on the ring wing since it is given in terms of integrals containing the pressure difference on the ring wing multiplied by an influence function dependent on two variables — the axial coordinate and the radius of the ring wing. The properties of these influence functions have been investigated and expressions convenient for numerical calculations are given.

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APPENDIX A.

Derivation of the pressure distribution on the inner side of a ring wing.

A. 1. The pressure-distribution-for-the-rotationally-symmetric-case-

If it is assumed that the flow is rotationally symmetric, the pressure on the inner side of a ring wing is given by an equation of the type (3.10). Thus the evaluation of the following expression has to be considered:

$$\frac{1}{2\pi i} \frac{\partial}{\partial x} \int_{q-i\infty}^{d+i\infty} \frac{R'(p)}{p\beta} e^{px} \frac{K_0'(p\beta)I_0(p\beta R) - I_0'(p\beta)K_0(p\beta R)}{K_0'(p\beta R) - I_0'(p\beta)K_0'(p\beta R)} dp.$$
 A(1)

Introduce the function

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$$P_{0}(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{K_{0}'(p\beta)I_{0}(p\beta R) - I_{0}'(p\beta)K_{0}(p\beta R)}{K_{0}'(p\beta)I_{0}'(p\beta R) - I_{0}'(p\beta)K_{0}'(p\beta R)} \frac{e^{px}}{p} dp \,. \tag{A(2)}$$

Differentiation of this expression under the integration sign gives an expression which is not defined in the ordinary sense, since the integral is diverging then. However it can be given a meaning by using the concept of distribution functions (ref. 5).

In this case the differentiation in eq. A(1) may be performed under the integration sign. Using the convolution theorem the result can be written as:

$$\frac{1}{\beta}\int_{0}^{x} r'(t)P_{0}'(x-t)dt. \qquad A(3)$$

Thus the properties of the following equation have to be investigated:

$$P_{0}'(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{K_{0}'(p\beta)I_{0}(p\beta R) - I_{0}'(p\beta)K_{0}(p\beta R)}{K_{0}'(p\beta R) - I_{0}'(p\beta)K_{0}'(p\beta R)} e^{px} dp. \qquad A(4)$$

Introducing $p = i\lambda$, eq. A(4) transforms into:

$$P_{0}'(x) = \frac{1}{2\pi i} \int_{-\alpha i - \infty}^{-\alpha i + \infty} e^{i\lambda x} \frac{J_{0}(\lambda\beta R) Y_{1}(\lambda\beta) - J_{1}(\lambda\beta) Y_{0}(\lambda\beta R)}{Y_{1}(\lambda\beta) J_{1}(\lambda\beta R) - J_{1}(\lambda\beta) Y_{1}(\lambda\beta R)} d\lambda.$$
 A(5)

This integral can be expressed formally as the sum of the residues of the poles of the integrand.

The numerator has a pole at $\lambda = 0$. By expanding the Bessel-functions around $\lambda = 0$, the residue is found to be

$$\frac{1}{2\pi i} \quad \frac{2R}{\beta(R^2-1)} \qquad A(6)$$

The other poles are located at the zeros of the denominator. If λ_n is the nth root of

$$Y_{1}(\lambda\beta)J_{1}(\lambda\beta R) - J_{1}(\lambda\beta)Y_{1}(\lambda\beta R) = 0$$
 A(7)

there is also a root $-\lambda_n$. All the zeros occur for real values of λ . The sum of the residues of the poles at λ_n and $-\lambda_n$ is given by

$$\frac{1}{2\pi i} \left\{ e^{i\lambda_{n,\ell}} \frac{\left[J_{0}(\lambda_{n}\beta R)Y_{1}(\lambda_{n}\beta) - J_{1}(\lambda_{n}\beta)Y_{0}(\lambda_{n}\beta R)\right]}{\frac{d}{d\lambda} \left[Y_{1}(\lambda\beta)J_{1}(\lambda\beta R) - J_{1}(\lambda\beta)Y_{1}(\lambda\beta R)\right]_{\lambda=\lambda_{n}}} + e^{-i\lambda_{n}x} - \frac{\left[J_{0}(\lambda_{n}\beta R)Y_{1}(\lambda_{n}\beta) - J_{1}(\lambda_{n}\beta)Y_{0}(\lambda_{n}\beta R)\right]}{\frac{d}{d\lambda} \left[Y_{1}(\lambda\beta)J_{1}(\lambda\beta R) - J_{1}(\lambda\beta)Y_{1}(\lambda\beta R)\right]_{\lambda=\lambda_{n}}} = \frac{2}{2\pi i} \cos \lambda_{n}x \frac{J_{0}(\lambda_{n}\beta R)Y_{1}(\lambda_{n}\beta) - J_{1}(\lambda_{n}\beta)Y_{0}(\lambda_{n}\beta R)}{\frac{d}{d\lambda} \left[Y_{1}(\lambda\beta)J_{1}(\lambda\beta R) - J_{1}(\lambda\beta)Y_{1}(\lambda\beta R)\right]_{\lambda=\lambda_{n}}} \right\}.$$
(8)

This expression can be greatly simplified by using eq. A(7) and the Wronskian relations (see for instance ref. 6)

$$J_{v}(z)Y_{v}'(z) - Y_{v}(z)J_{v}'(z) = \frac{2}{\pi z}$$

Eq. A(7) gives

$$\frac{J_1(\lambda_n\beta R)}{J_1(\lambda_n\beta)} = \frac{Y_1(\lambda_n\beta R)}{Y_1(\lambda_n\beta)} = k_n.$$
 (9)

The numerator of eq. A(3) can be written now as:

$$\frac{1}{k_n} \{ J_0(\lambda_n \beta R) Y_1(\lambda_n \beta R) - J_1(\lambda_n \beta R) Y_0(\lambda_n \beta R) \} = -\frac{2}{\pi \lambda_n \beta R k_n} .$$
 A(10)

Performing the differentiation and using the result of eq. A(9) and the relation $J_1(X)Y_1'(x) = -J_1'(X)Y_1(x) = \frac{2}{\pi x}$ one obtains

$$\frac{d}{d\lambda} \left[Y_1(\lambda\beta) J_1(\lambda\beta R) - J_1(\lambda\beta) Y_1(\lambda\beta R) \right]_{\lambda = \lambda_n} = \frac{2}{\pi \lambda_n K_n} \left\{ K_n^2 - 1 \right\}.$$
 A(11)

Collecting the results of eqs. A(6), A(8), A(10) and A(11) it is found that:

$$P_{o}'(x) = \frac{2R}{\beta(R^2 - 1)} - 2\sum_{1}^{\infty} \frac{1}{\beta R(K_n^2 - 1)} \cos \lambda_n x \, . \qquad A(12)$$

The singularities and discontinuities of $P_0'(x)$ have to be investigated now. This can be done by considering the asymptotic expansion of the infinite series. This means the asymptotic expansion of k_n and λ_n have to be derived. The following expressions hold:

$$J_{0}(x) = \left[\begin{array}{c} \frac{2}{\pi x} \left\{ \cos \left(x - \frac{1}{4} \pi \right) + \frac{1}{8 x} \sin \left(x - \frac{1}{4} \pi \right) \dots \right\} \right] \\ J_{1}(x) = \left[\begin{array}{c} \frac{2}{\pi x} \left\{ \cos \left(x - \frac{3}{4} \pi \right) - \frac{3}{8 x} \sin \left(x - \frac{3}{4} \pi \right) \dots \right\} \right] \\ Y_{0}(x) = \left[\begin{array}{c} \frac{2}{\pi x} \left\{ \sin \left(x - \frac{1}{4} \pi \right) - \frac{1}{8 x} \cos \left(x - \frac{1}{4} \pi \right) \dots \right\} \right] \\ Y_{1}(x) = \left[\begin{array}{c} \frac{2}{\pi x} \left\{ \sin \left(x - \frac{3}{4} \pi \right) + \frac{3}{8 x} \cos \left(x - \frac{3}{4} \pi \right) \dots \right\} \right] \\ \end{array} \right]$$

Using eqs. A(13) it can be proved that for a sufficiently large n the root λ_n of eq. A(7) is given by

$$\lambda_n = \frac{n\pi}{\beta(R-1)} + \frac{3}{8 n\pi\beta} \frac{R-1}{R} \dots A(14)$$

In deriving the asymptotic expansion for k we use the identity

$$k_{n} = -\frac{2}{\pi \lambda_{n} \beta R} \frac{1}{J_{0}(\lambda_{n} \beta R) Y_{1}(\lambda_{n} \beta) - J_{1}(\lambda_{n} \beta) Y_{0}(\lambda_{n} \beta)}.$$
 A(15)

The result can be written by using eqs. A(13) and A(14) as

$$k_{n} = \frac{1}{\sqrt{R}} \frac{1}{(-1)^{n} + 0\left(\frac{1}{n^{2}}\right)} = \frac{(-1)^{n}}{R} \left\{ 1 - 0\left(\frac{1}{n^{2}}\right) \dots \right\}$$
 A(16)

Substituting the expansions A(14) and A(16) in eq. A(12) gives

$$P_{0}'(x) = \frac{2R}{\beta(R^{2}-1)} + 2\sum_{1}^{\infty} \frac{1}{\beta(R-1)} \cos \frac{n\pi x}{\beta(R-1)} + 0\left(\sum_{1}^{\infty} \frac{\cos n\pi x}{n^{2}}\right).$$
 A(17)

As the term $0\left(\sum_{1}^{\infty} \frac{\cos n\pi x}{n^2}\right)$ does not possess singularities or discontinuities, all the singularities and discontinuities of $P_0'(x)$ are contained in the second term of eq. A(17). Considering the expansion of $\delta(\varphi)$ it is found that

$$\frac{2}{\beta(R-1)} \sum_{1}^{\infty} \cos \left(\frac{n\pi x}{\beta(R-1)} \right) = \frac{2\pi}{\beta(R-1)} \sum_{-\infty}^{+\infty} \delta \left\{ 2m\pi - \frac{\pi x}{\beta(R-1)} \right\} - \frac{1}{\beta(R-1)} \quad A(18)$$

Using eq. A(9), A(12) and A(18):

er. 1. 5. 24

$$P_{0}'(x) = \frac{1}{\beta(R+1)} + \frac{2\pi}{\beta(R-1)} \sum_{-\infty}^{+\infty} \delta\left(2m\pi - \frac{\pi x}{\beta(R-1)}\right) + Q_{0}(x).$$
 A(19)

 Q_0 is a continuous function which can be calculated by considering the equation:

$$Q_0(x) = \frac{2}{\beta R} \sum_{1}^{\infty} \frac{J_1^2(\lambda_n \beta)}{J_1^2(\lambda_n \beta) - J_1^2(\lambda_n \beta R)} \cos \lambda_n x - \frac{2}{\beta (R-1)} \sum_{1}^{\infty} \cos \frac{n \pi x}{\beta (R-1)}.$$
 A(20)

Substituting eq. A(19) in eq. A(3) the following relation is obtained finally:

$$\int_{0}^{x} r'(t) P_{0}'(x-t) dt = \frac{r(x) - r(0)}{\beta(R+1)} + r'(x) + 2 \sum_{n=1}^{n=m} r' \left\{ x - 2n\beta(R-1) \right\} + \int_{0}^{x} r'(t) Q_{0}(x-t) dt. A(21)$$

This expression holds for $2(m+1)\beta(R-1) > x > 2m\beta(R-1)$. The equation A(21) can be proved also by the following considerations. The integral A(4) is diverging in the ordinary sense.

However, using the asymptotic expansion of the integrand of eq. A(4) is is permitted to write:

$$\begin{split} P_{0}'(x) &= \frac{1}{2 \pi i} \int_{a-i\infty}^{a+i\infty} e^{p_{x}} \left\{ \frac{K_{0}'(p\beta)I_{0}(p\beta R) - I_{0}'(p\beta)K_{0}(p\beta R)}{K_{0}'(p\beta R) - I_{0}'(p\beta)K_{0}'(p\beta R)} - \frac{\cosh p\beta(R-1)}{\sinh p\beta(R-1)} \right\} dp + \\ &+ \frac{1}{2 \pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \frac{\cosh p\beta(R-1)}{\sinh p\beta(R-1)} dp \,. \end{split}$$

The first integral is convergent; the latter can be handled with distribution theory. Therefore:

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \frac{\cosh p\beta(R-1)}{\sinh p\beta(R-1)} dp = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \left\{ 1 + e^{-2p\beta(R-1)} \right\} \sum_{0}^{\infty} e^{-2np\beta(R-1)} dp = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left\{ e^{px} + 2\sum_{1}^{\infty} e^{p\left\{x-2n\beta(R-1)\right\}} \right\} dp = \delta_{+}(x) + 2\sum_{1}^{\infty} \delta \left\{ x-2n\beta(R-1) \right\}.$$

The quantity $\delta_+(x)$ is defined by $\int_0^\infty \delta_+(x) dx = 1$ and $\delta_+(x) = 0 |x| > 0$.

The first integral can be written as $(p = i\lambda)$

 $ai - \alpha$

$$\frac{1}{2\pi i}\int_{a_{1}+\infty}^{\infty}e^{i\lambda x}\left\{\frac{J_{0}(\lambda\beta R)Y_{1}(\lambda\beta)}{Y_{1}(\lambda\beta)J_{1}(\lambda\beta R)}-J_{1}(\lambda\beta)Y_{0}(\lambda\beta R)}-\frac{\cos\lambda\beta(R-1)}{\sin\lambda\beta(R-1)}\right\} d\lambda.$$

Now the integrand does possess a pole at $\lambda = 0$.

By expanding, the residue is found to be

$$\frac{1}{2\pi i} \quad \frac{1}{\beta(R+1)} \; .$$

The other poles are located at the zeros of $Y_1(\lambda\beta)J_1(\lambda\beta R) - J_1(\lambda\beta)Y_1(\lambda\beta R)$ and at the zeros of $\sin \lambda\beta(R-1)$. It can be seen by using the results already derived that:

$$P_{0}'(x) = \frac{1}{\beta(R+1)} + \frac{2}{\beta R} \sum_{1}^{\infty} \frac{J_{1}^{2}(\lambda_{n}\beta)}{J_{1}^{2}(\lambda_{n}\beta) - J_{1}^{2}(\lambda_{n}\beta R)} \cos \lambda_{n}x - \frac{2}{\beta(R-1)} \sum_{1}^{\infty} \frac{n\pi x}{\beta(R-1)} + \delta_{+}(x) + 2 \sum_{1}^{\infty} \delta \{x - 2n\beta(R-1)\}.$$

Substituting this expression into eq. A(3) it will be seen that the result A(21) can be obtained again. Then we have proved in this case, that formaly taking the residues in the poles of the integrands, without making use of the requirement of convergence of the integral, and making use of the concept of distribution theory, is equivalent to the more ordinary method of integration by splitting off the singularities.

A.2 The pressure distribution for the lifting cases.

. . .

If the ring wing axis has a certain curvature, and as a consequence is carrying lift, the pressure distribution on the ring wing innerside is given by an equation of the type (3.24)

$$\frac{1}{2\pi i} \quad \frac{\partial}{\partial x} \int_{a-i\infty}^{a+i\infty} e^{px} \quad \frac{K_1'(p\beta)I_1(p\beta R) - I_1'(p\beta)K_1(p\beta R)}{K_1'(p\beta R) - I_1'(p\beta)K_1'(p\beta R)} \quad \frac{A'(p)}{p\beta} dp \quad A(22)$$

Now, introduce the function:

$$P_{1}(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{K_{1}'(p\beta)I_{1}(p\beta R) - I_{1}'(p\beta)K_{1}(p\beta R)}{K_{1}'(p\beta R) - I_{1}'(p\beta)K_{1}'(p\beta R)} - \frac{e^{p_{2}}}{p} dp .$$
 A(23)

Using the same arguments as in section A.1 eq. A(22) can be written as

$$\int_{0}^{x} \alpha'(t) P_{1}'(x-t) dt \qquad A(24)$$

where

$$P_{1}'(x) = \frac{1}{2\pi i} \int_{a-ix}^{a+i\infty} \frac{K_{1}'(p\beta)I_{1}(p\beta R) - I_{1}'(p\beta)K_{1}(p\beta R)}{K_{1}'(p\beta R) - I_{1}'(p\beta)K_{1}'(p\beta R)} e^{px}dp.$$
(A(25)

The evaluation of this function is quite analogous to that of the function $P_0'(x)$.

Thus by introducing $p = i\lambda$, one finds:

$$P_{1}'(x) = -\frac{1}{2\pi i} \int_{-ai-x}^{-ai+x} \frac{J_{1}(\lambda\beta R)Y_{1}'(\lambda\beta) - Y_{1}(\lambda\beta R)J_{1}'(\lambda\beta)}{J_{1}'(\lambda\beta R)Y_{1}'(\lambda\beta) - Y_{1}'(\lambda\beta R)J_{1}'(\lambda\beta)} e^{i\lambda x} d\lambda.$$
 A(26)

The only poles of the integrand are located at the zero's of the denominator. If μ_n is the nth root of

$$J_{1}'(\lambda\beta R)Y_{1}'(\lambda\beta) - Y_{1}'(\lambda\beta R)J_{1}'(\lambda\beta)$$
 A(27)

then $-\mu_n$ is a root also. All the roots occur for real values of λ . The sum of the residues of the poles at μ_n and $-\mu_n$ is given by:

$$\frac{-1}{2\pi i} \begin{cases} e^{i\mu_{n}x} & \frac{[J_{1}(\mu_{n}\beta R)Y_{1}'(\mu_{n}\beta) - Y_{1}(\mu_{n}\beta R)J_{1}'(\mu_{n}\beta)]}{\frac{d}{d\lambda} [Y_{1}'(\lambda\beta)J_{1}'(\lambda\beta R) - Y_{1}'(\lambda\beta R)J_{1}'(\lambda\beta)]_{\lambda=\mu_{n}}} + \\
+ e^{-i\mu_{n}x} \frac{[J_{1}(-\mu_{n}\beta R)Y_{1}'(-\mu_{n}\beta) - Y_{1}(-\mu_{n}\beta R)J_{1}'(-\mu_{n}\beta)]}{\frac{d}{d\lambda} [Y_{1}'(\lambda\beta)J_{1}'(\lambda\beta R) - Y_{1}'(\lambda\beta R)J_{1}'(\lambda\beta)]_{\lambda=-\mu_{n}}} \end{cases} = \\
= \frac{-2}{2\pi i} \cos \mu_{n}x \frac{J_{1}(\mu_{n}\beta R)Y_{1}'(\mu_{n}\beta) - Y_{1}(\mu_{n}\beta R)J_{1}'(\mu_{n}\beta)}{\frac{d}{d\lambda} [Y_{1}'(\lambda\beta)J_{1}'(\lambda\beta R) - Y_{1}'(\lambda\beta R)J_{1}'(\lambda\beta R)]_{\lambda=-\mu_{n}}} . A(28)$$

This expression can be simplified by the same method as used in the simplification of eq. A(8). Eq. A(27) gives:

$$\frac{J_1'(\mu_n\beta R)}{J_1'(\mu_n\beta)} = \frac{Y_1'(\mu_n\beta R)}{Y_1'(\mu_n\beta)} = q_n.$$
 A(29)

The numerator of eq. A(28) can be written as:

$$\frac{1}{q_n} \{ J_1'(\mu_n \beta R) Y_1'(\mu_n \beta R) - Y_1(\mu_n \beta R) J_1'(\mu_n \beta R) \} = \frac{2}{\pi \mu_n \beta R q_n} .$$
 A(30)

Performing the differentiation gives:

$$\frac{d}{d\lambda} \left[Y_1'(\lambda\beta) J_1'(\lambda\beta R) - Y_1'(\lambda\beta R) J_1'(\lambda\beta) \right]_{\lambda=\mu_n} = \frac{2}{\pi\mu_n q_n} \left\{ q_n^2 \left(1 - \frac{1}{(\mu_n \beta)^2} \right) - 1 + \frac{1}{(\mu_n \beta R)^2} \right\}.$$
 A(31)

Using the results of eqs. A(30) and A(31):

$$P_{1}'(x) = -2 \sum_{1}^{\infty} \frac{\cos \mu_{n} x}{\beta R \left\{ q_{n}^{2} \left(1 - \frac{1}{(\mu_{n} \beta)^{2}} \right) - 1 + \frac{1}{(\mu_{n} \beta R)^{2}} \right\}}$$
 (32)

Considering the asymptotic expansions of q_n and μ_n , the properties of $P_1'(x)$ can be investigated.

The following expansions are needed:

$$\frac{J_{1}'(x)}{Y_{1}'(x)} = \frac{\frac{2}{\pi x} \left\{ \cos \left(x - \frac{1}{4} \pi \right) - \frac{7}{8x} \sin \left(x - \frac{1}{4} \pi \right) \dots \right\}}{\left\{ \frac{2}{\pi x} \left\{ \sin \left(x - \frac{1}{4} \pi \right) + \frac{7}{8x} \cos \left(x - \frac{1}{4} \pi \right) \dots \right\}} \right\}} A(33)$$

Using eq. A(33) it can be proved that the asymptotic expansion of μ_n given by:

$$\mu_n = \frac{n\pi}{\beta(R-1)} + \frac{7}{8 n\pi\beta} \quad \frac{R-1}{R} \quad A(34)$$

The asymptotic expansion of q_n can be obtained by considering the numerator of eq. A(28). The result is:

$$q_{n} = \frac{1}{\sqrt{R}} \frac{1}{(-1)^{n} + 0\left(\frac{1}{n^{2}}\right)} = \frac{(-1)^{n}}{\sqrt{R}} \left\{ 1 - 0\left(\frac{1}{n^{2}}\right) \right\}.$$
 A(35)

Substituting eqs. A(34) and A(35) in eq. A(32) gives:

$$P_{1}'(x) = 2 \sum_{1}^{\infty} \frac{\frac{n\pi x}{\beta(R-1)}}{\beta(R-1)} + 0 \left(\sum_{1}^{\infty} \frac{\cos n\pi x}{n^{2}}\right).$$
 A(36)

Using the result of eq. A(18) one can write:

$$P_{1}'(x) = \frac{2\pi}{\beta(R-1)} \sum_{-\infty}^{+\infty} \delta\left(2m\pi - \frac{\pi x}{\beta(R-1)}\right) - \frac{1}{\beta(R-1)} + Q_{1}(x), \qquad A(37)$$

where $Q_1(x)$ is given by:

$$Q_{1}(x) = 2 \sum_{i=1}^{\infty} \frac{\cos \mu_{n} x}{\beta R \left[1 - \frac{1}{(\mu_{n} \beta R)^{2}} - \left\{ \frac{J_{1}'(\mu_{n} \beta R)}{J_{1}'(\mu_{n} \beta)} \right\}^{2} \left(1 - \frac{1}{(\mu_{n} \beta)^{2}} \right) \right]}{\beta (R - 1)} - 2 \sum_{i=1}^{\infty} \frac{\cos \frac{\eta_{n} x}{\beta (R - 1)}}{\beta (R - 1)} = \Lambda(38)$$

Using eq. A(37), eq. A(24) can be written as:

$$\int_{0}^{x} \alpha'(t) P_{1}'(x-t) dt = -\frac{\alpha(x) - \alpha(0)}{\beta(R-1)} + \alpha'(x) + 2 \sum_{n=1}^{n=m} \alpha' \left\{ x - 2 n\beta(R-1) \right\} + \int_{0}^{x} \alpha'(t) Q_{1}(x-t) dt.$$
A(39)

This expression holds for $2(m+1)\beta(R-1) > x > 2m\beta(R-1)$.

APPENDIX B.

Evaluation of the functions used to calculate the pressure distributions.

The functions W_0 and W_1 .

The functions W_0 has been extensively discussed by Ward. However this treament will be given here again together with some remarks, as these evaluations form the basis of further considerations. W_0 is defined in eq. (3.8) as

$$W_{0} = \frac{1}{2\pi i} \int_{a=i\infty}^{a+i\infty} e^{px} \frac{K_{1}(p) - K_{0}(p)}{K_{1}(p)} dp. \qquad B(1)$$

From this definition it follows that $W_o(x) = 0$ for x < 0. K_1 has a branch point at $\lambda = 0$ and no zeros for $|\arg p| \le \pi$; the line of integration can be transformed therefore into the real negative axis



Using the relations

$$K_{0}(pe^{\pm i\pi}) = K_{0}(p) \mp i\pi I_{0}(p)$$

$$K_{1}(pe^{\pm i\pi}) = -K_{1}(p) \mp i\pi I_{1}(p)$$

B(2)

together with the Wronskian-relation one obtains

$$W_{0}(x) = \int_{0}^{\infty} \frac{e^{-px}}{K_{1}^{2}(p) + \pi^{2} I_{1}^{2}(p)} \frac{dp}{p}, \qquad B(3)$$

Using the asymptotic series of K_1 and I_1 , it can be shown that this provides the continuation into the strip -2 < R(x) < 0. To find the Taylor-expansion around x = 0 the asymptotic series for $K_0(\lambda)$ and $K_1(\lambda)$ will be used together with eq. B(1)

.

$$K_{0}(p) = \left(\frac{\pi}{2p}\right)^{\frac{1}{2}} e^{-p} \left[1 + \sum_{m=1}^{\infty} \frac{1}{(2p)^{m}} \frac{(m-\frac{1}{2})!}{m!(-m-\frac{1}{2})!}\right]$$
B(4)

$$K_{1}(p) = \left(\frac{\pi}{2p}\right)^{\frac{1}{2}} e^{-p} \left[1 + \sum_{m=1}^{\infty} \frac{1}{(2p)^{m}} \frac{(m+\frac{1}{2})!}{m!(-m+\frac{1}{2})!}\right]$$
B(5)

0° .

.

$$\begin{split} K_{\mathfrak{g}}(p) &= \left\{ 1 - \frac{1}{8\,p} + \frac{9}{128\,p^2} - \frac{75}{1024\,p^3} + \frac{75 \times 49}{1024 \times 32\,p^4} \dots \right\} \left(\frac{\pi}{2\,p} \right)^{\frac{1}{2}} e^{-p} \\ K_{\mathfrak{g}}(p) &= \left\{ 1 + \frac{3}{8\,p} - \frac{15}{128\,p^2} + \frac{105}{1024\,p^3} - \frac{105 \times 45}{1024 \times 32\,p^4} \dots \right\} \left(\frac{\pi}{2\,p} \right)^{\frac{1}{2}} e^{-p} \,. \end{split}$$

Substituting this in expression B(1) gives:

$$W_{0}(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \left\{ \frac{1}{2p} - \frac{3}{8p^{2}} + \frac{3}{8p^{3}} - \frac{63}{128p^{4}} + 0\left(\frac{1}{p^{5}}\right) \right\} dp$$
$$W_{0}(x) = \frac{1}{2} - \frac{3}{8}x + \frac{3}{16}x^{2} - \frac{21}{256}x^{3} + 0(x^{4}).$$
Bi(6)

or

This series will converge for |x| < 2 since $W_0(x)$ is singular at x = -2.

To obtain the asymptotic expansion of $W_0(x)$ the expansions of the Bessel-functions will be used in ascending powers of p.

$$K_0(p) = -\left(\ln\frac{p}{2} + \gamma\right) - \left\{\left(\gamma + \ln\frac{p}{2}\right) - 1\left\{\frac{1}{4}p^2 + \dots\right\}\right\}$$
B(7)

$$K_{1}(p) = \frac{1}{p} + \frac{1}{2} \left(ln \ \frac{p}{2} + \gamma - \frac{1}{2} \right) p + \left\{ \frac{1}{16} \left(\gamma + ln \ \frac{p}{2} \right) - \frac{5}{64} \right\} p^{3} + \dots$$
 B(8)

Substituting these expressions in eq. B(1) gives

$$W_{0}(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \left\{ 1 + (\gamma - \ln 2)p + p \ln p + \left(\frac{1}{2}(\gamma - \ln 2)(\gamma - \ln 2 - 1) - \frac{1}{4}\right)p^{3} + -\left(\gamma - \ln 2 - \frac{1}{2}\right)p^{3} \ln p - \frac{1}{2}p^{3}(\ln p)^{2} + 0 \dots \right\} dp.$$

Transforming the path of integration according to sketch 2 and on introducing $p = u e^{\pm i\pi}$, one gets:

$$W_{0}(x) = -\frac{1}{2\pi i} \int_{0}^{\infty} e^{-ux} \left\{ 1 - (\gamma - \ln 2)u - u(\ln u + i\pi) + \frac{1}{2} u^{3}(\ln u + i\pi)^{2} + \frac{1}{2\pi i} \int_{0}^{\infty} e^{-ux} \left\{ 1 - (\gamma - \ln 2)u - u(\ln u - i\pi) + \frac{1}{2} u^{3}(\ln u - i\pi)^{2} + \left(\gamma - \ln 2 - \frac{1}{2}\right)u^{3}(\ln u - i\pi) \right\} du$$

$$W_{0}(x) = \int_{0}^{\infty} e^{-ux}u \, du - \int_{0}^{\infty} e^{-ux}u^{3}\ln u \, du - \left(\gamma - \ln 2 - \frac{1}{2}\right)\int_{0}^{\infty} e^{-ux}u^{3}du + \dots$$

$$W_{0}(x) = \frac{1}{x^{2}} + \frac{6}{x^{4}} \ln 2x - \frac{8}{x^{4}} + \dots$$
B(9)

Equation B(3) can be used to calculate the numerical values of $W_0(x)$. However another expression can be derived by rotating the path of integration over the angle $\pi/2$. Thus $p = i\lambda$, and

$$W_0(x) = \frac{1}{2\pi} \int_0^\infty e^{i\lambda x} \frac{K_1(i\lambda) - K_0(i\lambda)}{K_1(i\lambda)} d\lambda + \frac{1}{2\pi} \int_0^\infty e^{-i\lambda x} \frac{K_1(-i\lambda) - K_0(-i\lambda)}{K_1(-i\lambda)} d\lambda .$$
 B(10)

Using the eqs.

$$K_{0}(i\lambda) = -\frac{1}{2}\pi i \{ J_{0}(\lambda) - i Y_{0}(\lambda) \}$$

$$K_{0}(-i\lambda) = -\frac{1}{2}\pi i \{ J_{0}(\lambda) + i Y_{0}(\lambda) \}$$

$$K_{1}(i\lambda) = -\frac{1}{2}\pi \{ J_{1}(\lambda) - i Y_{1}(\lambda) \}$$

$$K_{1}(-i\lambda) = -\frac{1}{2}\pi \{ J_{1}(\lambda) + i Y_{1}(\lambda) \}$$

the expression for $W_{0}(x)$ becomes:

$$W_{0}(x) = \frac{1}{\pi} \int_{0}^{\infty} \left\{ 1 - \frac{2}{\pi\lambda} \quad \frac{1}{J_{1}^{2}(\lambda) + Y_{1}^{2}(\lambda)} \right\} \cos \lambda x \, d\lambda + \frac{1}{\pi} \int_{0}^{\infty} \frac{J_{0}(\lambda)J_{1}(\lambda) + Y_{0}(\lambda)Y_{1}(\lambda)}{J_{1}^{2}(\lambda) + Y_{1}^{2}(\lambda)} \sin \lambda x \, d\lambda \cdot \mathbf{B}(11)$$

Since $W_0(x) = 0$ for x < 0, if defined as has been done here the result for $W_0(x)$ is either

$$W_{0}(x) = \frac{2}{\pi} \int_{0}^{\infty} \left\{ 1 - \frac{2}{\pi \lambda} \quad \frac{1}{J_{1}^{2}(\lambda) + Y_{1}^{2}(\lambda)} \right\} \cos \lambda x \, d\lambda \qquad B(12)$$

 $\mathbf{0}\mathbf{r}$

$$W_0(x) = \frac{2}{\pi} \int_0^\infty \frac{J_0(\lambda)J_1(\lambda) + Y_0(\lambda)Y_1(\lambda)}{J_1^2(\lambda) + Y_1^2(\lambda)} \sin \lambda x \, d\lambda \,. \tag{B}(13)$$

The function $W_1(x)$ is given by eq. (3.21)

$$W_{1}(x) = \frac{+1}{2\pi i} \int_{a-i\infty}^{u+i\infty} e^{px} \frac{K_{1}'(p) + K_{1}(p)}{K_{1}'(p)} dp. \qquad B(14)$$

 $K_{i}(p)$ has two zeros for $|\arg| \leq \pi$ and a branch-point at $\lambda = 0$. Transforming the line of integration into the real negative axis and using the relations

there is obtained

$$W_1(x) = + \int_0^\infty \frac{e^{-px}}{K_1'^2(p) + \pi^2 I_1'^2(p)} \frac{dp}{p} + 2\pi i \Sigma \operatorname{Res.}^*)$$
 B(16)

It can be shown that $W_1(x)$ according to this definition is singular at x = -2, just as in the case of $W_q(x)$.

In order to find the Taylor-expansion around x=0, the asymptotic series will be used:

$$K_n(p) = \left(\frac{\pi}{2p}\right)^{\frac{1}{2}} e^{-p} \left[1 + \frac{4n^2 - 1^2}{1!8p} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2!(8p)^2} + \dots\right]$$
B(17)

$$K_{n}'(p) = -\left(\frac{\pi}{2p}\right)^{\frac{1}{2}} e^{-p} \left[1 + \frac{4n^{2} + 1 \times 3}{1!8p} + \frac{(4n^{2} - 1^{2})(4n^{2} + 3 \times 5)}{2!(8p)^{2}} + \dots\right] B(18)$$

Substituting this in eq. B(14) gives

$$W_{1}(x) = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \left\{ -\frac{1}{2p} - \frac{1}{8p^{2}} + \frac{5}{8p^{3}} - \frac{121}{128} - \frac{1}{p^{4}} 0\left(\frac{1}{p^{5}}\right) \right\} dp$$

0**r**

$$W_1(x) = + \frac{1}{2} + \frac{1}{8}x - \frac{5}{16}x^2 + \frac{121}{768}x^3$$
. B(19)

*) The last term in eq. B (16) can be evaluated by observing that the differential equation for $K_1(p)$ is

$$\frac{d^{2}K_{1}}{dp^{2}} + \frac{1}{p} \frac{dK_{1}}{dp} - \left(1 + \frac{1}{z^{2}}\right)K_{1} = 0.$$

If the zeros of $K_1(\lambda)$ are given by a + bi and a - bi, one gets

$$2\pi i \Sigma Res = -\frac{e^{x(a+bi)}K_i(a+bi)}{\left(\frac{d}{dp} \{K_i'(p)\}\right)_{a+bi}} - \frac{e^{x(a-bi)}K_i(a-bi)}{\left(\frac{d}{dp} \{K_i'(p)\}\right)_{a-bi}}$$

Using the differential equation this gives:

$$2\pi i \Sigma Res = \frac{-2 e^{ax}}{(a^2 - b^2 + 1)^2 + 4 a^2 b^2} \left[\left\{ (a^2 - b^2)(a^2 - b^2 + 1) + 4 a^2 b^2 \right\} \cos xb - 2 ab \sin xb \right].$$

The numerical values for a and b are given in ref. 4.

As can be proved from eq. B(16) this expansion is valid for |z| < 2. It can be remarked here that the function V(z) introduced by Ward is equal to:

Performing the integration .

$$V(z) = 1 - \frac{1}{2} z - \frac{1}{16} z^2 + \frac{5}{48} z^3 - \frac{121}{4 \times 768} z^4.$$
 B(21)

Ward gives three terms of the expansion, which are in accordance with those given above.

To obtain the asymptotic expansion of $W_1(x)$ the following expansion of Bessel-functions in ascending powers of p is used:

$$K_{n}(p) = (-1)^{n+1} \left(\gamma + \ln \frac{x}{2} \right) I_{n}(p) + \frac{1}{2} \sum_{p=0}^{n-1} \frac{(-1)^{p}(n-p-1)!}{p!} \left(\frac{x}{2} \right)^{2p-n} + \cdots$$

$$+ \frac{(-1)^{n}}{2} \sum_{p=0}^{\infty} \frac{1}{p!(n+p)!} \left(\frac{x}{2}\right)^{2p+n} \left\{ 1 + \frac{1}{2} \dots \frac{1}{p} + 1 + \frac{1}{2} \dots \frac{1}{n+p} \right\}$$
 B(22)

$$K_n'(p) = -\frac{n}{p} K_n - K_{n-1}. \qquad B(23)$$

Substitution of this expression gives

$$W_{1}(x) = + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \left\{ -p - p^{s} \ln p - (\gamma - \ln 2)p^{s} - \frac{1}{2} p^{s} (\ln p)^{2} - \left(\gamma - \ln 2 + \frac{1}{2}\right)p^{s} \ln p + p^{s} \dots \right\} dp.$$

Transforming the path of integration, as has been done for $W_0(x)$:

$$W_{1}(x) = + \int_{0}^{\infty} -e^{-ux} x^{3} du - \int_{0}^{\infty} e^{-ux} u^{5} \ln u \, du - \left(\gamma - \ln 2 + \frac{1}{2}\right) \int_{0}^{\infty} e^{-ux} u^{5} \, du + \dots$$
$$W_{1}(x) = -\frac{6}{x^{4}} + 120 \frac{\ln 2x}{x^{6}} - \frac{334}{x^{6}} + \dots$$
B(24)

or

To derive equations for the function $W_1(x)$ as a Fourier-sine or cosine-transform, the path of integration will be rotated over the angle $\pi/2$ such that $p = i\lambda$ and the following eqs. will be applied:

$$K_{n}\left(\lambda e^{\pm i\frac{\pi}{2}}\right) = \pm \frac{1}{2} \pi i e^{\mp \frac{1}{2}n\pi i} \left[-J_{n}(z) \pm iY_{n}(z)\right]$$
$$K_{n'}\left(\lambda e^{\pm i\frac{\pi}{2}}\right) = \pm \frac{1}{2} \pi i e^{\mp \frac{1}{2}n\pi i} \left[-J_{n'}(z) \pm iY_{n'}(z)\right] \cdot e^{\mp i\pi}.$$

The result is:

$$W_{1}(x) = \frac{+1}{2\pi} \int_{0}^{\infty} e^{i\lambda x} \frac{K_{1}'(i\lambda) + K_{1}(i\lambda)}{K_{1}'i\lambda} d\lambda + \frac{1}{2\pi} \int_{0}^{\infty} e^{-i\lambda x} \frac{K_{1}'(-i\lambda) + K_{1}(-i\lambda)}{K_{1}(-i\lambda)} d\lambda \qquad B(25)$$

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$$W_{1}(z) = + \frac{1}{\pi} \int_{0}^{\infty} \left\{ 1 - \frac{2}{\pi\lambda} \frac{1}{J^{\prime 2}(\lambda) + Y_{1}^{\prime 2}(\lambda)} \right\} \cos \lambda \, xd \, \lambda - \frac{1}{\pi} \int_{0}^{\infty} \frac{J_{1}(\lambda)J_{1}^{\prime}(\lambda) + Y_{1}(\lambda)Y_{1}^{\prime}(\lambda)}{J_{1}^{\prime}(\lambda)^{2} + Y_{1}^{\prime}(\lambda)^{2}} \cdot \frac{\sin \lambda x \, d\lambda}{\sin \lambda x \, d\lambda}$$

Hence the final result is:

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$$W_{1}(x) = + \frac{2}{\pi} \int_{0}^{\infty} \left\{ 1 - \frac{2}{\pi \lambda} \frac{1}{J_{1}'(\lambda)^{2} + Y_{1}'(\lambda)^{2}} \right\} \cos \lambda x d\lambda \qquad B(26)$$

$$W_1(x) = -\frac{2}{\pi} \int_0^\infty \frac{J_1'(\lambda)J_1(\lambda) + Y_1'(\lambda)Y_1(\lambda)}{J_1'(\lambda)^2 + Y_1'(\lambda)^2} \sin \lambda x d\lambda . \qquad B(27)$$

The Taylor-expansions of $P_0'(x)$ and $P_1'(x)$ at x=0 and in particular those of $Q_0(x)$ and $Q_1(x)$ will be given. It will be shown that there is a simple relationship between $Q_0(x)$ and $W_0(-x)$ and between $Q_1(x)$ and $W_1(-x)$ as long as $x \leq 2$.

To derive the Taylor-expansion the asymptotic expansion of the integrand is needed:

1.
$$P_{\mathfrak{g}}'(x) \coloneqq \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{K_{\mathfrak{g}}'(p\beta)I_{\mathfrak{g}}(p\beta R)}{K_{\mathfrak{g}}'(p\beta)I_{\mathfrak{g}}'(p\beta R)} \stackrel{I_{\mathfrak{g}}'(p\beta)K_{\mathfrak{g}}(p\beta R)}{\leftarrow I_{\mathfrak{g}}'(p\beta)K_{\mathfrak{g}}'(p\beta R)} e^{px}dp$$

or

$$P_{0}'(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{K_{1}(p\beta)I_{0}(p\beta R) + I_{1}(p\beta)K_{0}(p\beta R)}{K_{1}(p\beta R) - I_{1}(p\beta)K_{1}(p\beta R)} e^{px}dp.$$
 B(28)

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If
$$U_n(z) = 1 - \frac{4n^2 - 1^2}{1! 8z} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2!(8z)^2} - \dots$$
 etc.
then $I_n(z) \approx \frac{e^2}{(2-x)^{\frac{1}{2}}} U_n(z) + \frac{e^{-2+}(n + \frac{1}{2}\pi i)}{(2-x)^{\frac{1}{2}}} U_n(-z) \left(\frac{1}{2} - \pi < \arg z < 1\right)$

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 $f \in X$

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$$I_n(z) \approx -\frac{e^2}{(2\pi z)^{\frac{1}{2}}} U_n(z) + \frac{e^{-z+}(n+\frac{1}{2}\pi i)}{(2\pi z)^{\frac{1}{2}}} U_n(-z) \left(\frac{1}{2} - \pi < \arg z < \frac{3}{2}\pi K_n(z) \approx \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} U_n(-z).$$

Introducing this into eq. B(28):

$$P_{0}'(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{-p\beta}U_{1}(-p\beta) \left\{ e^{p\beta R} U_{0}(p\beta R) + ie^{-p\beta R}U_{0}(-p\beta R) \right\} + ie^{-p\beta R}U_{0}(-p\beta R) \left\{ e^{p\beta U_{1}(-p\beta)} - ie^{-p\beta R}U_{1}(-p\beta R) \right\} - ie^{-p\beta R}U_{0}(-p\beta R) \left\{ e^{p\beta}U_{1}(p\beta) - ie^{-p\beta}U_{1}(-p\beta) \right\} - ie^{-p\beta R}U_{0}(-p\beta R) \left\{ e^{p\beta}U_{1}(p\beta) - ie^{-p\beta}U_{1}(-p\beta) \right\}$$

or ,...

$$P_{0}'(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{p\beta} \frac{e^{p\beta(R-1)} U_{1}(-p\beta)U_{0}(p\beta R) + e^{-p\beta(R-1)} U_{1}(p\beta)U_{0}(-p\beta R)}{e^{p\beta(R-1)} U_{1}(-p\beta)U_{1}(-p\beta R) - e^{-p\beta(R-1)} U_{1}(p\beta)U_{1}(-p\beta R)} dp.$$

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This can be written as

$$P_{0}'(x) = \frac{+1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \frac{U_{1}(-p\beta)U_{0}(+p\beta R) + 0 \left\{ e^{-2p\beta(R-1)} \right\}}{U_{1}(-p\beta)U_{1}(p\beta R) + 0 \left\{ e^{-2p\beta(R-1)} \right\}} dp = \frac{+1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \frac{U_{0}(p\beta R)}{U_{1}(p\beta R)} + 0 \left\{ e^{-2p\beta(R-1)} \right\}}{1 + 0 \left\{ e^{-2p\beta(R-1)} \right\}} dp.$$
B(29)

Assuming now that $x < 2\beta(R-1)$, the terms of $0 \{ e^{-2\beta(R-1)} \}$ are very small compared to the first term and asymptotically negligible. The Taylor-expansion for $x < 2\beta(R-1)$ around x = 0 is obtained by considering:

$$P_{0}'(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \frac{U_{0}(p\beta R)}{U_{1}(p\beta R)} dp$$

This can be written as

$$P_{0}'(x) = \delta_{+}(x) + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \frac{U_{0}(p\beta R) - U_{1}(p\beta R)}{U_{1}(p\beta R)} dp.$$

This can be transformed into:

$$P_{0}'(x) = \delta_{+}(x) - \frac{1}{\beta R} \cdot \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\lambda y} \frac{U_{1}(\lambda) - U_{0}(\lambda)}{U_{1}(\lambda)} d\lambda \qquad B(30)$$

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where:

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$$y = \frac{x}{\beta R}$$
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An expansion is needed for:

$$T_{0}(y) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+ix} e^{\lambda y} \frac{U_{1}(\lambda) - U_{0}(\lambda)}{U_{1}(\lambda)} d\lambda$$

Comparing this with the formula for $W_0(y)$, one finds:

$$W_0(y) =: \frac{1}{2 \pi i} \int_{a-i\infty}^{a+i\infty} e^{\lambda y} \frac{U_1(-\lambda) - U_0(-\lambda)}{U_1(-\lambda)} d\lambda.$$

The following expansion exists

$$\frac{U_1(-\lambda)-U_0(-\lambda)}{U_1(-\lambda)} = \sum_{1}^{\infty} (-)^n \frac{a_n}{\lambda \dot{n}}.$$

From this equation it can be concluded that:

$$\frac{U_1(\lambda) - U_0(\lambda)}{U_1(\lambda)} = \sum_{1}^{\infty} \frac{a_n}{\lambda n}$$

since by substituting $(-\lambda)$ in the second formula, the first must be obtained. This gives the result

if
$$W_0(y) = \sum_{0}^{\infty} b_n y^n$$
 for $y < 2 \frac{R-1}{R}$, then
 $T_0(y) = \sum_{0}^{\infty} (-)^n b_n y^n$ for $y < 2 \frac{R-1}{R}$.
This means that:

This means that:

$$W_0(-y) = -T_0(y)$$
 for $y < 2 \frac{R-1}{R} < 2$. B(31)

Therefore it has been proved that:

$$P_{\mathfrak{o}}'(x) = \delta_{+}(x) + \frac{1}{\beta R} W_{\mathfrak{o}}\left(-\frac{x}{\beta R}\right) \text{ if } x < 2\beta(R-1). \tag{B(32)}$$

This gives with eq. A(19):

$$Q_{\mathfrak{o}}(x) = -\frac{1}{\beta(R+1)} + \frac{1}{\beta R} W_{\mathfrak{o}}\left(-\frac{x}{\beta R}\right) \text{ if } x < 2\beta(R-1).$$
 B(33)

Thus if the pressure distribution on the outer side of the ring wing is given by

$$c_{poo} = \frac{2}{\beta} \left\{ r'(x) - \frac{1}{\beta R} \int_{0}^{x} W_{1}\left(\frac{n-t}{\beta R}\right) r'(t) dt \right\}, \qquad B(34)$$

the pressure on the inside is given by

$$c_{poi} = \frac{2}{\beta} \left\{ r'(x) + \frac{1}{\beta R} \int_{0}^{x} W_{0}\left(\frac{t-x}{\beta R}\right) r'(t) dt \right\} \text{ as long as } x < 2(R-1).$$
 B(35)

Physically this means that the result is valid as long as the ring wing innerside is not influenced by the presence of the fuselage. Therefore the pressure on this side is the same as if the ring were without fuselage. The result B(35) was already quoted by Ward in the case of a free ring. His expression , for $P_o'(x)$ is therefore less complicated.

2. The Taylor-expression of $P_1'(x)$ around x=0. Consider:

$$P_{1}'(x) = + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{K_{1}'(p\beta)I_{1}(p\beta R) - I_{1}'(p\beta)K_{1}(p\beta R)}{K_{1}'(p\beta)I_{1}'(p\beta R) - I_{1}'(p\beta)K_{1}'(p\beta R)} e^{px}dp \qquad B(36)$$

on using the relations:

$$K_{1}'(z) = -K_{0}(z) - \frac{1}{z} K_{1}(z)$$

$$I_{1}'(z) = I_{0}(z) - \frac{1}{z} I_{1}(z)$$

$$P_{1}'(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \frac{-e^{p\beta(R-1)}U_{1}(p\beta R) \left\{ U_{0}(-p\beta) + \frac{1}{p\beta} U_{1}(-p\beta) \left\{ + \frac{1}{p\beta} U_{1}(-p\beta) U_{1}(p\beta R) - \frac{1}{p\beta} \left\{ \frac{U_{0}(-p\beta)U_{1}(p\beta R)}{R} - \frac{1}{p\beta} \left\{ \frac{U_{0}(-p\beta)U_{1}(p\beta R)}{R} - \frac{1}{p^{2}\beta^{2}R} U_{1}(p\beta R) U_{1}(-p\beta) \right\} + 0 \left\{ e^{-p\beta(R-1)} \right\}} dp$$

The numerator can be written as:

$$-e^{-p\beta(R-1)}\left\{U_{0}(-p\beta)+\frac{1}{p\beta}U_{1}(-p\beta)\right\}\left\{U_{0}(p\beta R)-\frac{1}{p\beta R}U_{1}(p\beta R)\right\}+0e^{-p\beta(R-1)}$$

Thus we may write

$$P_{1}'(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \left\{ \frac{U_{1}(p\beta R)}{U_{0}(p\beta R) - \frac{1}{p\beta R} U_{1}(p\beta R)} + 0 \ (e^{-2p\beta(R-1)}) \right\} dp.$$

In order to obtain the Taylor-expansion of $P_1'()$ around x = 0, for $x < 2\beta(R-1)$, consider the following expressions

$$P_{1}'(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} \frac{U_{1}(p\beta R)}{U_{0}(p\beta R) - \frac{1}{p\beta R}} U_{1}(p\beta R) dp. \qquad B(38)$$

This can be written as:

$$P_{1}'(x) = \delta_{+}(x) - \frac{1}{2\pi i} \cdot \frac{1}{\beta R} \int_{a-i\infty}^{a+i\infty} e^{\lambda y} \frac{U_{0}(\lambda) - \frac{1}{\lambda} U_{1}(\lambda) - U_{1}(\lambda)}{U_{0}(\lambda) - \frac{1}{\lambda} U_{1}(\lambda)} d\lambda. \qquad B(39)$$

Introducing

$$T_{1}(y) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\lambda y} \frac{U_{0}(\lambda) - \frac{1}{\lambda} U_{1}(\lambda) - U_{1}(\lambda)}{U_{0}(\lambda) - \frac{1}{\lambda} U_{1}(\lambda)} d\lambda$$

and comparing this with the formula for $W_1(y)$, the result is:

$$W_1(y) = + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\lambda y} \frac{-U_0(-\lambda) - \frac{1}{\lambda} U_1(-\lambda) + U_1(-\lambda)}{-U_0(-\lambda) - \frac{1}{\lambda} U_1(-\lambda)} d\lambda .$$

By the same arguments as in the previous case, one can state therefore:

$$W_1(-y) = -T_1(y)$$
 for $y < 2 \frac{R-1}{R} < 2$. B(40)

Hence:

$$P_{1}'(x) = \delta_{+}(x) + \frac{1}{\beta R} W_{1}\left(-\frac{x}{\beta R}\right) \text{ if } x < 2\beta(R-1).$$
 B(41)

With eq. A(37) this results in:

$$Q_1(x) = \frac{1}{\beta(R-1)} + \frac{1}{\beta R} W_1\left(-\frac{x}{\beta R}\right) \text{ if } x < 2\beta(R-1).$$
 B(42)

Thus if the pressure on the outer side of the ring wing is given by:

$$c_{p}\bullet_{00} = \frac{2}{\beta} \left\{ \alpha'(x) - \frac{1}{\beta R} \int_{0}^{x} W_{1}\left(\frac{x-t}{\beta R}\right) \alpha'(t) dt \right\} \cos \vartheta$$

the pressure on the ring wing inner side is given by:

$$c_{p} *_{0i} = -\frac{2}{\beta} \left\{ \alpha'(x) + \frac{1}{\beta R} \int_{0}^{\infty} W_{1}\left(\frac{t-x}{\beta R}\right) \alpha'(t) dt \right\} \cos \vartheta \qquad B(43)$$

as long as $x < 2\beta(R-1)$.

The function W_0 .

This function is defined according to eq. (4.18) by

$$W_{0}(y,R) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{py} \frac{\sqrt{Re^{i(R-i)}K_{0}(pR) + K_{0}'(p)}}{\sqrt{RK_{0}'(p)}} dp. \qquad B(44)$$

This function has already been considered by Nielsen in ref. 7. However no further investigation of the function is given there.

To find the Taylor-expansion use is made of the asymptotic series for K_0 and K_1 . Substitution of these expressions gives:

$$W_{a}(y,R) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{py}}{\sqrt{R}} \left\{ \frac{1+3R}{\beta R} \cdot \frac{1}{p} - \frac{33R^{2}+6R+9}{128R^{2}} \cdot \frac{1}{p^{2}} + \frac{177R^{3}+33R^{2}+99R+75}{1024R^{3}} \cdot \frac{1}{p^{3}} \cdot \cdot \right\} dp$$

or

$$W_0(y,R) = \frac{1}{\sqrt{R}} \left\{ \frac{1+3R}{8R} - \frac{3R^2 + 6R + 9}{128R^2} y + \frac{177R^3 + 33R^2 + 99R + 75}{2048R^3} y^2 + 0(y^3) \dots \right\}.$$
 B(45)

To obtain the asymptotic series of $W_{0}(y, R)$ the integrand has to be expanded around the singularity at p = 0. Using the equations B(7) and B(8):

$$W_{0}(y,R) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{py} \left\{ \frac{1}{\sqrt{R}} + \left(1 + p(R-1) + \frac{p^{2}}{2} (R-1)^{2} \dots \right) \left(ap + p \ln p + bp^{3} + \left(\gamma - \ln 2 - \frac{1}{2} \ln R - \frac{1}{2} \right) p^{3} \ln p - \frac{1}{2} p^{3} (\ln p)^{2} + \dots \right) \right\} dp.$$

Only the terms with lnp and $(lnp)^2$ are given with their coefficients since these are the terms that contribute to the asymptotic series.

Integration gives:

$$W_0(y,R) = \frac{1}{x^2} - \frac{2(R-1)}{x^3} + \frac{6}{x^4} \ln 2x + \frac{3}{x^4} \ln R - \frac{8}{x^4} + \frac{3(R-1)^2}{x^4} + \dots \qquad B(47)$$

To obtain expressions suitable for numerical integration, the path of integration is rotated over an angle $\frac{\pi}{2}$.

Thus $p = i\lambda$, and

$$W_{0}(y,R) = \frac{1}{2\pi} \int_{0}^{\infty} e^{i\lambda x} \frac{-\sqrt{Re} i\lambda(R-1)}{\sqrt{R}} K_{0}(i\lambda R) + K_{1}(i\lambda)} + \frac{1}{2\pi} \int_{0}^{\infty} e^{-i\lambda x} \frac{-\sqrt{Re} i\lambda(R-1)}{\sqrt{R}} K_{0}(-i\lambda R) + K_{1}(-i\lambda)} d\lambda.$$

Using equations similar to those used to derive eq. B(11) one gets:

$$\begin{split} W_{0}(y,R) &= \\ 2\pi \int_{0}^{\infty} \frac{J_{0}(\lambda R)J_{1}(\lambda) + Y_{0}(\lambda R)Y_{1}(\lambda)\}\cos\lambda(R-1) - \{J_{0}(\lambda R)Y_{1}(\lambda) - Y_{0}(\lambda R)J_{1}(\lambda)\}\sin\lambda(R-1)}{J_{1}^{2}(\lambda) + Y_{1}^{2}(\lambda)} \sin\lambda(R-1) \sin\lambda x \, d\lambda \, . \\ B(48b) \end{split}$$

Either of the two equations can be used to calculate the numerical values of $W_0(y, R)$.

The functions Z_0 and Z_1 .

The following complex integral has to be considered according to eq. (4.17)

$$\frac{1}{2\pi i}\int_{\overline{a}-i\infty}^{a+i\infty} e^{\overline{p}(l+y-i)} \frac{K_0'(\overline{p}\beta R)}{K_0'(\overline{p}\beta)} d\overline{p}.$$
 B(49)

This integral can be written as

$$\frac{1}{2\pi i}\int_{a=i\infty}^{a+i\infty}e^{p\left\{\frac{j+a-j}{\beta}-(R-1)\right\}}\cdot\frac{e^{r(R-1)}}{\beta}\frac{K_{0}'(pR)}{K_{0}'(p)}dp$$

or

$$\frac{1}{2\pi i\beta} \int_{a-i\infty}^{a+i\infty} e^{p} \left\{ \frac{l+y-t}{\beta} - (R-1) \right\} \frac{\sqrt{Re} p^{p(R-1)} K_{0}'(pR) - K_{0}'(p)}{\sqrt{R} K_{0}'(p)} dp + \frac{1}{2\pi i\beta} \int_{a-i\infty}^{a+i\infty} e^{p} \left\{ \frac{l+y-t}{\beta} - (R-1) \right\} dp . \quad B(50)$$

As can be seen from the asymptotic expansion of the Bessel-functions, the first integral of eq. B(50) is now convergent, while the second can be expressed as a delta function. In the function $Z_0(y, R)$ is defined as follows:

$$Z_{0}(y,R) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{py} \frac{\sqrt{R}e^{j(R-1)}}{\sqrt{R}K_{0}'(p)} \frac{K_{0}'(p)}{\sqrt{R}K_{0}'(p)} dp \qquad B(51)$$

eq. B(50) can be written as

$$\frac{1}{\beta} Z_{\circ} \left(\frac{y+l-t-\beta(R-1)}{\beta}, R \right) + \frac{\delta_{+} \{ y+l-t-\beta(R-1) \}}{\sqrt{R}}.$$
 B(52)

In this section the properties of the function $Z_o(y, R)$ will be investigated. To find the Taylor-expansion around y=0, the asymptotic series for $K_o'(p) = -K_1(p)$ will be used, as given by eq. B(5). Substitution of this expression into eq. B(49) gives:

$$Z_{0}(y,R) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1-R}{R^{3/2}} \left\{ \frac{3}{8p} - \frac{3}{128p^{2}} \frac{5+11R}{R} + \frac{3}{1024p^{3}} \frac{35+50R+83R^{2}}{R^{2}} + \dots \right\} e^{py} dp$$

or

$$Z_{0}(y,R) = \frac{1-R}{R^{3}} \left\{ \frac{3}{8} - \frac{3}{128} - \frac{5+11R}{R} y + \frac{3}{2048} - \frac{35+50R+83R^{2}}{R^{2}} y^{2} + \dots 0(y^{3}) \right\}$$
B(53)

To get the asymptotic series of $Z_{q}(y, R)$ the integrand will be expanded around p = 0. Using eq. B(8):

$$\begin{aligned} Z_{a}(y,R) &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{py} \left[\left\{ 1 + p(R-1) + \frac{p^{2}}{2} (R-1)^{2} \right\} \left[\frac{1}{R} + \frac{1}{2} - \frac{R^{2}-1}{R} p^{2} \ln p + \dots \right] \\ &+ \left\{ \frac{1}{2} \left(\gamma - \ln 2 - \frac{1}{2} \right) \frac{R^{2}-1}{R} + \frac{R}{2} \ln R \right\} p^{2} - \frac{1}{4} - \frac{R^{2}-1}{R} p^{4} (\ln p)^{2} + \dots \right] \\ &+ \left\{ \frac{1}{16} - \frac{R^{4}-1}{R} - \frac{1}{2} \left(\gamma - \ln 2 - \frac{1}{2} \right) \frac{R^{2}-1}{R} - \frac{1}{2} \left(\frac{R^{2}-1}{R} - \frac{1}{2} \right) \frac{R^{2}-1}{R} - \frac{1}{2} \left(\frac{$$

Since only the terms with ln p or $(ln p)^2$ contribute to the asymptotic series this can be written as:

$$\begin{split} Z_0(y,R) &\approx \frac{1}{2\pi i} \int\limits_{u-i\infty}^{u+i\infty} e^{py} \left\{ \frac{1}{2} - \frac{R^2 - 1}{R} p^2 \ln p + \frac{1}{2} - \frac{(R-1)(R^2 - 1)}{R} p^3 \ln p - \frac{1}{4} - \frac{R^2 - 1}{R} p^4 (\ln p)^2 + \right. \\ &+ \left\{ \frac{1}{16} - \frac{R^4 - 1}{R} - \frac{1}{2} \left(\gamma - \ln 2 - \frac{1}{2} \right) \frac{R^2 - 1}{R} - \frac{R}{4} \ln R + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} p^4 \ln p + \frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)(R - 1)^2}{R} \right\} p^4 \ln p + \dots \left\{ -\frac{1}{4} \frac{(R^2 - 1)$$

Performing the integration, the result is:

$$Z_{0}(y,R) \approx -\frac{R^{2}-1}{R} \frac{1}{y^{3}} + \frac{3(R-1)(R^{2}-1)}{R} \frac{1}{y^{4}} - 12 \frac{R^{2}-1}{R} \frac{\ln 2y}{y^{5}} + \dots + \left\{ 19 \frac{R^{2}-1}{R} - \frac{3}{2} \frac{R^{4}-1}{R} + 6R \ln R - 6 \frac{(R^{2}-1)(R-1)^{2}}{R} \right\} \frac{1}{y^{5}} \dots B(54)$$

Since the formulas for $Z_0(y, R)$ given by eq. B(49) is not in a form convenient for numerical conputations, integral expressions will be derived for $Z_0(y, R)$ by rotating the path of integration, or

$$Z_{0}(y,R) = \frac{1}{2\pi} \int_{0}^{\infty} e^{i\lambda y} \left\{ e^{i\lambda (R-1)} \frac{K_{0}'(i\lambda R)}{K_{0}'(i\lambda)} - \frac{1}{VR} \right\} d\lambda + \frac{1}{2\pi} \int_{0}^{\infty} e^{-i\lambda y} \left\{ e^{-i\lambda (R-1)} \frac{K_{0}'(-i\lambda R)}{K_{0}'(-i\lambda)} - \frac{1}{VR} \right\} d\lambda.$$

Using the formulae given with eq. B(10) this can be transformed into

$$\begin{split} Z_0(y,R) &= \frac{1}{\pi} \int_0^\infty \left[\frac{J_1(\lambda R) J_1(\lambda) + Y_1(\lambda R) Y_1(\lambda)}{J_1^2(\lambda) + Y_1^2(\lambda)} \cos \lambda \left\{ y + R - 1 \right\} - \frac{1}{\sqrt{R}} \cos \lambda y \right] d\lambda + \\ &+ \frac{1}{\pi} \int_0^\infty \frac{Y_1(\lambda R) J_1(\lambda) - J_1(\lambda R) Y_1(\lambda)}{J_1^2(\lambda) + Y_1^2(\lambda)} \sin \lambda \left\{ y + R - 1 \right\} d\lambda . \ B(55) \end{split}$$

Since $Z_{o}(y, R) = 0$ for y < 0 this leads to:

$$Z_{0}(y, R) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\{J_{1}(\lambda R)J_{1}(\lambda) + Y_{1}(\lambda R) | Y_{1}(\lambda)\} \cos \lambda (R-1) + \{Y_{1}(\lambda R)J_{1}(\lambda) - J_{1}(\lambda R)Y_{1}(\lambda)\} \sin \lambda (R-1)}{J_{1}^{2}(\lambda) + Y_{1}^{2}(\lambda)} + \frac{1}{\sqrt{R}} \cos \lambda y \, d\lambda \qquad B(56a)$$

or

$$Z_{0}(y, R) = \frac{2}{\pi} \int_{0}^{\pi} \frac{\left[-\left\{-Y_{1}(\lambda R)J_{1}(\lambda)-J_{1}(\lambda R)-Y_{1}(\lambda)\right\}\cos\lambda(R-1)-\left\{J_{1}(\lambda R)J_{1}(\lambda)+Y_{1}(\lambda R)Y_{1}(\lambda)\right\}\sin\lambda(R-1)\right]}{J_{1}^{2}(\lambda)+Y_{1}^{2}(\lambda)} \cdot \frac{\sin\lambda y \,d\lambda}{\operatorname{B}(56b)}$$

Consider now the function Z_1 , which is defined according to eqs. (4.27) and (4.28) by:

$$Z_{1}(y,R) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{py} \frac{\sqrt{Re}^{p(R-1)} K_{1}'(pR) - K_{1}'(p)}{\sqrt{R} K_{1}'(p)} dp. \qquad B(57)$$

To obtain the Taylor-expansion around y=0, the asymptotic expansion of $K_1'(p)$ will be used:

$$K_{\tau}'(p) \approx \left(\frac{\pi}{2p}\right)^{\frac{1}{2}} e^{-p} \left\{-1 - \frac{7}{8p} - \frac{57}{128p^2} + \frac{195}{1024p^3} + \dots\right\}$$
 B(58)

Therefore:

$$Z_{1}(y,R) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{R-1}{R^{3/2}} e^{py} \left\{ -\frac{7}{8p} - \frac{1}{128p^{2}} \frac{57-41R}{R} + \frac{1}{1024p^{3}} \frac{195+594R+307R^{2}}{R^{2}} + \dots \right\} dp$$

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$$Z_1(y,R) = \frac{1-R}{R^{3/2}} \left\{ \frac{7}{8} + \frac{1}{128} \quad \frac{57-41}{R} y - \frac{1}{2048} \quad \frac{195+594}{R^2} + \frac{307}{R^2} y^2 + \dots \right\}.$$
 B(59)

The asymptotic expansion can be obtained in the same way as has been done for Z_0 .

$$\begin{split} Z_1(y,R) &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+ix} e^{py} \left| 1 + p(R-1) + \frac{p^2}{2} (R-1)^2 \right| \left| \frac{1}{R^2} + \frac{1}{2} \frac{1-R^2}{R^2} p^2 \ln p + \right| \\ &+ \left| \frac{1}{2} \left(\gamma - \ln 2 + \frac{4}{2} \right) \frac{1-R^2}{R^2} - \frac{1}{2} \ln R \right| p^2 + \frac{1}{4} \frac{1-R^2}{R^2} p^4 (\ln p)^2 + \right| \\ &+ \left| \frac{3}{16} \frac{1-R^2}{R^2} + \frac{1}{2} \frac{1-R^2}{R^2} \left(\gamma - \ln 2 + \frac{1}{2} \right) - \frac{1}{4} \ln R \right| p^4 \ln p - \frac{1}{\sqrt{R}} + \dots \right] dp \,. \end{split}$$

Retaining only terms with lnp and $(lnp)^2$:

$$\begin{split} Z_1(y,R) &\approx \frac{1}{2\pi i} \int\limits_{a=i\infty}^{a+i\infty} e^{py} \left\{ \frac{1}{2} - \frac{1-R^2}{R^2} p^2 \ln p + \frac{1}{2} - \frac{(1-R^2)(R-1)}{R^2} p^3 \ln p + \frac{1}{4} - \frac{1-R^2}{R^2} p^4 (\ln p)^2 + \right. \\ &+ \left\{ \frac{3}{16} - \frac{1-R^2}{R^2} + \frac{1}{2} - \frac{1-R^2}{R^2} \left(\gamma - \ln 2 + \frac{1}{2} \right) - \frac{1}{4} \ln R + \frac{1}{4} \frac{(1-R^2)(R-1)^2}{R^2} \right\} p^4 \ln p \dots \left. \right\} dp \\ Z_1(y,R) &\approx -\frac{1}{x^3} - \frac{1-R^2}{R^2} + \frac{3}{x^4} - \frac{(1-R^2)(R-1)}{R^2} + 12 \frac{1-R^2}{R^2} - \frac{\ln 2x}{x^5} + \frac{6}{x^5} \ln R + \\ &- \frac{1-R^2}{2R^2} \left\{ 21R^2 - 24R + 83 \right\} \frac{1}{x^5} + \dots \quad B(60) \end{split}$$

To obtain integral representations of $Z_1(y, R)m$ which are convenient for numerical computations, intro-duce $p = i\lambda$ in eq. B(57). Thus

$$Z_{1}(y,R) = \frac{1}{2\pi} \int_{0}^{\infty} e^{i\lambda y} \left[\frac{e^{i\lambda(R-1)} K_{1}'(i\lambda R)}{K_{1}'(i\lambda)} - \frac{1}{\sqrt{R}} \right] d\lambda + \frac{1}{2\pi} \int_{0}^{\infty} e^{-i\lambda y} \left[\frac{e^{-i\lambda(R-1)}K_{1}'(-i\lambda R)}{K_{1}'(-i\lambda)} - \frac{1}{\sqrt{R}} \right] d\lambda$$

Since $K_{1}'(i\lambda) = \frac{1}{2}\pi i \{ J_{1}'(\lambda) - i Y_{1}'(\lambda) \}$, and

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$$K_1'(-i\lambda) = -\frac{1}{2}\pi i \left\{ J_1'(\lambda) + i Y_1'(\lambda) \right\}$$

this can be written as:

$$\begin{split} Z_{1}(y,R) &= \frac{1}{\pi} \int_{0}^{\infty} \left\{ \frac{J_{1}'(\lambda R) J_{1}'(\lambda) + Y_{1}'(\lambda R) Y_{1}'(\lambda)}{\{J_{1}'(\lambda)\}^{2} + \{Y_{1}'(\lambda)\}^{2}} \cos \lambda \left\{ y + R - 1 \right\} - \frac{\cos \lambda y}{\sqrt{R}} \right\} d\lambda + \\ &+ \frac{1}{\pi} \int_{0}^{\infty} \frac{Y_{1}'(\lambda R) J_{1}'(\lambda) - J_{1}'(\lambda R) Y_{1}'(\lambda)}{\{J_{1}'(\lambda)\}^{2} + \{Y_{1}'(\lambda)\}^{2}} \sin \lambda \left\{ y + R - 1 \right\} d\lambda \end{split}$$

since $Z_1(y, R) = 0$ for y < 0, the result is:

$$Z_{1}(y,R) = \frac{2}{\pi} \int_{0}^{\infty} \left[\frac{\{J_{1}'(\lambda R)J_{1}'(\lambda) + Y_{1}'(\lambda R)Y'(\lambda)\}\cos\lambda(R-1) + \{Y_{1}'(\lambda R)J_{1}'(\lambda) - (J_{1}'(\lambda))\}^{2} + \frac{-J_{1}'(\lambda R)Y_{1}'(\lambda)\}\sin\lambda(R-1)}{+\{Y_{1}'(\lambda)\}^{2}} - \frac{1}{\sqrt{R}} \right] \cos\lambda y \, d\lambda \quad \mathbf{B}(61a)$$

or

$$Z_{1}(y,R) = \frac{2}{\pi} \int_{0}^{\pi} \frac{\left[\left\{ Y_{1}'(\lambda R) J_{1}'(\lambda) - J_{1}'(\lambda R) Y_{1}'(\lambda) \right\} \cos \lambda (R-1) - \left\{ J_{1}'(\lambda R) J_{1}'(\lambda) + \left\{ J_{1}'(\lambda) \right\}^{2} + \frac{+Y_{1}'(\lambda R) Y_{1}'(\lambda) \sin \lambda (R-1) \right]}{\left\{ Y_{1}'(\lambda) \right\}^{2}} \sin \lambda y \, d\lambda \quad B(61b)$$

APPENDIX C.

Derivation of the expression for the potential behind the wing trailing edge.

· C.1 The first case.

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The coefficients a(p), $a^{*}(p)$ and $b^{*}(p)$ of the following equations have to be determined: $\varphi_{ao} = \overline{a}(p)K_{o}(p\beta r)$

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$$\overline{\varphi_{ai}} = a^*(p)I_o(p\beta r) + b^*(p)K_o(p\beta r).$$

Using eq. (4.10), the following relation is found:

 $\exists \in \mathbb{C}^{n'_{1}}$

$$b^{*}(p) = -a^{*}(p) \frac{I_{0}'(p\beta)}{K_{0}'(p\beta)}.$$
 C(1)

Using eq. (4.8) together with the eqs. (3.2) and (3.4) the following expression is obtained:

$$2 R'(p) e^{pl} - 2 e^{pl} \int_{0}^{1} e^{-pxr'(x)} dx = p\beta \left[a^{*}(p) \frac{I_{0}'(p\beta R)K_{0}'(p\beta)}{K_{0}'(p\beta)} - \frac{K_{0}'(p\beta R)I_{0}'(p\beta)}{K_{0}'(p\beta)} - \overline{a}(p)K_{0}'(p\beta R) \right].$$
Solving for $\overline{a^{*}(p)}$.

$$\overline{a}(p) = a^{*}(p) \frac{I_{0}'(p\beta R)K_{0}'(p\beta) - K_{0}'(p\beta R)I_{0}'(p\beta)}{K_{0}'(p\beta R)} - 2 \frac{R'(p)}{p\beta} \cdot \frac{e^{pl}}{K_{0}'(p\beta R)} + \frac{2}{p\beta} \frac{e^{pl}}{K_{0}'(p\beta R)} \int_{0}^{l} \frac{e^{-pxr'}(x)dx}{p\beta} \cdot C(2)$$

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When using eqs. (3.6) and (3.9) and denoting $\frac{\partial \varphi_0}{\partial x} - \frac{\partial \varphi_i}{\partial x}$ by g(x) the condition given by cq. (4.9) becomes:

$$e^{pl} \frac{R'(p)}{\beta} \left[\frac{K_0(p\beta R)}{K_0'(p\beta R)} + \frac{K_0'(p\beta)I_0(p\beta R) - I_0'(p\beta)K_0(p\beta R)}{K_0'(p\beta R) - I_0'(p\beta)K_0'(p\beta R)} \right] - e^{pl} \int_0^l e^{-px} g(x)dx = \\ = \left\{ a^*(p) \frac{I_0(p\beta R)K_0'(p\beta) - K_0(p\beta R)I_0'(p\beta)}{K_0'(p\beta)} - \bar{a}(p)K_0(p\beta R) \right\} p. \quad C(3)$$

Substituting eq. C(2) into eq. C(3) and using the Wronskian-relations the following result is obtained:

$$-\frac{e^{pl}R'(p)}{p\beta^{2}R} \frac{K_{\mathfrak{o}'}(p\beta R) \left[K_{\mathfrak{o}'}(p\beta)I_{\mathfrak{o}'}(p\beta R) - I_{\mathfrak{o}'}(p\beta)K_{\mathfrak{o}'}(p\beta R)\right]}{\beta R} - e^{pl} \int_{0}^{t} e^{-px}g(x)dx =$$

$$= -\frac{a^{*}(p)}{\beta R} \frac{1}{K_{\mathfrak{o}'}(p\beta R)} - \frac{2e^{pl}}{\beta} \frac{K_{\mathfrak{o}}(p\beta R)}{K_{\mathfrak{o}'}(p\beta R)} \int_{0}^{t} e^{-px}r'(x)dx . \quad C(4)$$

Using the eqs. C(1), C(2) and C(4):

$$a^{*}(p) = \frac{e^{p!}R'(p)}{p\beta} \frac{K_{0}'(p\beta)}{K_{0}'(p\beta R) - I_{0}'(p\beta)K_{0}'(p\beta R)} + \beta R K_{0}'(p\beta R)e^{pl} \int_{0}^{l} e^{-px}g(x)dx + -2R K_{0}(p\beta R)e^{pl} \int_{0}^{l} e^{-px}r'(x)dx \quad C(5)$$

$$b^{*}(p) = -\frac{e^{pl}R'(p)}{p\beta} \frac{I_{0}'(p\beta)}{K_{0}'(p\beta)I_{0}'(p\beta R) - I_{0}'(p\beta)K_{0}'(p\beta R)} - \beta R \frac{K_{0}(p\beta R)I_{0}'(p\beta)}{K_{0}'(p\beta)} e^{pl} \int_{0}^{t} e^{-px}g(x)dx + 2R \frac{K_{0}(p\beta R)I_{0}'(p\beta)}{K_{0}'(p\beta)} e^{pl} \int_{0}^{t} e^{-px}r'(x)dx \quad C(6)$$

$$\begin{split} \widetilde{a}(p) &= -\frac{e^{pl}R'(p)}{p\beta} \quad \frac{1}{K_{o}'(p\beta R)} + \beta Re^{pl} \frac{I_{o}'(p\beta R)K_{o}'(p\beta) - K_{o}'(p\beta R)I_{o}'(p\beta)}{K_{o}'(p\beta)} \int_{0}^{l} e^{-px}g(x)dx + \\ &+ 2 Re^{pl} \frac{K_{o}(p\beta R)I_{o}'(p\beta) - I_{o}'(p\beta R)K_{o}'(p\beta)}{K_{o}(p\beta)} \int_{0}^{l} e^{-px}r'(x)dx \,. \end{split}$$

The additional potential is thus given by:

$$\begin{split} \tilde{\varphi}_{ao} &= -\frac{ep^{l}R'(p)}{p\beta} \cdot \frac{K_{0}(p\beta R)}{K_{0}'(p\beta R)} + \beta Rep^{l} \frac{I_{0}'(p\beta R)K_{0}'(p\beta) - K_{0}'(p\beta R)I_{0}'(p\beta)}{K_{0}'(p\beta)} \cdot K_{0}(p\beta R) \int_{0}^{l} e^{-px}g(dx) + \\ &+ 2 Rep^{l} \frac{K_{0}(p\beta R)I_{0}'(p\beta) - I_{0}(p\beta R)K_{0}'(p\beta)}{K_{0}'(p\beta)} K_{0}(p\beta r) \int_{0}^{l} e^{-px}r'(x)dx \quad C(8) \end{split}$$

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$$\begin{split} \bar{\varphi}_{ai} &= \frac{e^{pl}R'(p)}{p\beta} \quad \frac{K_{0}'(p\beta)I_{0}(p\beta r) - I_{0}'(p\beta)K_{0}(p\beta r)}{K_{0}'(p\beta R) - I_{0}'(p\beta)K_{0}'(p\beta R)} + \beta R \frac{K_{0}'(p\beta R)}{K_{0}'(p\beta)} \\ & \cdot \left\{ K_{0}'(p\beta)I_{0}(p\beta r) - I_{0}'(p\beta) K_{0}(p\beta r) \right\} e^{pl} \int_{0}^{l} e^{-px}g(x)dx + \\ & - 2R \frac{K_{0}(p\beta R)}{K_{0}'(p\beta)} \left\{ K_{0}'(p\beta)I_{0}(p\beta r) - I_{0}'(p\beta)K_{0}(p\beta r) \right\} e^{pl} \int_{0}^{l} e^{-px}r'(x)dx. \quad C(9) \end{split}$$

If eq. C(8) is compared with eq. (3.6), it is seen that the inversion of the first term of eq. C(8) will give precisely the opposite of eq. (3.6). Thus these two terms cancel each other. The remaining part of the expression is then valid in the region aft of the leading edge Mach-wave of the ring wing. A similar result holds for eq. C(9) and eq. (3.9). The expressions for the potential valid in the whole region are therefore given by

$$\begin{split} \bar{\varphi}_{0} &= \beta R \ e^{pl} \frac{I_{0}'(p\beta R) K_{0}'(p\beta) - K_{0}'(p\beta R) I_{0}'(p\beta)}{K_{0}'(p\beta)} \ . K_{0}(p\beta r) \int_{0}^{l} e^{-px}g(x) dx + \\ &+ 2 R \ e^{pl} \frac{K_{0}(p\beta R) I_{0}'(p\beta) - I_{0}(p\beta R) K_{0}'(p\beta)}{K_{0}'(p\beta)} \ K_{0}(p\beta R) \int_{0}^{l} e^{-pxr'}(x) dx \quad C(10) \\ \bar{\varphi}_{i} &= \beta R \ e^{pl} \frac{K_{0}'(p\beta R)}{K_{0}'(p\beta)} \left\{ K_{0}'(p\beta) I_{0}(p\beta r) - I_{0}'(p\beta) K_{0}(p\beta r) \right\} \int_{0}^{l} e^{-pxg}(x) dx + \\ &- 2 R \ e^{pl} \frac{K_{0}(p\beta R)}{K_{0}'(p\beta)} \left\{ K_{0}'(p\beta) I_{0}(p\beta r) - I_{0}'(p\beta) K_{0}(p\beta r) - I_{0}'(p\beta) K_{0}(p\beta r) \right\} \int_{0}^{l} e^{-pxr'}(dx). \quad C(11) \end{split}$$

A check on these equations can be made by considering

$$\frac{\partial \varphi_0}{\partial x} = \frac{\partial \varphi_i}{\partial x}$$

For r = R this is equal to:

$$\frac{\partial \varphi_0}{\partial x} - \frac{\partial \varphi_i}{\partial x} = \frac{1}{2\pi i} \beta R \int_{a-i\infty}^{a+i\infty} \frac{pepx}{p\beta R} \int_0^t e^{-pt}g(t)dt dp = x = y+l$$
$$= \int_0^t g(t) \frac{i}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{p(x-t)}dt = \int_0^t g(t)\delta_+(x-t)dt \qquad C(12)$$

If
$$l > x = t > o$$
 this gives $\frac{\partial \varphi_0}{\partial x} - \frac{\partial \varphi_i}{\partial x} = g(x)$
and if $x = t > 1$ this gives $\frac{\partial \varphi_0}{\partial x} - \frac{\partial \varphi_i}{\partial x} = 0.$ (13)

This check therefore yields the correct values for r = R.

C 2. The second case.

The coefficients $\overline{a_1}(p) a_1^*(p)$ and $b_1^*(p)$ of the following equations have to be determined:

$$\overline{\varphi'_{ao}} = \overline{a_1}(p) K_0(p\beta r)$$

$$\overline{\varphi_{ai}} = a_1^*(p) I_0(p\beta r) + b_1^*(p) K_0(p\beta r).$$

Using eq. (4.10) it is found that: (see eq. C(1))

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$$b_1^*(p) = -a_1^*(p) \frac{I_0'(p\beta)}{K_0'(p\beta)}.$$
 C(14)

Substitution of the eqs. (3.12) and (3.13) in eq. (4.8) gives:

$$= p\beta \left[a_1^{*}(p) \ \frac{I_0'(p\beta R)K_0'(p\beta) - K_0'(p\beta R)I_0'(p\beta)}{K_0'(p\beta)} - \overline{a}_1(p)K_0'(p\beta R) \right]$$
C(15)

or

$$\overline{a_1}(p) = a_1^*(p) \frac{I_0'(p\beta R)K_0'(p\beta) - K_0'(p\beta R)I_0'(p\beta)}{K_0'(p\beta R)K_0'(p\beta R)}.$$
C(16)

Using eqs. (3.14, a) and (3.14, c) together with eq. (4.9) and denoting $\frac{\partial \varphi_{10}}{\partial x} - \frac{\partial \varphi_{1i}}{\partial x}$ by $g_i(x)$ leads to:

$$e^{pl} \frac{C'(p)}{\beta} \left[\frac{K_{o}(p\beta R)}{K_{o}'(p\beta R)} - \frac{K_{o}'(p\beta)I_{o}(p\beta R)}{K_{o}'(p\beta R)} - \frac{I_{o}'(p\beta)K_{o}(p\beta R)}{I_{o}'(p\beta R)} \right] - e^{pl} \int_{0}^{t} e^{-px}g_{1}(x)dx =$$

$$= \left\{ a_{1}^{*}(p) \frac{I_{o}(p\beta R)K_{o}'(p\beta)}{K_{o}'(p\beta)} - \frac{K_{o}(p\beta R)I_{o}'(p\beta)}{K_{o}'(p\beta)} - \tilde{a}(p)K_{o}(p\beta R) \right\} p. \quad C(17)$$

Elimination of $\overline{a_1}(p)$ with the aid of eq. C(16) and using the Wronskian-relation gives:

$$a_1^*(p) = -\frac{e^{plC'(p)}}{p\beta} \frac{K_{0'}(p\beta)}{K_{0'}(p\beta)I_{0'}(p\beta R) - I_{0'}(p\beta)K_{0'}(p\beta R)} + \beta R K_{0'}(p\beta R)e^{-px} \int_0^{\infty} e^{pl}g_1(x)dx \quad C(18,a)$$

$$b_1^*(p) = \frac{e^{pl}C'(p)}{p\beta} \frac{I_0'(p\beta)}{K_0'(p\beta)I_0'(p\beta R) - I_0'(p\beta)K_0'(p\beta R)} - \beta R \frac{K_0'(p\beta R)I_0'(p\beta)}{K_0'(p\beta)} e^{pl} \int_0^1 e^{-px}g_1(x)dx$$

$$C(18, b)$$

$$\bar{a}_{1}(p) = -\frac{e^{pl}C'(p)}{p\beta K_{0}'(p\beta R)} + \beta R \frac{I_{0}'(p\beta R)K_{0}'(p\beta) - K_{0}'(p\beta R)I_{0}'(p\beta)}{K_{0}'(p\beta)} e^{pl} \int_{0}^{1} e^{-px}g_{1}(x)dx.$$
 C(18, c)

The additional potentials are thus given by

$$\begin{split} \bar{\varphi'}_{ao} &= -\frac{e^{pl}C'(p)}{p\beta} \frac{K_{0}(p\beta r)}{K_{0}'(p\beta R)} + \beta Rlp^{l} \cdot \frac{I_{0}'(p\beta R)K_{0}'(p\beta) - K_{0}'(p\beta R)I_{0}'(p\beta)}{K_{0}'(p\beta)} K_{0}(p\beta r) \int_{0}^{t} e^{-px}g_{1}(x)dx \\ \bar{\varphi'}_{ai} &= -\frac{e^{pl}R'(p)}{p\beta} \frac{K_{0}'(p\beta)I_{0}(p\beta r) - I_{0}'(p\beta)K_{0}(p\beta r)}{K_{0}'(p\beta R) - I_{0}'(p\beta)K_{0}'(p\beta R)} + \\ \beta R \frac{K_{0}'(p\beta R)}{K_{0}'(p\beta)} \left\{ K_{0}'(p\beta)I_{0}(p\beta r) - I_{0}'(p\beta)K_{0}(p\beta r) - I_{0}'(p\beta)K_{0}(p\beta r) \right\} e^{pl} \int_{0}^{t} e^{-px}g_{1}(x)dx. \quad C(19,b) \end{split}$$

With the same arguments as used in section C.1 it can be shown when comparing eqs. C(19, a) and C(19, b) with eqs. (3.14, a) and (3.14, c) that the first terms of eqs. C(19, a) and C(19, b) cancel each other. The remaining expressions are valid in the whole region and are given by:

$$\overline{\varphi_{o}}' = \beta Re^{pl} \frac{K_{o}(p\beta r)}{K_{o}'(p\beta)} \{ I_{o}(p\beta R) K_{o}'(p\beta) - K_{o}'(p\beta R) I_{o}'(p\beta) \} \int_{0}^{t} e^{-px} g_{1}(x) dx \qquad C(20, a)$$

$$\bar{\varphi_i}' = \beta Re^{pl} \frac{K_0'(p\beta)}{K_0'(p\beta R)} \{ K_0'(p\beta)I_0(p\beta r) - I_0'(p\beta)K_0(p\beta r) \} \int_0^t e^{-px}g_1(x)dx \}.$$
 C(20, b)

C.3 The third case.

In this case the coefficients $\overline{f}(p)$, $f^*(p)$ and $g^*(p)$ of the eqs. (4.25, a) and (4.25, b) have to be determined. Applying eq. (4.10) it is found that:

 $f^{\ast}(p) = -g^{\ast}(p) \frac{I_{1}'(p\beta)}{K_{1}'(p\beta)}$ C(21)

1

,

Using eq. (4.8) together with eqs. (3.16) and (3.18,b) gives

$$o = p\beta \left[g^{\bullet}(p) \ \frac{I_{1}'(p\beta R)K_{1}'(p\beta) - K_{1}'(p\beta R)I_{1}'(p\beta)}{K_{1}'(p\beta)} - \bar{f}(p)K_{1}'(p\beta R) \right].$$
 C(22)

Solving for $\overline{f}(p)$ gives:

$$\bar{f}(p) = g^{*}(p) \frac{I_{1}'(p\beta R)K_{1}'(p\beta) - K_{1}'(p\beta R)I_{1}'(p\beta)}{K_{1}'(p\beta R)}.$$
C(23)

Substitution of eqs. (3.20, (3.23), (4.25, a) and (4.25, b) in eq. (4.9) and denoting $\frac{\partial \Phi_{ao}}{\partial x} - \frac{\partial \Phi_{oi}}{\partial x}$ by $g_2(x)$ gives:

$$e^{pl} \frac{A'(p)}{\beta} \left[\frac{K_{1}(p\beta R)}{K_{1}'(p\beta R)} - \frac{K_{1}'(p\beta)I_{1}(p\beta R) - I_{1}'(p\beta)K_{1}(p\beta R)}{K_{1}'(p\beta R) - I_{1}'(p\beta)K_{1}'(p\beta R)} \right] - e^{pl} \int_{0}^{t} e^{-px}g_{2}(x)dx = \\ = \left\{ g_{1}^{*}(p) \frac{I_{1}(p\beta R)K_{1}'(p\beta) - K_{1}(p\beta R)I_{1}'(p\beta)}{K_{1}'(p\beta)} - \bar{f}(p)K_{1}(p\beta R) \right\} p.$$
 C(24)

Using eq. C(23) and the Wronskian-relations the final result for $g_i^*(p)$ is:

$$g_{1}^{\bullet}(p) = -\frac{e^{pl}A'(p)}{p\beta} \frac{K_{1}'(p\beta)}{K_{1}'(p\beta)I_{1}'(p\beta R) - I_{1}'(p\beta)K_{1}'(p\beta R)} + \beta R K_{1}'(p\beta R)e^{pl} \int_{0}^{1} e^{-px}g_{2}(x)dx. \quad C(25)$$

Insert this result into eqs. C(21) and C(23):

$$f^{\ast}(p) = \frac{e^{pl}A'(p)}{p\beta} \frac{I'(p\beta)}{K_{1}'(p\beta)I_{1}'(p\beta R) - I_{1}'(p\beta)K_{1}'(p\beta R)} - \beta R \frac{K_{1}'(p\beta R)I_{1}'(p\beta)e^{pl}}{K_{1}'(p\beta)} \int_{0}^{1} e^{-px}g_{2}(x)dx$$

$$C(26, a)$$

$$\bar{f}(p) = -\frac{e^{pl}A'(p)}{p\beta} \frac{1}{K_1'(p\beta R)} + \beta Re^{pl} \frac{I_1'(p\beta R)K_1'(p\beta) - K_1'(p\beta R)I_1'(p\beta)}{K_1'(p\beta)} \int_0^t e^{-px}g_2(x)dx. \quad C(26, b)$$

The expressions for the additional potential thus become:

$$\overline{\Phi'}_{ao} = - \frac{e^{pl}A'(p)}{p\beta} \frac{K_1(p\beta r)}{K_1'(p\beta R)} + \beta Re^{pl} \left\{ I_1'(p\beta R)K_1'(p\beta) - K_1'(p\beta R)I_1'(p\beta) \right\} \frac{K_1(p\beta r)}{K_1'(p\beta)} \int_0^r e^{-px}g_2(x)dx$$

$$C(27, a)$$

$$\begin{split} \widetilde{\Phi'}_{ai} &= -\frac{e^{pt}A'(p)}{p\beta} \frac{K_{1}'(p\beta)I_{1}(p\beta r) - I_{1}'(p\beta)K_{1}(p\beta r)}{K_{1}'(p\beta R) - I'_{1}(p\beta)K_{1}'(p\beta R)} + \\ &+ \beta R \frac{K_{1}'(p\beta R)}{K_{1}'(p\beta)} \{ K_{1}'(p\beta)I_{1}(p\beta r) - I_{1}'(p\beta)K_{1}(p\beta r) \} e^{pt} \int_{0}^{t} e^{-px}g_{2}(x)dx. \end{split}$$

Comparing eqs. C(27, a) and C(27, b) with eqs. (3.20) and (3.23) and remembering that x = y + l it can be seen that eq. (3.20) consists the first term of eq. C(27, a) in the whole field. In the same way eq. (3.23) is cancelled by the first term of eq. C(27, b). The resulting expressions thus become:

$$\Phi_o = \beta Re^{pl} \left\{ I_1'(p\beta R) K_1'(p\beta) - K_1'(p\beta R) I_1'(p\beta) \right\} \frac{K_1(p\beta r)}{K_1'(p\beta)} \int_0^r e^{-px} g_2(x) dx \qquad C(28,a)$$

$$\overline{\Phi_i} = \beta Re^{pl} \left\{ I_1(p\beta r) K_1'(p\beta) - K_1(p\beta r) I_1'(p\beta) \right\} \frac{K_1'(p\beta R)}{K_1'(p\beta)} \int_0^t e^{-px} g_2(x) dx \qquad C(28, b)$$

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REPORT NLR-TR G. 30

On the determination of optium shapes with finite nose angles

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P. J. ZANDBERGEN

Summary

This report presents a method for the determination of axially-symmetric shapes with a given base area that are optimum with respect to wave drag in supersonic flow and which have finite nose angles. Use has been made of the exact non-linear differential equations for supersonic flow together with the shock equations.

The computed results indicate that the optimum bodies with a finite nose angle have a lower drag than those with cusped noses. This would make their practical application of a certain significance. A number of such shapes have been presented in this report.

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List of symbols

- speed of sound a
- total velocity non dimensionalized with U_{∞} а
- radial co-ordinate r
- axial velocity component u
- radial velocity component v
- x axial co-ordinate
- D drag
- M Mach number

- P ratio of stagnation pressures of disturbed and undisturbed flow
- radius of base area
- radius of intersection of shockwave and aftcharacteristic surface
- velocity of undisturbed stream

$$\beta = \sqrt{M^2 - 1}$$

- ratio of the specific heats
- variation in v
- semi-top angle of conical nose
- semi-top angle of shockwave
- constant multiplier
- multiplier
- density; variation in u

èS.

- refers to surface; initial value
- refers to shockwave
- refers to undisturbed flow

1 Introduction

During recent years, a number of investigations have been carried out to determine the shape of axiallysymmetric configurations which are optimum with respect to wave drag. Such shapes can be determined only when a sufficient number of constraints are defined. The most commonly used constraints are prescribed values for the base area, the length of the body and the Mach number of the undisturbed stream. These problems were first studied in the light of linearized potential theory (ref. 1). But when it became apparent that this theory, especially for interference problems,

This investigation has been performed under contract with the Netherlands Aircraft Development Board.

yields unreliable results, it was decided to use a more accurate theory for the description of the flow properties.

Fortunately, the method used in ref. 1 can be generalized easily and made applicable to cases for which the more exact non-linear differential equations are used. This is possible, due to the particular scheme followed in defining and solving the optimum problem. The scheme is based on the assumption that the body lies in a volume enclosed by a control surface. This control surface consists of two parts, namely the surface separating the regions of disturbed and undisturbed flow and the forward facing characteristic surface emanating from the base, the so called aft-characteristic surface (fig. 1).



Fig. 1 Optimum body embedded in a control volume.

It is demonstrated that the wave drag and the basearea can be expressed as integrals of a function dependent on the velocities along this aft-characteristicsurface. If the base area is assumed to have a prescribed value, while also the radius of the intersecting circle of the two parts of the control surface is given* together with the Mach number of the undisturbed stream, the optimum velocity distribution may be determined by solving a set of two ordinary differential equations. These equations are obtained by using variational theory together with the appropriate characteristic equation along the aft-characteristic surface.

For the case of isentropic flow inside the control surface the solution of the problem has been obtained along these lines in ref. 2. Since no shock-waves can form inside the control surface, the body must have a cusped nose generating a compression fan which will converge upon the circumference of a certain circle outside the region considered. This circle is found to coincide with the intersection of the two parts of the control surface. This case has a more theoretical than practical value, however, because of the unrealistic nose shape.

Therefore it seems recommendable to develop a method for finding optimum configurations with a nose of finite top angle. Contrary to the case of zero top angle treated in ref. 2 a shock wave will be generated by this nose and the flow inside the control surface in general ceases to be isentropic. Only in the case of a conical shock wave extending at least to the intersection of the two parts of the control surface the flow will still be isentropic inside this surface. However, the expressions for the drag and the base area will change, since the ratio of the stagnation pressures of the disturbed and the undisturbed flow is no longer equal to unity, due to the increase in entropy across the shock wave.

It is this particular case that will be investigated in the present report. Before proceeding to its solution a few general remarks will be made on the interesting features of this problem.

In the first place it is the purpose of this investigation to provide an answer to the question whether or not the minimum drag will be lower when a shock wave is allowed to exist inside the control volume as compared to the minimum drag for a cusped-nosed body. Or in other words, how does the minimum drag value behave as a function of the ratio of the stagnation pressures.

In the second place it should be mentioned that for a given Mach number and given values of the radii of base and outer edge of the control surface, the present scheme does not present a physically adequate solution for arbitrary values of the cone angle. This is due to the fact that the solution itself provides a set of boundary conditions, which in essence require the existence of a shockwave followed by a compression fan converging at the juncture of the two parts of the control surface (fig. 2). As long as the strength of this compression fan is positive a solution exists. The limiting value of the cone angle is obtained when this strength becomes zero. Any smaller value of the cone angle hence gives a solution of physical significance. When the top-angle of the cone becomes greater than the limiting value, a solution according to the present scheme



Fig. 2 General view of optimum configuration.

[•] It should be observed that this constraint is identical with the constraint of a given length when linearized theory is used. For the non-linear theory this is no longer true and the length is obtained as a result of the computations.

is only possible if an expansion fan starts at the outer edge of the control surface. This, however, could occur only if there were a fixed boundary at the outer edge, which in general will not be the case. Although we will refrain from considering this in more detail, a solution can be obtained by allowing a discontinuity in the slope of the contour of the optimum body, giving rise to an expansion fan. The problem then becomes much more complicated due to the fact that part of the boundary conditions can only be given in numerical form.

The present report consists of three main parts.

The first presents the mathematical formulation of the problem. Since most of the necessary formulae have been derived elsewhere, no detailed derivations will be given. Only the essential differences with earlier work will be discussed in some detail.

In the second part a review will be given of the numerical methods used in the actual computation. The schemes according to which the computations have been performed will be illustrated by flow-diagrams contained in appendix A.

A discussion of the results obtained will be presented in the third part of the report. In fact these results and their implications form the nucleus of the presentation, and it is believed that this is the first time that a number of optimum shapes with finite nose-angle and continuous slopes are given which are determined by fully exact-methods.

In conclusion a few words will be said about further generalizations of the way of approach given here. Instead of having a conical nose of such a length that the ratio of stagnation pressures is constant along the aft-characteristic surface, the problem could be considered with this ratio as a given function of the radial co-ordinate along this surface. This means that the nose of the body will have a finite radius of curvature. It should indeed be possible then to obtain blunt-nosed optimum shapes. This would constitute a subject for further investigations which is highly interesting. The numerical computations will be very complicated, however, and will require much research in order to become practical.

2 The mathematical formulation and the solution of the optimum problem

In this section the mathematical formulation of the optimization problem will be given, together with its solution by means of variational theory. It is assumed that we are considering an axially-symmetric configuration which is oriented along the x-axis. The radial coordinate of a point of the flow-field is given by r, while the axial and radial components of the velocity at that point are given by u and v respectively. Furthermore the body of revolution is assumed to have a coni-

cal nose of such a length that the flow inside the control-surface, consisting of the shockwave and the aftcharacteristic surface, is isentropic. In addition the Mach number M_{∞} of the free stream, the radius R_B of the base area and the radius R_C of the intersecting circle of shockwave and aft-characteristic surface have prescribed values. The geometry of the problem has been outlined in fig. 1.

2.1 The mathematical formulation

Under the circumstances discussed above the wave drag D experienced by the body can be expressed as the integral of a certain function taken along the aft-characteristic surface. According to ref. 2 the following expression is valid

$$\frac{D}{\varrho_{\infty} U_{\infty}^2} = 2\pi \int_{R_B}^{R_c} \frac{1}{\gamma M_{\infty}^2} \left[1 - (a^2 M_{\infty}^2)^{\gamma/(\gamma-1)} P \right] r dr + 2\pi \int_{R_B}^{R_c} (u-1) (a^2 M_{\infty}^2)^{1/(\gamma-1)} P \frac{q^2}{u-\beta v} r dr$$
(2.1)

In this equation a number of dimensionless quantities occur that will be defined subsequently. The axial and radial velocity components u and v are made dimensionless with the free stream velocity U_{∞} . The dimensionless total velocity q is given by

$$q = \sqrt{u^2 + v^2} \tag{2.2}$$

and the local velocity of sound a, also dimensionless can be expressed by the formula

$$a^{2} = \frac{1}{M_{\infty}^{2}} + \frac{\gamma - 1}{2} - \frac{\gamma - 1}{2} q^{2}$$
(2.3)

As is usual γ is the ratio of the specific heats, while β is given by

$$\beta = \sqrt{M^2 - 1} \tag{2.4}$$

where the local Mach number M is defined by the ratio of q and a.

Moreover the quantity P, which is the ratio of the stagnation pressures of the disturbed and the undisturbed flow, occurs in formula (2.1). If this quantity is smaller than unity a shock wave exists ahead of the aft-characteristic surface. In accordance with the assumption that the body has a conical nose of a certain length, it will be assumed that the shockwave generated by this nose remains at least conical to the intersecting circle of the control surface, and hence that P as occurring in eq. (2.1) has a given constant value, which will be smaller than or at most equal to unity.

In order to minimize the wave drag given by eq. (2.1), that is, to find such a distribution of u and v along the aft-characteristic surface that D attains its minimum value the accessory conditions inherent to the problem will have to be satisfied.

The conservation of mass requires that the mass flowing through the shock wave surface be equal to the mass flowing through the aft-characteristic surface. The equation may be written as (ref. 2)

$$R_{C^{2}} = 2 \int_{R_{B}}^{R_{C}} (a^{2} M_{\infty}^{2})^{1/(\gamma-1)} \frac{q^{2}}{u-\beta v} Pr dr \qquad (2.5)$$

Furthermore the flow along the aft-characteristic surface has to satisfy the appropriate characteristic equation given by

$$(u\beta - v)\frac{\mathrm{d}u}{\mathrm{d}r} + (u + \beta v)\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{v}{r}\frac{q^2}{u - \beta v} - \frac{a^2\beta}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r} = 0$$
(2.6)

Since P is assumed to have a constant value along the aft-part of the control surface, equation (2.6) reduces to

$$(u\beta - v)\frac{\mathrm{d}v}{\mathrm{d}r} + (u + \beta v)\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{v}{r}\frac{q^2}{u - \beta v} = 0 \quad (2.7)$$

The problem of finding the minimum value for the wave drag under the constraints represented by eqs. (2.5) and (2.7) is particularly adapted to solution by means of variational theory. With reference to the boundary conditions to be imposed at the end points R_B and R_C of the internal considered, the following can be observed.

At the circumterence of the base no physical boundary condition is present and hence the variational procedure itself must yield a so-called "natural" boundary condition. At the intersecting circle of fore and aftpart of the control surface the shock wave conditions must be fulfilled.

Given the quantity P, the semi-top angle ϑ_w of the conical shock wave can be determined from the following well-known formula

$$P = \left[\frac{(\gamma+1)M_{\infty}^{2}\sin^{2}\vartheta_{w}}{(\gamma-1)M_{\infty}^{2}\sin^{2}\vartheta_{w}+2}\right]^{\gamma/(\gamma-1)} \times \left[1 + \frac{2\gamma}{\gamma+1}(M_{\infty}^{2}\sin^{2}\vartheta_{w}-1)\right]^{-1/(\gamma-1)}$$
(2.8)

From the value of ϑ_w , the velocity components u and v just behind the shock wave can be computed by using the following equations (ref. 2)

$$u-1 = \frac{2}{\gamma+1} \frac{1-M_{\infty}^2 \sin^2 \vartheta_w}{M_{\infty}^2}$$
(2.9)

and

$$v = -(u-1)\cot\vartheta_w \tag{2.10}$$

It will be discussed furtheron, how these equations have to be used when determining the solution of the variational problem; this solution will be derived in the following section.

2.2 Solution of the variational problem

The determination of the minimum value of the wave drag, given by eq. (2.1) and under the conditions specified by eqs. (2.5) and (2.7), is a particular case of a variational problem considered by Bolza (ref. 3). This problem can be solved by using the method of multipliers, a system of auxiliary constants and functions which enables an easy formal derivation of the solution. We then have to consider the following function, combining the integrands of eqs. (2.1) and (2.5) and the characteristic equation

$$F = \left\{ \frac{1}{\gamma M_{\infty}^{2}} \left[1 - (a^{2} M_{\infty}^{2})^{\gamma/(\gamma-1)} P \right] + (u-1)(a^{2} M_{\infty}^{2})^{1/(\gamma-1)} \frac{q^{2} P}{u-\beta v} \right\} r + \lambda (a^{2} M_{\infty}^{2})^{1/(\gamma-1)} \frac{q^{2} P r}{u-\beta v} + \mu(r) \left\{ (\beta u - v) \frac{\mathrm{d}u}{\mathrm{d}r} + (u+\beta v) \frac{\mathrm{d}v}{\mathrm{d}r} + \frac{v}{r} - \frac{q^{2}}{u-\beta v} \right\}$$
(2.11)

where λ is a constant multiplier and μ is a multiplier which is a function of r.

The necessary conditions for a minimum are found by considering the variation of the integral over the function F and to require that this variation is zero, or

$$\delta \int_{R_B}^{R_C} F\left(r, u, v, \frac{\mathrm{d}u}{\mathrm{d}r}, \frac{\mathrm{d}v}{\mathrm{d}r}, \mu, \lambda\right) \mathrm{d}r = 0 \qquad (2.12)$$

As a result of this variation, a set of two differential equations is obtained, known as Euler's equations. If a prime denotes differentiation with respect to r, these equations can be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\partial F}{\partial u'}\right) - \frac{\partial F}{\partial u} = 0 \qquad (2.13)$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\partial F}{\partial v'}\right) - \frac{\partial F}{\partial v} = 0 \qquad (2.14)$$

Moreover the variational procedure yields boundary conditions which, when ρ and η denote the variations in *u* and *v* respectively, can be written as

$$\varrho \frac{\partial F}{\partial u'} + \eta \frac{\partial F}{\partial v'} = 0 \quad \text{in } r = R_B \text{ and } r = R_C \text{ respectively}$$
(2.15)

Carrying out the operations indicated in eqs. (2.13) and (2.14) and replacing μ by $\bar{\mu}P$, a set of equations results which are homogeneous in P and hence can be divided by this quantity. This means that the equations reduce to the same form as in the purely isentropic case considered in ref. 2.

As has been demonstrated there the resulting equation, after elimination of the function $\bar{\mu}$, takes the following form

$$\left[uZ + \beta^2 a^4 \left\{ u - \beta v + (u - 1 - \lambda) \right\} \right] \frac{\mathrm{d}u}{\mathrm{d}r} + \left[vZ + \beta^2 a^4 \left\{ 4v - \beta(u - 1 - \lambda) - 2vb \frac{X}{Y} \right\} \right] \frac{\mathrm{d}v}{\mathrm{d}r} + 2 \frac{v}{r} \beta a^4 \left[-2\beta v - (u - 1 - \lambda) \left\{ (1 - \beta^2) + \alpha_2 \left(\frac{u\beta a^2 - vb}{\beta a^4} \right) \right\} \right] = 0$$
(2.16)

in which the quantities X, Y and Z may be written as

$$X = 2v^{2} + (u - 1 - \lambda)(u - \beta v)$$
(2.17)a

$$Y = v^2 b - \beta^2 a^4$$
 (2.17)b

$$Z = Y + a^2 \frac{X}{Y} \left\{ v^2 b(\gamma + 1 - \gamma \beta^2) + \beta^4 a^4 \right\} - (u - 1 - \lambda)\beta v b$$
(2.17)c

and

$$b = \frac{1}{M_{\infty}^2} + \frac{\gamma - 1}{2}, \quad \alpha_2 = \frac{q^2}{u - \beta v}$$
 (2.18)

The complicated equation (2.16) derived by the variational procedure forms together with the characteristic equation (2.7) a system of two non-linear first order differential equations for the functions u and v along the aft-characteristic surface. This system is identical to the one considered in ref. 2 for the case of zero noseangle. Differences occur, however, as will be shown in the following, when the two boundary conditions necessary to solve the system are written down. These boundary conditions will be considered next.

At the circumference of the base, given by $r = R_B$, no physical boundary conditions are present, as already stated. This means that eq. (2.15) has to be satisfied for arbitrary variations ρ and η . Hence we have

$$\frac{\partial F}{\partial u'} = \frac{\partial F}{\partial v'} = 0 \quad \text{for } r = R_B \tag{2.19}$$

According to ref. 2 this condition means that $\bar{\mu}$ has to be zero, which can be expressed by the following equation giving the boundary condition for $r = R_B$

$$-2\beta v - (u-1-\lambda)\left\{(1-\beta^2) - \alpha_2\left[\frac{u\beta a^2 - vb}{\beta a^4}\right]\right\} = 0$$
(2.20)

It is evident now that only one boundary condition can be present at the intersection circle of fore and aftpart of the control surface, i.e. $r = R_C$. It can be found by writing eq. (2.16) in the following form

$$(\beta u - v) du + (\beta v + u) dv = 0$$
(2.21)

This equation, which is readily recognized as the differential-equation for the Prandtl-Meijer compression fan, has the following solution

$$V \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \frac{\sqrt{\frac{\gamma - 1}{\gamma + 1}} (\beta - \beta_s)}{1 + \frac{\gamma - 1}{\gamma + 1} \beta \beta_s} + \tan^{-1} \frac{\beta - \beta_s}{1 + \beta \beta_s} + \tan^{-1} \frac{v u_s - u v_s}{u u_s + v v_s} = 0 \quad (2.22)$$

in which s refers to a known initial condition.

In order to interpret this expression, the following should be remarked. The expression has to be valid at the point $r=R_C$; this point is multivalued unless $\beta=\beta_s$, $u=u_s$ and $v=v_s$. Then in general this point will be the point of convergence of a compression fan generated somewhere on the body. Its behaviour is found by the condition that in the point $r=R_C$ the velocities are given by eqs. (2.9) and (2.10). This can be arranged by selecting these values of the velocities as the initial conditions u_s and v_s in eq. (2.22).

The whole problem can be summarized as follows now:

The condition for the wave drag as given by eq. (2.1) to attain its minimum value can be satisfied by specifying a certain velocity distribution along the aft-characteristic surface. This distribution can be found by solving the two differential equations (2.7) and (2.16) for the boundary conditions given by eqs. (2.20) and (2.22). The unknown multiplier λ occurring in these expressions has to be determined by using the mass-flow conditions as given by eq. (2.5).

It will be apparent that the differences between this system and the one solved in ref. 2 for the case P=1, occur only in the boundary condition (2.22) and the mass-flow condition (2.5). In fact the problem considered here is a generalization of the procedure given in ref. 2 and it contains the solution of the problem solved there as a special case. With P equal to unity, $u_s=1$ and $v_s=0$, eqs. (2.22) and (2.5) are exactly identical to those used in ref. 2.

We will now proceed with a review of the numerical schemes and methods used in the solution of the problem.

3 The numerical evaluation of the problem

Although the two differential equations (2.7) and (2.16) are not amenable to an analytical solution, because of their non-linear character, they can be solved without any difficulty, since a variety of methods exists for the integration of such equations by numerical means. However, the amount of time required for the computation of a solution depends to a great extent on the form in which the initial conditions are expressed. The most simple case is that in which these conditions

are given in one of the edge points of the region considered. Then one integration will suffice to determine the solution. This is no longer true when for both edgepoints boundary values are prescribed. In that case the non-linear character of the differential equations requires an iteration procedure, to obtain the solution. A closer examination of eqs. (2.20) and (2.22) reveals that this case is present here. Moreover, in one of the differential equations and in the boundary condition at the circumference of the base the unknown multiplier λ occurs. Its value cannot be determined in advance, since it is governed by the mass-flow condition eq. (2.5). This in itself leads to a new iteration procedure. Hence, in order to solve the problem set forth in the preceding section a double iteration procedure is required. The details of the method will be described hereafter with the aid of appendix A, which contains the flow diagrams used in the computation.

First the case will be considered in which the values for P, M_{∞} , R_C and R_B are given. With the given value for P, the value of the shock-wave angle ϑ_w can be determined from eq. (2.8). This is not so simple, however, since for the determination of the parameter $M_{\infty} \sin \vartheta_w$ with a prescribed accuracy, P has to be known with a far greater accuracy. When performing the calculations with a computer such a problem has to be treated with some care. The way in which this problem can be solved has been outlined in appendix B. Once ϑ_w has been determined the velocity components u_s and v_s just behind the shockwave can be obtained from eqs. (2.9) and (2.10). In order to be able to integrate the differential equations the values for u, v and λ for say $r = R_c$ have to be known. By estimating the value of u, the velocity v can be determined from eq. (2.22). When also estimating λ^* , the integration of the differential equations can be performed. Use has been made of the Runge-Kutta method in the version of Gill (ref. 4). When the integration has proceeded until $r = R_B$, in general neither eq. (2.20) nor eq. (2.5) will be satisfied. Then an inner iteration cycle is performed by using the same estimated value for u but a different value for λ , until eq. (2.5) is satisfied. Then in general, still eq. (2.20) will not be satisfied yet. Thus an outer iteration cycle will have to be performed by taking a new estimated value for u and hence for v, going through the process already described till finally both eq. (2.20) and eq. (2.5) are satisfied. Then a solution of the problem has been determined. All the computations were performed with a digital computer except the iteration procedures for λ and u, since in advance no estimate could be made about their variation for different values of P for given values of M_{∞} , R_B and R_C . Computation time was saved by using a larger interval Δr in the initial phase of the iteration procedure when u and λ are not known accurately.

As has been put forward in the introduction not every value of P gives rise to a physically significant solution. As long as $u < u_s$ in the point $r = R_c$, the strength of the compression-fan is positive; in other words the point $r = R_C$ can really be considered as the point of convergence of a compression fan generated by the body itself. For $u > u_s$, an instantaneous expansion should occur at R_c and this is not possible unless a fixed boundary is present, which is not the case. It is clear now that a limiting value of P exists for which $u = u_s$ (and thus $v = v_s$) holds true. This means that no compression fan is generated by the body and that therefore the semi-top angle of the cone is the largest slope occurring. Since this case is a special one, it has to be treated separately. The scheme of the computations is given in the flow diagram b) of appendix A, Instead of performing a double iteration with respect to u and λ , a double iteration with respect to P and λ is performed in essentially the same way as in the program discussed above.

Once the distribution of the velocities along the aftcharacteristic surface has been determined the only problem which remains is the determination of the shape of the optimum configuration itself. This can be achieved by using the conical-flow solution and the method of characteristics. The procedure is as follows. First, from the known values of M_{∞} , u_s and v_s and the fact that the flow over the nose has to be conical, the flow and the value of the semi-top angle ϑ_s of the cone can be calculated by using the known formula for conical flow (see f.i. ref. 2). Then the velocity distribution along the last characteristic of this conical region, which passes through $r = R_c$, is also known.

Next the flow inside the compression fan has to be determined. This can be achieved by observing that R_c is a point which is multi-valued and that hence a set of characteristics converges in this point. The flow region inside the compression fan can then be computed by using the method of characteristics together with the velocity distribution along the last characteristic of the conical region. The contour of the body is determined by the condition that it should be a stream surface expressed by

$$\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{v}{u} \tag{3.1}$$

Before the flow over the aft-part of the configuration and the shape of its contour can be constructed, the shape of the aft-characteristic should be known first. This can be determined by using the equation for the characteristic direction

$$\frac{\mathrm{d}r}{\mathrm{d}x} = -\frac{u-\beta v}{\beta u+v} \qquad (3.2)$$

[•] A good estimate for u and λ can be found by using the linearized theory of ref. 1.

In combination with the now known velocity distribution along the last characteristic of the compression fan the flow over the aft-part of the configuration can be computed. This completes the determination of an optimum configuration. It should be remarked that the length of the configuration is determined by integrating eq. (3.2) using eq. (2.10). There follows that

$$l = -\frac{v_s}{u_s - 1} R_C - \int_{R_C}^{R_B} \frac{\beta u + v}{u - \beta v} \, \mathrm{d}r \qquad (3.3)$$

For P = 1, this becomes

$$l = \beta_{\infty} R_C - \int_{R_C}^{R_B} \frac{\beta u + v}{u - \beta v} \, \mathrm{d}r \tag{3.4}$$

A sketch of the different regions and notions together with the resulting body is given in fig. 2.

4 Discussion of the results

Computations have been performed for a number of cases with different Mach numbers and for several values of the ratio of the radii R_C and R_B . In all cases considered the value of R_B has been taken equal to 0.1. First for all ratios of R_c and R_B the limiting value of P has been obtained as described in the foregoing section, after which the purely isentropic case was considered for P=1. The results are given in table 1. It contains besides the given values of M_{∞} and R_C , the values of the multiplier λ , the quantity P (being either the limit value or equal to unity), the length *l*, and the dimensionless value of the drag $Dl^2/\rho_{\infty} U_{\infty}^2 R_B^4$. Some remarkable features are exhibited by this table. In the first place it is apparent that the length of the bodies which cusped noses (P=1) is materially greater than that of the bodies with a conical nose for the same value of R_C/R_B . This, however, could be expected, since shock waves tend to have much larger tangents than the characteristics of the undisturbed flow. Far more interesting is the behaviour of the drag, since its value appears to be smaller for the cases with a conical nose. It should be observed that according to the linearized theory (ref. 1) the value of the dimensionless drag considered here is independent of Mach number and can be represented by

$$\frac{Dl^2}{\varrho_{\infty} U_{\infty}^2 R_B^4} = 2\pi \frac{(1 - R_B/2R_c)^2}{1 - \left(\frac{R_B}{R_c}\right)^2}$$
(4.1)

The values obtained by using this formula are also given in table 1.

The results obtained for the drag indicate a rather large difference between the results for P = 1 and for P equal to its limiting value. This difference tends to become larger for smaller body lengths and higher Mach numbers.

Thus the question whether or not the drag of an optimum configuration with a finite nose has a lower drag than that with a cusped nose has been partially answered. But how does the drag behave as a function of the ratio of the stagnation pressure, which is a direct measure of the shock wave strength? This question has

M_{∞}	R_C/R_B	Р	λ	l/R_B	Dl^2	Dl ² (eq. 4.1)
					$\overline{\varrho_{\infty}U_{\infty}^2R_B^4}$	$\overline{\varrho_{\infty} U_{\infty}^2 R_B^4}$
2	3.4641016	0.99994719	-0.02915115	10.22948	4.89435	5.018493
	2.4748737	0.99955040	-0.06078521	6.70366	4.46678	4.781506
		1.0000000	-0.06061171	7.10101	4.92111	
	2.0655911	0.99848399	0.09326397	5.19287	4.18319	4.714458
3	3.4641016	0.99996877	0.01079695	16.83735	4.91886	5.018493
	2.4748737	0.99974104	0.02210580	11.16637	4.52702	4.781506
		1.00000000	0.02207138	11.61946	4.84295	
	2.0655911	0.99915510	0.03317775	8.77514	4.28331	4.714458
		1.00000000	0.03306169	9.33174	4.75345	
4	3.4641016	0.99997342	0.00574590	23.09443	4.92528	5.018493
	2.4748737	0.99978039	-0.01172619	15.35467	4.54250	4.781506
		0000000.1	-0.01171088	15.91093	4.82571	
	2.0655911	0.99928674	-0.01753491	12.10126	4.30853	4.714458
	1.7000000	0.99729412	-0.02783722	9.12277	4.03187	4.787189
		1.00000000	-0.02763601	9.98577	4.69860	

TABLE 1

The drag as function of the stagnation pressures for a certain case

$M_{\rm oc} = 3 R_{\rm C}/R_{\rm B} = 2.0656$				
P	l/R_B	$D_0 l^2$	acc. to ref. 1	
		$\overline{\varrho_{\infty} U_{\infty}^2 R_B^4}$		
0.999155	8.77514	4.28331		
0.999400	8.83393	4,31738		
0.999700	8.93492	4.38720		
0.999950	9.11170	4.53702		
1.000000	9.33174	4.75347	4.71446	

TABLE 2

been answered in table 2. For a number of values of P between the limiting value and P=1 the drag has been computed for a specific case ($M_{\infty}=3$, $R_C/R_B=2.0656$).

Its behaviour is shown in fig. 3. This figure shows the remarkable fact that the drag is sharply increasing when P approaches unity, and although no theoretical justification is available, it suggests an infinite slope for the drag curve at that point. It should be observed that this figure gives all the information which can be



Fig. 3 The drag as a function of the ratio of the stagnation pressures for a certain case.

gained by the present theory for this particular case. However, it raises as many questions as it provides answers. For instance, how does this curve behave continued for smaller values of the quantity P, a case which can be solved only by allowing a discontinuity in the slope of the body contour and which would involve a far more complicated analysis. Instead of speculating about the answer to this question, it seems appropriate to give more insight into the significance of the present solution by discussing some further data. So it is interesting to know whether there is a large difference in the velocity distribution along the aftcharacteristic surface between the cases for P unequal and equal to unity. As can be seen from table 3 this difference is very small indeed, and thus the shape of this characteristic differs only slightly. Yet this small difference gives rise to a rather important change in model configuration.

Another point is the determination of the optimum shape. Since in ref. 2 certain cases for cusped-nose bodies have been studied, the attention is focussed on

TABLE 3

Optimum velocity distributions along aft-characteristics for different P values and for $M_{\infty} = 3$ and $R_c/R_B = 2.06559$

$M_{\infty} = 3$	P = 1.000000		$= 1.000000 \qquad P = 0.999155$	
r	u	v	u	v
0.206559	0.982290	0.045505	0.982195	0.045562
0.204783	0.982363	0.045736	0.982268	0.045793
0.201231	0.982511	0.046207	0.982416	0.046265
0.197679	0.982663	0.046693	0,982569	0.046751
0.194127	0.982820	0.047192	0.982726	0.047252
0.190575	0.982982	0.047707	0,982889	0.047767
0.187023	0.983149	0.048237	0.983056	0.048298
0.183471	0.983322	0.048784	0.983228	0.048846
0,179919	0.983499	0.049348	0,983407	0.049411
0.176367	0.983683	0.049931	0.983591	0.049995
0.172815	0.983873	0,050533	0.983781	0.050597
0.169263	0.984070	0.051155	0,983978	0.051221
0.165711	0.984274	0.051799	0.984182	0.051865
0.162159	0.984485	0.052465	0.984394	0.052533
0.158608	0.984704	0.053156	0.984613	0.053224
0.155056	0.984932	0.053871	0.984841	0.053941
0.151504	0.985168	0.054614	0.985078	0.054685
0.147952	0.985414	0.055385	0.985325	0.055457
0.144400	0,985671	0.056187	0.985581	0.056260
0.140848	0.985938	0.057021	0,985849	0.057095
0.137296	0.986217	0.057889	0.986129	0.057965
0.133744	0,986509	0.058795	0.986421	0.058872
0.130192	0.986815	0.059739	0,986727	0.059818
0.126640	0.987135	0.060726	0.987048	0.060806
0.123088	0.987472	0.061758	0.987385	0.061840
0.119536	0.987825	0.062839	0.987740	0.062922
0.115984	0.988198	0.063972	0.988113	0.064057
0.112432	0,988592	0.065161	0.988507	0.065248
0.108880	0.989008	0.066412	0.988924	0.066501
0.105328	0.989449	0.067729	0.989366	0.067820
0.101776	0.989917	0.069118	0.989835	0.069211
0.100000	0.990163	0.069842	0.990081	0.069936

the construction of a number of optimum configurations with finite nose angles. For the Mach numbers 2, 3 and 4 such configurations together with their slope and pressure distribution are given in figs. 4, 5 and 6. The ratio of R_C and R_B has been chosen such that bodies of a practical shape occur. Only the case for the limiting value of P has been considered. This will be evident by considering the slope along the contour, which shows a kink at the end of the conical region. In order to check the accuracy of the computation, the drag has been determined by integrating the axial force along the configuration by means of the following equation:

$$\frac{D}{\rho_{\infty} U_{\infty}^2} = \pi \int_0^l c_p r \frac{\mathrm{d}r}{\mathrm{d}x} \,\mathrm{d}x \tag{4.2}$$

The results for the case considered in the figures 4, 5 and 6 have been collected in table 4. As can be seen the values differ by a few units in the fifth significant figure which indicates that the accuracy can be qualified as very good. This is confirmed by the construction of the contour, which should pass through the point $r = R_B$ of the aft-characteristic surface and which misses this point only by a very little. The last point which has been investigated tries to give an indication whether or not an appreciable reduction in drag is obtained when using these optimum configurations. This has been done by calculating the drag for cones of the same length and the same base area. The results are presented in table 5 which shows that indeed a reasonable reduction is obtained and indicates that it might become less at higher Mach numbers.

TABLE 4

Comparison of the drag as calculated by integrating the axial force along the body contour and by integrating along the aft characteristic surface

$Drag$ as found along the aft-characteristic M_{∞} $D_0 l^2$		Drag as found by inte- grating along the fuselage $D_0 l^2$
	$\rho_{\infty} U_{\infty}^2 R_B^4$	$\underline{\varrho_{\infty} U_{\infty}^2 R_B^4}$
2	4,46678	4.46669
3	4.28331	4.28324
4	4.03187	4.03212

TABLE 5

The drag for an optimum configuration as compared to that of a conical configuration with the same length and base area

	$D_0 l^2$	$D_0 l^2$	"optimum" drag
M_{∞}	$\frac{\varrho_{\infty} U_{\infty}^2 R_B^4}{(\text{optimum})}$	$\frac{\overline{\varrho_{\infty} U_{\infty}^2 R_B^4}}{\text{(conical)}}$	referred to "conical" drag
	4.46678	5.60191	79.7%
3	4.28331	5.22698	81.9%
. 4	4.03187	4.73226	85.2%



Fig. 4a Shape and slope of an optimum configuration with a conical nose for $M_{\infty} = ?$



Fig. 4b Pressure distribution along an optimum configuration with a conical nose for $M_{\infty} = 2$.



Fig. 5a Shape and slope of an optimum configuration with a conical nose for $M_{\infty} = 3$.



Fig. 5b Pressure distribution along an optimum configuration with a conical nose for $M_{\infty} = 3$.



Fig. 6a Shape and slope of an optimum configuration with a conical nose for $M_{cc} = 4$.



Fig. 6b Pressure distribution along an optimum configuration with a conical nose for $M_{\infty} = 4$.

5 Conclusions

In this report a generalization has been given of a method previously described by the author to determine optimum shapes for minimum wave drag by applying the exact differential equations (ref. 2). The analysis leads to the construction of optimum shapes with a finite nose angle. In general the flow has to be conical along a certain part of the body, succeeded by a region which gives rise to a compression fan behind which a gradual expansion occurs.

The interesting result obtained is that the drag has its minimum value, in the case considered, for the strongest possible shock wave, that is when the strength of the compression fan becomes zero. The interdependence of the drag and the shock strength suggests that this function has an infinite slope for vanishing shock strength.

To our knowledge this is the first time that optimum shapes with continuous slopes and of practical significance have been obtained by using the exact nonlinear differential equations. The results for this particular case of noses with an infinite radius of curvature cover, however, only a small part of the problem where noses with a variable curvature are considered. Further research on the latter would lead to a family of shapes, as for instance those with blunt noses.

The question whether or not such shapes are of real practical significance will make it necessary to study a number of other problems as for instance the dependence of the drag on volume and base drag, while also the behaviour of the drag in off-design conditions will be a major problem.

6 Acknowledgement

The author should like to thank Mr. J. Wouters and Miss A. Wessel for their kind co-operation. Mr. J. Wouters prepared the flow-diagrams contained in appendix A and supervised the programming, which was executed by Miss A. Wessel.

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APPENDIX A (contributed by J. Wouters)

Flow diagrams

a) Flow diagram for the case of given values for P, M_{∞} , R_c and R_B .





b) Flow diagram for the case $u = u_s$, $v = v_s$ at $r = R_c$ and given

values for M_{∞} , R_C and R_B .

APPENDIX B

The calculation procedure for ϑ_w for a given value of P

The value of ϑ_w has to be computed from eq. (2.8), which reads by introducing the quantity $z = M_{\infty}^2 \sin^2 \vartheta_w - 1$ as follows:

$$P = \left[\frac{1+z}{1+\frac{\gamma-1}{\gamma+1}z}\right]^{\gamma/(\gamma-1)} \left[1+\frac{2\gamma}{\gamma+1}z\right]^{-1/(\gamma-1)} B(1)$$

By expanding the right-hand side of this equation into a power series, and by inverting this power series, the following result is obtained

$$z = \frac{\gamma + 1}{\sqrt[3]{-\frac{2}{3}\gamma(\gamma^2 - 1)}} a^{\frac{1}{4}} + \frac{\sqrt[3]{\frac{9}{4}\gamma(\gamma^2 - 1)}}{\gamma - 1} a^{\frac{3}{4}} + \dots$$
(B2)

where

$$a = (v - 1) \ln P$$
 B(3)

With this formula an estimate of z denoted by z_b can be found; this estimate can be made better by application of the following formula, which determines the correction Δz which has to be added to z_b to obtain a new and better value for z itself

$$a-a_b=-\frac{2\gamma(\gamma-1)}{(\gamma+1)^2}$$

$$\times \frac{z_b^2}{(1+z_b)\left(1+\frac{\gamma-1}{\gamma+1}z_b\right)\left(1+\frac{2\gamma}{\gamma+1}z_b\right)}\Delta z \qquad B(4)$$

where a_b is given by

$$a_b = \gamma \ln \frac{1+z_b}{1+\frac{\gamma-1}{\gamma+1}z_b} - \ln \left(1+\frac{2\gamma}{\gamma+1}z_b\right) \quad B(5)$$

To ensure the required accuracy it is necessary to calculate a and a_b by applying the following series

$$a = -2(\gamma - 1) \sum_{0}^{\infty} \frac{1}{2n+1} \left(\frac{\Delta P}{2 - \Delta P}\right)^{2n+1} \qquad B(6)$$

$$a_{b} = -\frac{4\gamma(\gamma-1)}{(\gamma+1)^{2}} \frac{z_{b}^{3}}{(2+z_{b})\left(2+\frac{\gamma-1}{\gamma+1}z_{b}\right)\left(2+\frac{2\gamma}{\gamma+1}z_{b}\right)} + \sum_{1}^{\infty} \frac{2\gamma}{2n+1} \left[\left(\frac{z_{b}}{(2+z_{b})}\right)^{2n+1} - \left(\frac{\frac{\gamma-1}{\gamma+1}z_{b}}{2+\frac{\gamma-1}{\gamma+1}z_{b}}\right)^{2n+1} - \frac{1}{\gamma} \left(\frac{\frac{2\gamma}{\gamma+1}z_{b}}{2+\frac{2\gamma}{\gamma+1}z_{b}}\right)^{2n+1} \right]$$

$$B(7)$$

By a few iteration steps z can be then determined with a given accuracy.

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REPORT NLR-TR M.2100

The transition of fatigue cracks in alclad sheet

by

D. Broek, P. de Rijk and P. J. Sevenhuysen

Summary

Some fatigue-crack-propagation tests with constant nett-stress amplitude on 2024 alclad sheet specimens were carried out in order to study the conditions for the transition of the crack. For different values of the stress amplitude the transition was found to occur at a constant crack rate for constant sheet thickness and testing frequency.

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List of symbols

a — constant

- c constant
- F gross cross sectional area of specimen (including the notch)
- F_n nett cross sectional area of specimen (area of unbroken part of the specimen)
- K_t theoretical stress concentration factor
- Half crack length from tip to tip (mm)
- *n* number of cycles

 $\frac{dl}{dn}$ — crack rate (mm/kc)

 $n_y - n_x$ — number of cycles to extend the crack from l = x mm to l = y mm

$$S_m$$
 — gross mean stress kg/mm²

$$S_{mn} - \text{net mean stress (kg/mm^2)}$$

$$S_{mn} = \frac{\text{Mean Load}}{F_n}$$

$$S_a - \text{gross-stress amplitude (kg/mm^2)}$$

$$S_a = \frac{\text{Load Amplitude}}{F}$$

$$S_{an} - \text{nett-stress amplitude (kg/mm^2)}$$

$$S_{an} = \frac{\text{Load Amplitude}}{F_n}$$

$$S_{max n} - S_{mn} + S_{an}$$

$$S_u - \text{ultimate tensile strength (kg/mm^2)}$$

$$S_{0.2} - 0.2\% \text{ yield stress (kg/mm^2)}$$

$$\delta - \text{elongation}$$

$$1 \text{ kc} = 1000 \text{ cycles, 1c. p.m.} = 1 \text{ cycle per minute}$$

1 Introduction

Fatigue cracks in light alloy sheet material consistently show a zone where the crack surface rotates around the growing direction as an axis (refs. 1, 2, 3). Along the first part of the crack the failure surface is perpendicular to the sheet surface; at a certain crack length the failure surface rotates until the angle with the sheet surface is about 45° , as it is for static failure in sheet (fig. 7a). In ref. 2 it was suggested that this transition takes place if the crack rate exceeds a certain value. It was the aim of the present investigation to confirm this suggestion.

According to Weibull (refs. 4, 5, 6) a constant crack rate is obtained if in a crack propagation test the nettstress amplitude is kept constant. This has to be realized by decreasing the loads at increasing crack length. For identical specimens the crack length is claimed to be a function of the nett stress amplitude only. If this is true such tests would be very suitable for the present investigation, since, depending on the stress amplitude the transition should occur either immediately (above a certain amplitude i.e. crack rate) or not at all (below that amplitude). Therefore this preliminary investigation consisted of five tests with constant nett stress amplitude. Also two conventional tests (constant gross stress amplitude) have been carried out.

In the present report the test results are given and discussed.

2 Experimental details

The specimens (fig. 7c) were cut to a size of $345 \times 160 \text{ mm}$ from 2 mm 2024 alclad ($S_u = 48.5 \text{ kg/mm}^2$; $S_{0.2} = 36.9 \text{ kg/mm}^2$; $\delta(2'') = 16\%$). A small central notch (ref. 4) initiated visible fatigue cracks after a small number of cycles. The tests were carried out in a vertical Schenck pulsator, type PVQ 002 S.

The nett stress amplitude and the nett mean stress were kept constant by decreasing the load amplitude and the mean load after every 2 mm of crack propagation. The new loads were chosen in such a way that the average nett stresses during the subsequent crack extension of 2 mm had the desired values. Adjustment of the loads occurred within 15 seconds without stopping the fatigue machine.

At the start of a test the cycling frequency was in the order of 2000 c.p.m., at the end of the test this value had decreased to ca. 1500 c.p.m. due to the decrease of the loads. This frequency is in accordance with previous tests (refs. 1, 2, 3) which allows a comparison to be made. For the same reason a mean stress was adopted in accordance with the previous tests ($S_{mn} = 8.5 \text{ kg/mm}^2$, in previous tests $S_m = 8.18 \text{ kg/mm}^2$).

Recording of the crack growth occurred at one side of the specimen only. Fine lines at 1 mm spacing were inscribed in the specimen and the use of a large magnifying glass made it possible to read the crack length to an accuracy of 0.1 mm. Recording was stopped as soon as either the loads could not be adjusted further due the limitation of the fatigue machine or the total crack length including the notch exceeded 100 mm.

3 Test results

In four tests with $S_{an} = 5.7$, 3.4, 2.9 and 2.5 kg/mm² respectively a transition was observed. In a fifth test with $S_{an}=2.0$ kg/mm² no transition occurred. (The stresses $S_{an}=5.7$, 3.4 and 2.5 kg/mm² correspond with $S_a = 5.49$, 3.27 and 2.41 kg/mm² resp. from previous tests).

Numerical test results are given in table 1 and the crack propagation curves in figs. 1 to 5 incl. The transition points have been indicated.



Microscopical examination of the failure surfaces revealed growth lines in the specimen with $S_{an} =$ 5.7 kg/mm² only. In the other tests the crack rate was too low for such an observation to be made. This is in agreement with previous experiences (ref. 2).

4 Discussion

The crack rates as calculated from the test results have been plotted in fig. 6. In this figure also the transition points are indicated. It is seen that Weibull's hypothesis of constant crack rate at constant S_{an} is not fulfilled in these tests.



Disregarding for a moment the test at $S_{an} = 5.7$ kg/mm² it can be concluded that, in accordance with the suggestion made in Ref. 2, the transition occurs if the crack rate exceeds a certain value, this value being about 0.2 mm/kc. Therefore no transition occurred in the test with the lowest stress amplitude.

At $S_{an} = 5.7$ kg/mm² the transition took place at a much higher crack rate. The crack rate here is almost immediately much higher that 0.2 mm/kc. Therefore the crack should start at an angle of 45° and a transition should not be observed at all. However, the geometrical shape of the notch forces the crack to start in a plane perpendicular to the sheet surface; the transition can only follow as soon as possible. The transition is not a discontinuity, but it occurs gradually during a crack extension of several millimeters. Since in the figures the end of the transition is indicated as the transition point, see fig. 7a, the transition at $S_{an} =$ 5.7 kg/mm² seems to take place at 4 mm from the notch, but actually it will have started together with the beginning of the crack growth.

In order to prove the above reasoning another two tests were carried out. A small artificial crack of ca. 0.5 mm length was made in the root of the notch at an angle of 45° with the sheet surface by making a cut with a fine frett saw. One of the specimens was tested at $S_a = 5.49 \text{ kg/mm}^2$ the other at $S_a = 2.41 \text{ kg/mm}^2$. For reasons of convenience the tests were performed at constant load amplitude, $S_a = \text{constant}$.

At $S_a = 5.49$ kg/mm² no transition was observed now; from the start to the end the failure surface was at 45° with the sheet surface. On the contrary at $S_a =$ 2.41 kg/mm², the crack, although forced to the start at an angle of 45°, immediately rotated to a plane perpendicular to the sheet surface and at a crack rate of 0.2 mm/kc the usual transition occurred. This is shown in fig. 7b. Frost and Dugdale (ref. 7) also carried out such a type of test. They too observed an immediate rotation of the failure surface to a plane





perpendicular to the sheet surface, thus confirming part of the transition on the crack rate.

A comparison with previous constant load-amplitude tests has been made in fig. 8. With the exception of the curves for $S_a = 5.49$ kg/mm², as previously explained, fig. 8 confirms the transition to occur at dl/dn = 0.2 mm/kc.

In refs. 1 and 2 for which the same specimens were used as in the present investigation, it was shown that the crack rate for transition depends upon the cycling frequency and increases with decreasing frequency. For 20 c.p.m. the transition occurred at $dl/dn \approx 0.4$ mm/kc. Also environmental circumstances and sheet thickness may have an influence.

In ref. 7, for 2024 sheet of 1 mm thickness at 4000 c.p.m., the transition was found to occur at dl/dn = 0.08 mm/kc, if tested in air as a medium and at dl/dn =

The suggestion of fig. 9b that for a certain sheet curves for constant Sa and constant San respectively. evident as indicated by the different slopes of the constant $K_t S_{an}$, the importance of the stress history is Although the transition in fig. 9b occurs at almost meter, neither for the crack rate nor for the transition.

The theoretical or physical meaning of this factor is not be considered more or less as an empirical factor. phenomenon, since K. as defined in ref. 9 has to tribute very much to the explanation of the transition $K_i S_{an}$ (same order of magnitude as S_u) does not conmaterial the transition occurs at a constant value of

clear.



trends of fig. 9a. for R = 0 which explains the fact they did not find the Evily and Illg checked this theory only with test results sufficient, since also the stress history plays a role. Mc. probably $K_t S_{an}$ is a better parameter although it is not function of K₁S_{max n} only. It appears from fig. 9 that Illg (ref. 9). The theory states that the crack rate is a Fig. 9a is a check of the theory of Mc. Evily and

to be a more or less fortuitous result. crack rates as observed by Weibull must be considered crack rate was not a constant. It is felt that the consant true, so it is not strange that in the present tests the independent of the crack length. This can hardly be history would be of no importance and if K_t would be equation could be expected to be valid only if the stress constant crack rate at constant San. However, this propagation: $dl/dn = CS_{an}a$. This assumption leads to a Weibull supposes for his theory on fatigue crack

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tion of fatigue cracks in 2024 alclad sheet material A preliminary study of the conditions for the transi-

> transition points in their test results, so no more data kc. Unfortunately most investigators do not indicate the the transition occurred at a crack rate of about 0.5 mm/ panels from 2024 material of 3 mm thickness. There (ref. 1) reports crack propagation tests on stiffened 0.1 mm/kc if the test was carried out in water. Schijve





governed by two parameters: that both the failure mechanism and crack rate are mechanism before and after the transition. It is felt failure in sheet. This might point to a different failure fatigue and after the transition it is similar to a static the transition the crack is more or less typical for parameters are responsible for the transition. Before actually not the crack rate, but more fundamental crack rate for the transition it must be expected that Since it turned out that many variables influence the

I The instantaneous peak stress at the tip of the crack

crack. 2 The stress history of the material at the tip of the

fig. 9a that the peak stress cannot be the only parastress amplitude respectively in fig. 9. It is seen from the maximum peak stress in a load cycle and the peak scribed in ref. 9 the crack rate has been plotted versus concentration factor K, according to the method denumber of parameters to one. Calculating the stress instantaneous peak stress are identical. This reduces the ple assumption is made that the stress history and the cycle only will be decisive. In order words the very simeffect, it is now assumed that the stress in the last approach. Although the stress history must have some The second parameter is not amenable to a simple

allows the following conclusions:

- 1 For a certain sheet thickness and certain testing conditions the transition occurs at constant crack rate independent of the stress amplitude. If forced by the geometrical shape of the notch the transition may take place at higher crack rates.
- 2 The crack rate at which the transition takes place depends upon cycling frequency, sheet thickness and environmental circumstances, and possibly also upon other testing variables.
- 3 For a certain sheet thickness the transition also occurs at a more or less constant value of the peak stress amplitude at the tip of the crack.
- 4 Some shortcomings of the theory of Weibull and the theory of Mc. Evily and Illg emerged from the present tests.

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TEST RESULTS					
1	$n_l \rightarrow n_3$ (kc)				
(mm)	$S_{an} = 5.7 \text{ kg/mm}^2$	$\overline{S_{an}} = 3.4 \text{ kg/mm}^2$	$S_{an} = 2.9 \text{ kg/mm}^2$	$S_{an} = 2.5 \text{ kg/mm}^2$	$S_{an} = 2.0 \text{ kg/mm}^2$
5	6.8	21.3	38.2	48.1	149.0
7	11.4	35.8	62.9	80.4	222.7
9	14.6	47.0	80.2	103.9	272.7
11	17.0	56.9	93.7	122.3	308.6
13	19.5	64.5	105.5	137.0	339.0
15	21.3	71.2	116.1	150.9	367.0
17	22.9	75.9	125.5	162.6	391.5
19	24.5	80.5	133.5	173.1	411.7
21	25.8	84.7	141.3	182.6	430.7
23	27.0	89.4	149.0	192.4	449.9
25	28.2	92.9	155.6	201.6	467.1
27	29.5	96.6	162.1	209.6	483.4
29	30.5	100.1	168.7	218.0	499.8
31	31.6	103.9	174.8	225.8	516.0
33	32.4	107.1	181.5	233.7	529.2
35		110.3	187.5	242.0	544.8
37			194.0	249.8	560.8
39	•		199.8	258.4	576.6
41			206.0	267.3	594.0
43			212.0	275.1	610.5
45			218.0	284.6	625.9
47			225.1	294.8	642.0
49			232.5	304.6	662.4
51			240.7	313.8	

TABLE I

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NLR-MP.218

Some experimental investigations on runway waviness¹⁾

by F. J. Plantema and J. Buhrman

Summary: A review is given of the work concerned with surface waviness of runways and taxi-tracks carried out under the auspices of the Structures and Materials Panel of AGARD. The collection of statistical data and the advantages and disadvantages of various forms of presentation of these data are discussed. Runway roughness criteria are dealt with very briefly.

After a discussion of the shortcomings of existing systems and methods for measuring the range of wavelengths from 4 ft to 200 ft the design of an NLR slope measuring system is explained in some detail. A description of a provisional realization of this system is given and the results of comparative measurements with the provisional system and with precision level-and-rod apparatus are presented. It is concluded that the NLR system shows favourable characteristics as a rapid and simple measuring system. A few current and possible future extensions of the work are briefly dealt with.

Contents

1. Introduction.

2. Collection of statistical data of runways and taxitracks.

3. Runway roughness criteria.

- 4. Existing runway or road measuring systems.
- 5. Choice of the principle of the NLR measuring system.
- 6. Provisional NLR measuring system and measuring results.
- 7. Current and possible future extensions of the work. 8. References.
 - Acknowledgements.
 - 13 figures.

1 Introduction

In this paper a brief review will be given of the work concerned with surface waviness of runways and taxitracks (runway roughness), that has been carried out under the auspices of the Structures and Materials Panel of the Advisory Group for Aeronautical Research and Development (S/M-Panel of AGARD) over a period of more than five years.

The problem was first felt in the U.S. where already in 1954 results of measurements of a few runways were published as NACA TN 3305. In this paper the frequency of occurrence of large load applications in routine ground airline operations was mentioned as the incentive to carry out the measurements. In an introductory paper for the S/M-Panel in October 1958 Dr. Houbolt of the NASA mentioned the following difficulties encountered as a consequence of runway roughness:

1.1 Structural failures of certain large aircraft carrying heavy masses on outboard regions of wings, such as engines, tanks and missiles.

¹) This report was submitted to the Structures and Materials Panel of AGARD in partial Fulfilment of a contract granted to the NLR upon recommendation of this Panel.

- 1.2 Difficulties in reading panel instruments in the cockpit.
- 1.3 Concern about the fatigue life of the aircraft structure.
- 1.4 Pilot complaints concerning taxiing behaviour, such as porpoising and a tendency to become prematurely airborne.

Factors contributing to the increased severity of the problem have been the use of outboard masses mentioned under 1.1, the use of higher pressure tires and the increased taxiing speeds.

Since the problem was considered to be mainly of importance from the point of view of aircraft loads it was included on the programme of work of the S/M-Panel. Up to now the following aspects have been studied:

- Collection of statistical data for a number of runways and taxi-tracks in various NATO countries.
- An attempt to establish criteria for runways, either newly constructed or in need of repair, based on a
- correlation of the statistical data with the operational experience on a number of runways.
- A study of systems for measuring the waviness properties of runways and taxi-tracks relatively quickly and inexpensively, followed by the design of a new measuring system and the testing of a simplified prototype system. For the work under this item the National Aero- and Astronautical Research Institute (NLR), Amsterdam was granted with a few contracts and this work has now nearly been completed.

This report is not concerned with the determination of the aircraft loads following from the runway roughness input; it is felt that this problem contains several aspects deserving further study.

2 Collection of statistical data of runways and taxi-tracks

The NASA had kindly offered to evaluate the results of

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all the measurements obtained from an AGARD cooperative programme in the same way as it had evaluated previous U.S. measurements. Consequently, the measurement technique used in the U.S. was also adopted by the other NATO countries contributing to the collection of statistical data. Use was made of the standard level-androd apparatus; a vertical rod with scale division was moved along a line parallel to the axis of the runway with a measuring interval of 2 ft, and was read by means of a horizontal surveyor's level. In this way the elevation of the runway surface was determined with respect to a horizontal reference plane.

The measuring interval of 2 ft was selected as half of the shortest wavelength one desired to include ²). This shortest wavelength L_{\min} is determined by the taxiing speed V and the highest resonant frequency f likely to be excited by the runway roughness. Taking V = 100 miles per hour (160 km/h) and f = 35 Herz it follows that $L_{\min} = 4$ ft approximately. The longest wavelength considered in the NASA evaluation of the measurements was $L_{\max} = 160$ ft (50 m approx.). The readings were made to an accuracy of 0.001 ft (0.3 mm), the last decimal being estimated. This accuracy seems to be exaggerated if it is observed that incidental surface irregularities are likely to be of the order of 0.01 ft. According to U.S. data the average speed of measuring amounted to 250 ft (75 m) per hour approximately.

The results of the measurements for 34 runways are summarized in Ref. [1]; detailed tabulated data are given in a series of AGARD Research Memoranda. Up to 1961 a total of about 60 runways and taxi-tracks in the U.S. and Europe were processed. In Ref. [1] the results are presented graphically in the form of surface profiles and power spectra.

The surface profile is useful to indicate locations where the runway is of good or bad quality. It can also be used to determine the deviations from imaginary straight edges which are commonly used as a criterion for runway construction. For example, a standard criterion is that there shall be no gap exceeding 0.1'' or 0.125'' under a straight edge of 10 ft length placed anywhere on the runway surface.

The power spectrum is generally used nowadays in the treatment of stochastic phenomena because it forms part of a modern mathematical theory of such phenomena. The theory was developed some twenty years ago and



Fig. 1. Discrete energy or power spectrum.

²) Sampling a disturbance at intervals of one-half the shortest wavelength present specifies the disturbance.

the power spectrum was first used for aeronautical applications in the U.S. about 10 years ago. Some insight into the significance of the power spectrum can be obtained in the following manner.

A periodic function y(x) of x with period L_o can be written in the form of a Fourier series

$$y(x) = \sum_{n=1}^{\infty} A_n \sin \Omega_n x + \sum_{n=0}^{\infty} B_n \cos \Omega_n x,$$

where $\Omega_n = 2\pi n/L_0$ (1)

The constants A_n and B_n can be determined by means of the standard procedures of Fourier analysis. The components with the frequency Ω_n can also be written as

 $A_n \sin \Omega_n x + B_n \cos \Omega_n x = C_n \sin (\Omega_n x + \varphi_n)$

where

$$C_n = \sqrt{A_n^2 + B_n^2}$$
and
(2)

 $\varphi_n = \arctan B_n / A_n$

The series thus consists of sines which have different phases. If y(x) is supposed to be the displacement of a vibrating system (x being the time), then C_n is the amplitude of the component having the frequency Ω_n and C_n^2 is a measure of the energy contribution due to this component. The total energy of the system is equal to the sum of the energy contributions of the components (which is not true for the amplitudes). The bar graph of Fig. 1, where $C_n^2 L_0/4\pi$ has been depicted as a function of Ω_n , is called the discrete energy, or 'power spectrum' of y(x). In fig. 1 the area of the column between Ω_n and Ω_{n+1} is equal to $\frac{1}{2}C_n^2$ and the total hatched area is equal to the average value of $y^2(x)$ over the period L_0 . The expression power spectrum is also used for other phenomena, such as runway roughness, where no real energy is involved.

If now the function y(x) is non-periodic then the limiting case $L_0 \to \infty$ must be considered. The discrete spectrum then becomes a continuous power spectrum having as abscissa $\Omega = 2\pi/L$, where Ω and L are continuous variables. The ordinate is usually called the power spectral density function and denoted as $\Phi(\Omega)$. $\Phi(\Omega) d\Omega$ now is a measure of the energy contribution of the components having frequencies between Ω and $\Omega + d\Omega$, i.e. wavelengths between $2\pi/\Omega$ and $2\pi/(\Omega + d\Omega)$. If $\Phi(\Omega)$ is finite everywhere then the energy of the component having one discrete frequency Ω is equal to zero.

An example of a power spectrum, relating to the elevation of a runway surface, is given in fig. 2. ³) This spectrum was computed for $0.0349 \leq \Omega \leq 2.094$ and thus covers wavelengths ranging from 3 to 180 ft.

If it is assumed that the mean value of the elevation has been reduced to zero, then the standard deviation, or root-meansquare (r.m.s.) value of y(x) can in principle be computed by integration of the power spectrum

$$\sigma = \sqrt{\operatorname{Ave} \cdot (y^2)} = \left[\int_{-\infty}^{\infty} \Phi(\Omega) \, d\Omega \right]^{1/2}.$$
(3)

In ref. [1] the power spectra of runway elevation are given for wavelengths ranging from 4 ft to 160 ft ($\pi/80 \le \Omega \le \pi/2$) and, in addition, the values

$$\sigma' = \left[\int_{\pi/80}^{\pi/2} \Phi(\Omega) \, d\Omega\right]^{1/2}$$

are presented. It is suggested in ref. [1] that σ is a good measure of the average roughness of a runway.

3) Figs 2 and 3 will be discussed in more detail in section 6.



Fig. 2. Power spectra of runway elevations.

It is easy to show that the magnitude of σ may give a completely wrong impression of the runway quality. This is apparent if a good runway is considered having a slope with respect to a horizontal plane. Such a runway will show a power spectrum with a pronounced peak near $\Omega = 0$ and can have a large value of σ . This peak is cut off in the calculation of σ '. However, fig. 2 indicates, and this was confirmed by a recalculation for one of the runways of ref. [1], that the magnitude of σ ' is nearly completely determined by the part of the integral for values of Ω near the lower boundary (wavelengths near the upper boundary). This means that σ ' has little to do with the shorter-wavelength com-

ponents and cannot be considered as a measure of the average roughness. Since it will strongly depend on the chosen lower limit of Ω , it will even be of dubious value as a basis of comparison for the long-wavelength roughness ⁴).

The power spectral theory has the important feature that it is possible to compute from a given input spectrum (e.g. gust or runway waviness spectrum) the output spec-

⁴) σ and σ ' will only give a good overall impression on the runway quality if they are computed for a power spectrum that is approximately a white spectrum. This may be the case for the power spectrum of the runway slopes.

trum of the aircraft response (e.g. acceleration or stress) if certain aircraft characteristics are known, and a few simplifying assumptions are approximately satisfied 5). For this reason the power spectrum was considered in ref. [1] to be the most important property of a runway. It is generally considered to yield a good overall picture of the quality of the runway. It is, however, impossible to deduce from the power spectrum the local properties of a runway, e.g. the existence and location of parts in need of repair.

Recently, even the reliability of the power spectrum as an indication of the overall quality of a runway seems to be in doubt. In ref. [2] the responses of a simulated aircraft on two runways having nearly the same power spectra were determined by means of an analog computer. It appeared that the two runways caused appreciably different aircraft responses; the response was defined as the magnitudes and number of acceleration peaks exceeding 0.5 g in absolute value. It is therefore recommended in ref. [2] to determine the quality of a runway and the location of places in need of repair by means of such analog computer studies.

The method recommended in ref. [2] has not yet been considered in the work carried out under the auspices of AGARD. Another method of presentation of the measuring results, proposed by the Canadian Panel Member A. H. Hall, has however been used. This method has certain advantages over the power spectrum but it cannot be used for load predictions. The ends of straight lines

⁵) For readers not familiar with the subject the concise and clear summary of power spectral techniques given in ref. [4] is recommended.

of various lengths l are put on the runway surface, beginning at one end of the runway, and the lines are then shifted one measuring interval each time until the other end of the runway is reached. For each position of the straight line the vertical distance d between the middle of the line and the corresponding point of the runway surface is determined (fig. 3) 6). For each length l the frequency distribution of the absolute values of the deviations d falling within consecutive intervals of 0.01 ft (i.e. 0 to 0.01 ft, 0.01 to 0.02 ft, etc.) are then computed. Fig. 3 gives an example of the results so obtained. The percentages for each interval of 0.01 ft are given in the middle of the interval (at l = 48 ft 20 % of the deviations d are between 0.02 ft and 0.03 ft). The method of presentation of fig. 3 gives more information on local properties (e.g. maximum deviations exceeding tolerable limits) than the power spectrum, although the location of bad parts of the surface does not appear from the final results.

3 Runway roughness criteria

The problem of the establishment of criteria which should be satisfied by newly constructed runways or which can be used to determine if a runway is in need of repair has been treated in various papers (a.o. refs. [3], [4] and [5]). It has been attempted to base the criteria on a correlation of the results obtained by measuring the run-

⁶) Hence, if the runway length is L_o and the measuring interval *a* then the total number of values of *d* for a straight line of length *l* is equal to (1/a) $(L_o - l) + 1$.





= results of measurements with NLR measuring system (run 2).

N.B. In the left-hand figure both curves coincide.

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way with the operational experience from the use of the runway. Proposals for criteria in the form of power spectra ($\Phi(\Omega) = \text{constant} \times \Omega^{-2}$), maximum departures from straight edges of various lengths, and maximum and r.m.s. values of the deviations d defined in fig. 3 have been made.

Serious difficulties arose, however, when the proposed AGARD criteria were submitted for consideration to the NATO Airfields Section, because they were of an entirely different form than the criteria commonly used by runway builders. The NATO criteria for runway construction specify a maximum deviation from the theoretical design profile (which consists of straight lines and transition curves with a specified minimum radius) and maximum departures from a 10 ft straight edge placed on the runway surface. The main objection against the AGARD proposals was that they were impracticable for checking a runway during construction, in particular as long as time-consuming measurements had to be made. Upon request of the Executive of the S/M-Panel the authors made an attempt to correlate the two sets of criteria. They reached the preliminary conclusion that by a few changes of the numerical values contained in the NATO criteria it would become highly probable that a runway built to these criteria would also conform to the AGARD proposals. No further action has as yet been taken, also because the S/M-Panel wished to reconsider the proposed criteria in the light of some new evidence.

4 Existing runway of road measuring systems

The measurement of runways by means of the classical level-and-rod method takes a long time, viz. about two days per 3,300 ft (1 km) with experienced personnel. This fact and the expectation that in future a periodic check of NATO runways in Europe would be necessary, gave rise to the desire of having available a means for obtaining the required data more quickly. At the end of 1959 the principal design requirements for such a measuring system were considered to be:

4.1 Measuring speed of the order of walking speed or more.

- 4.2 A range of wavelengths from 4 ft (1.2 m) to 200 ft (60 m) should be covered.
- 4.3 The system should be relatively simple, inexpensive and foolproof in operation.
- 4.4 Preferably, the system should be easily transportable by air.
- 4.5 It should be simple to evaluate the measuring data by means of a digital computer.
- 4.6 The primary final data produced should be the power spectrum of runway elevations. Later on, it was also required to obtain frequency distributions of deviations from straight lines of various lengths. The accuracy of these results should be of the same order as that of the data obtained from the classical method.
- 4.7 The surface profile of a runway need not be obtained to a great degree of accuracy but it should be reproduced 'without loss of wavelengths' in the range mentioned under 4.2.

Although a review of measuring systems given in ref. [6] had already shown that a system satisfying most of these requirements was unlikely to exist, it was considered useful to review the existing measuring systems again, in particular the European apparatus used for runways and roads, before making a design for a new system.

The conclusions of the study carried out at the NLR were that several European systems enabled a satisfactory measurement of wavelengths up to about 33 ft (10 m) but that no system existed for measuring longer wavelengths. Two of the said systems were the French 'Viagraphe' and the very similar British 'Profilometer'. The latter is shown in fig. 4. It consists of four 4-wheeled carriages and a central box with recording apparatus, which remains at a constant height above the average level of the 16 wheels. A measuring wheel can slide freely up and down in the central box and the relative displacement of box and measuring wheel is recorded on a rotating drum. The total length of the Profilometer is 22'6'' (about 7 m). From a communication by the Road Research Laboratory it was learnt that up to wavelengths of 25 ft (7.5 m) the ratio between the recorded amplitude and the actual amplitude for sine waves is approximately equal to unity; for longer wavelengths, however, this



Fig. 4. Profilometer of the Road Research Laboratory.



Fig. 5. Transfer factor of Viagraphe $A = 1 - \cos \frac{1}{2} \alpha \cos \alpha \cos 2 \alpha$, where $\alpha = 2 \pi a/L$

ratio decreases and is about 0.5 at L = 40 ft (12 m). The measuring speed of the Profilometer is about 1 ft/sec (1 km/h).

The French 'Viagraphe' and a 'Mauzin' measuring coach built by the French railways are very similar in principle to the Profilometer. The 'Viagraphe' has one row of 8 equally spaced wheels (spacing 1.43 m = 4.7 ft) and a central measuring wheel. For this system the ratio of measured amplitude to actual wave amplitude when running over a sinusoidal surface of wavelength L is given in fig. 5. It will be seen that reasonable results are obtained for wavelengths in the range from about 5.25 ft to 45 ft (1.6 m to 14 m), but that large errors occur both at shorter and longer waves.

Other measuring systems are based on the recording of the relative displacement of a mass supported by a weak spring in a running cart or of the acceleration of a wheel following the runway surface, but these systems were not considered sufficiently promising to warrant further study.

Interesting information was also obtained on a few systems under development in the U.S. A paper design of a simple cart measuring a quantity related to the slope of the runway surface had been made by the NASA (see ref. [4]). The same principle had meanwhile been adopted at the NLR (see section 5). The NASA design was not developed further and the dimensions of the proposed cart were too limited to make it satisfactory for fulfilling the requirements 4.2 and 4.6 mentioned before, but this information strongly encouraged the further evaluation of the NLR system.

Already in 1957 the Wright Air Development Center, now Aeronautical Systems Division (A.S.D.), Wright Field, had started the development of a system measuring the elevation of the runway surface with the aid of a horizontal light beam (refs. [6] and [7]). The principle is indicated in fig. 6. The system consists of two carts, a stationary one producing the light beam and a running cart carrying the recording apparatus. The 'light cannon' produces a collimated light beam of 3 inch height and 4 inch width (truncated circle) by means of a zirconium element (point source) and a special 'unique' lens. At 1,500 ft distance these dimensions have grown to 3.75 inch \times 10.5 inch. Under favourable circumstances the beam can be used up to 2,000 ft distance; the normal distance is 1,000 ft (300 m). The running cart (speed up to 5 miles per hour) carries a battery of 2×5 photocells which automatically centres itself vertically on the light beam, and a profile follower wheel running on the ground surface. The mutual distance between the battery and the wheel is recorded in digital form on a magnetic tape to an accuracy of 0.03 inch (0.75 mm); the smallest measuring interval is 6 inches (15 cm).

The ASD profilometer was not ready for practical application until 1961 and a few interesting results are included in ref. [7]. From a comparison with standard level-and-rod measurements over a distance of 300 ft (90 m) it appeared that 84 % of the profilometer measurements were within ± 0.2 inch (5 mm) of the level-and-rod data. A comparison of 10 profilometer runs over a distance of 600 ft showed that 67 % of the measured elevations reproduced within ± 0.1 inch (2.5 mm).

For the purposes of AGARD the ASD profilometer (apart from the question whether the design would be successfully completed) was considered to be too complicated, too vulnerable and much too expensive.

5 Choice of the principle of the NLR measuring system

When in the beginning of 1960 the results of the study of existing systems were available it was considered which measuring principle was the most promising for fulfilling



Fig. 6. Principle of measuring system of Aeronautical Systems Division, Wright-Patterson A. F. Base. 61



Fig. 7. Transfer factor of simple slope measuring cart.

A = amplitude ratio of slope of cart and slope of runway surface.

the requirements mentioned in section 4. It had been suggested that the best choice might be the use of an instrumented aircraft, which would have a big advantage owing to its easy transportability. This matter was therefore considered first, and discussed with various bodies where taxiing tests of instrumented aircraft had been carried out or were being planned. It then appeared that such tests were considered to be useful for special purposes, in particular for obtaining data on the transfer function of the tested aircraft or on loads on the aircraft or a similar one. For general purposes, and especially for the measurement of runway roughness properties an instrumented aircraft was unanimously considered unsuitable. The main disadvantages were formed by the following features;

- 5.1 The evaluation of the measured accelerations or strains is very difficult and uncertain owing to the non-linear properties of an ordinary landing gear. Even when a simple cantilever spring-type undercarriage was used the aircraft properties appeared to depend in an unpredictable way upon the taxi-speed and the nature of the runway.
- 5.2 The aircraft responds mainly to disturbances having a frequency equal to one of its resonance frequencies and tends to filter out all other frequencies. Hence for measuring a wide range of wavelengths a runway should be measured at a number of taxi speeds and possibly a few different aircraft would have to be used.

Finally, the advantage of easy transportability was thought to be illusory and the costs of using an instrumented aircraft high.

Disadvantage 5.2 also applies to other systems based on measuring accelerations of a spring-mass combination. A relatively simple method of direct measurement of runway elevations over the large range of wavelengths



Fig. 8. Principle of NLR slope measuring system.

from about 4 ft to 200 ft was not thought to be possible. The requirements 4.6 and 4.7 led to the investigation of the suitability of a slope-measuring system, since the first step in the calculation of the power spectrum is the determination of the differences of successive elevations (prewhitening), which are used further. This means that in essence the power spectrum of the runway slopes is computed and converted into the power spectrum of the elevations as a final step (postdarkening). It was concluded that all requirements could be satisfactorily met by the use of the slope measuring principle.

Fig. 7 presents a graph of the transfer factor A of a cart having a wheel base a, running over a sinusoidal profile. The transfer factor is defined as the ratio between the amplitude of the slope of the cart and the amplitude of the profile slope. The figure shows that a/L is an important parameter that should not exceed 0.3 to 0.4 in order to keep the measuring errors within acceptable limits. For measuring the slope of the cart a horizontal reference is required and a satisfactory solution is obtained in the form of a distant light source photographed by a camera mounted on the cart. However, the use of a stationary light source at a great distance has several disadvantages, so that the properties of the measuring system of fig. 8 were investigated. This system consists of two carts at a constant distance n.a, the one carrying the light source being towed by the measuring cart carrying the camera. Various combinations of nand a were investigated and for the required range of wavelengths (4 ft < L < 200 ft) the values n = 100 and a = 1.5 ft were selected as the most appropriate combination. The transfer factor of this system is given in fig. 9; for a/L > 0.3 the curve coincides with that given in fig. 77).

The suitability of this design was further investigated by carrying out a number of calculations concerning a paper measurement of a known runway by means of a system having n = 50 and a = 2 ft (which is less accurate). It is to be noted that the deviations d according to fig. 3 and the surface profile must be obtained by integration of the measuring results, so that in principle a cumulation of errors occurs. The results of the calculations showed, however, that both the power spectrum

7) Fig. 9 relates in fact to a = 45 cm = 1.475 ft, which was later rounded off to 1.5 ft.



Fig. 9. Transfer factor of NLR system with n = 100.

and the frequency distributions of the deviations from straight lines of various lengths were in good agreement with those obtained from the data of level-and-rod measurements. It was therefore decided to build a provisional measuring system and to carry out a number of comparative measurements with the NLR system and precision level-and-rod apparatus.

The accuracy of the slope measurement aimed at was 0.01° , which was considered to be satisfactory on the basis of the calculations carried out. By taking a number of samples of known runway measurements it was concluded that a measuring range of $\pm 3^{\circ}$ would be quite sufficient.

6 Provisional NLR measuring system and measuring results

The very simple provisional measuring system, which was

intended to evaluate the actual characteristics of the proposed system, is shown in figs 10 to 13 incl. As was already shown in fig. 8 the light source (a flash light with a cross) is photographed by a camera mounted on the small measuring cart proper. The boundary of the image window on the film is used as a reference line, the distance between this reference line and the image of the light source (a small cross) being a measure of the angle between the optical axis of the camera and the light ray. In order to obtain a light flash every 1.5 ft the circumference of the wheels of the camera cart has been made equal to 1.5 ft. A microswitch (fig. 13, foreground) is actuated once per revolution and gives an electric signal, transmitted along the towing cable to the flash light. At the same time a counter is actuated counting the number of revolutions (fig. 13, right).

The camera used was a continuous camera with variable film speed built at the NLR for other purposes.



Fig. 10. Cart carrying flash light.



Fig. 11. Camera cart dismounted from its supporting cart. .

The lens is always open, which is not objectionable if the measurements are made in the dark or with a clouded sky. However, in order to enable the measurements to be made under sunny weather conditions a butterfly shutter was added driven in a simple mechanical way. Just before the light flashes the shutter is removed laterally and leaves the lens free during a short time (fig. 13).

In fig. 11 the camera cart is shown dismounted from its supporting cart. It is mounted by means of a long pin and two adjustable spring dynamometers are used to press the camera cart to the ground with a predetermined force (see fig. 12). The wheels of the camera cart are provided with hard rubber tires (70° shore).

A series of runs with the NLR measuring system were made on a part of a runway of 3,300 ft (1 km) length at the Air Force Base De Peel. The same stretch was measured with precision level-and-rod apparatus of the 'Rijkswaterstaat'. The cart of figs. 11 and 12 was towed at a constant speed ranging from 1.6 to 6 ft/sec. (1.8 to 6.6 km/h) and the force exerted by the spring dynamometers on the camera cart was 22 lbs (10 kg) or 66 lbs

(30 kg). The cart of fig. 10 was towed by the other one by means of a cable also containing the electrical leads. and corrective steering action was taken if necessary. The measurements were made within ± 6 inches (15 cm) approximately from the centre line of the runway and the shots were taken near markings at 1.5 ft spacing on this line, which were also used for making the level and rod measurements. The records on the films (consisting of about 2,200 crosses and the reference line) were converted into a punched tape by means of a Benson-Lehner Oscar digitizer and the further calculations were carried out on the X-1 digital computer of the NLR. The level and rod data were also evaluated on the X-1 after having been punched on a digital tape.

The results of the measurements are presented in figs 2 and 3. Power spectra were computed for five runs and they showed no systematic effect of the measuring speed or the magnitude of the dynamometer force. All five power spectra showed exactly the same trends, such as the S-shaped parts at $\Omega = 0.3$ and $\Omega = 0.8$. Hence, only the scatterband of the five runs and the power spectrum of

> Eig. 13. Detail showing camera, butterfly shutter, counter and measuring wheel with microswitch.



Fig. 12. Detail of mounting of camera cart.





run 2 are presented in fig. 2 in comparison with the power spectrum obtained from the level-and-rod data. Run 2 was selected more or less arbitrarily for computing the frequency distributions of fig. 3; it was, however, avoided to select a run which might present a too optimistic comparison with the level-and-rod data.

The scatter between the five runs in fig. 2 is considered to be small for power spectra, especially if it is observed that the various runs were only approximate repetitions of each other. Up to $\Omega = 0.59$ the maximum scatter factor (maximum $\Phi(\Omega)$ over minimum $\Phi(\Omega)$) is 1.30 and for larger Ω , i.e. smaller wavelengths, the maximum scatter factor is 1.91 at $\Omega = 1.92$. The 'level and rod' power spectrum falls outside the envelope of the 5 runs only at a few places. The large discrepancy near $\Omega = 0.035$ is not quite clear, but it is known that at the upper limit of the range of wavelengths ($\Omega = 0.035$) inaccuracies due to the method of calculation may occur, which cannot be considered to be real errors of the measuring system. The most serious discrepancies occur at $\Omega = 0.38$ and $\Omega = 1.12$ where the 'level and rod' results are 14 % and 15 % less than the lower boundary of the 5 runs respectively.

In view of the foregoing it may be concluded that the results of the comparison are quite satisfactory. The same conclusion can be drawn from fig. 3 for the comparison of the frequency distributions.

7 Current and possible future extensions of the work

The work carried out at the NLR for AGARD is now being rounded off. Principal design drawings of a definite version of the measuring system have already been prepared and the preparation of detail drawings will be completed shortly. The definite version of the system has been designed such that one or two small tractors can be used to tow the measuring system. A camera suitable for use on the camera cart has recently become commercially available. A final report containing more detailed information on the design of the measuring system will be submitted in the middle of 1963.

It has been experienced that the most cumbersome part of the evaluation of the measuring data is the conversion of the film into a digital tape. If many sets of measurements would have to be evaluated in the future then it might be worthwhile to modify the recording apparatus in such a way that the end product of the measuring system is a punched tape instead of a film. Some preliminary thought was given to this problem.

Another problem that was considered provisionally is an extension of the range of wavelengths to a higher upper limit, say about 300 ft. Recent information indicates that components of more than 200 ft wavelength are gaining importance for large modern aircraft. It will be seen from fig. 9 that the transfer factor of the present set-up is satisfactory up to wavelengths of about 500 ft. However, the inclusion of longer waves will necessitate a reconsideration of the programming for the computation of the power spectra, in respect of the amount of work and the accuracy of the calculation. Another method would be to increase the wheel base a (say to 2 ft) in which case there would be less increase of the computational labour but some loss of information in the region of short waves (see fig. 7). Finally, it could be considered to adapt both a and n to the desired properties of the system. It is thought that in each particular case the best compromise should be determined in view of the importance attached to the various aspects of the problem.

In ref. [8] some ingenious measurements of the deflection across a runway at Schiphol airport are described when a dead weight of 100 tons was run along the runway at a speed of 5 km/h. These tests were considered of interest in view of the weak structure of the ground in the western part of the Netherlands. One objection that can be made to these measurements is that they give no information on the runway roughness caused by the deflections. The NLR system would be well suited to be towed along the runway, both without and with a similar dead weight or a large aircraft, and could thus give an indication of the importance of runway deflections under, the load of the aircraft itself for the problem of runway roughness.

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REPORT NLR MP. 220

Models for helicopter dynamic stability investigations

by

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Summary

A mathematical model is described, which has been devised for improving the physical understanding of helicopter dynamic instability in hovering (two degrees of freedom). A demonstration model has been built according to this principle. The influence of parameters. artificial stabilization and sling load on the dynamic characteristics is shown.

Page

A short 16 mm film is available.

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Fig. 1 to 5

Lecture given at the 23rd meeting of the AGARD Flight Mechanics Panel, Athens, July 1963.

List of symbols

See fig. 1 for positive directions.

- *a* angle between thrust and rotor shaft centreline
- a_q damping in pitch or roll=change of a per unit of angular velocity of the rotor head, including the effect of artificial rate stabilization
- $a'_q = a_q + ha_u$
- a_u speed stability = change of a per unit of linear velocity of the rotor head
- g acceleration due to gravity

- *h* distance of the rotor head above the helicopter centre of gravity
- I helicopter moment of inertia about axis through centre of gravity
- M helicopter mass
- q θ
- u horizontal velocity of helicopter centre of gravity
- θ attitude angle of fuselage
- λ linear scale of mechanical model with respect to mathematical model
- μ mass scale of mechanical model with respect to mathematical model

1 Introduction

It is known from experience, that the motion of a normal helicopter without special provisions is dynamically unstable. This is also proven by several theoretical considerations, in which it is pointed out that certain criteria are not met or unstable roots appear. Only relatively seldom a mechanical approach is used in order to improve the physical understanding of the behaviour.

The main object of this paper is to give a contribution in this direction, firstly by making use of a rather simple mathematical helicopter model. Thereafter it is explained how this model can be built at a reduced size for demonstration purposes.

2 Theoretical aspects

2.1 General

Before dealing with these models, it is necessary to

recall some important theoretical aspects of helicopter and rotor dynamics. In order not to confuse the derivations with too many details, it has been attempted to consider only essential quantities. This has resulted in the adoption of some rather drastic simplifications:

- hovering helicopter

- small deviations of the helicopter in two degrees of freedom (roll and side slip or pitch and forward speed).

Other assumptions are mentioned in the next two chapters.

2.2 The helicopter

The helicopter quantities which are important for the analysis are:

- the helicopter mass M,
- the distance h of the centre of gravity below the rotor
 head,
- the moment of inertia I about the axis through the centre of gravity. The aircraft is free to rotate about this axis.

Some additional assumptions are:

- helicopter centre of gravity on the shaft centreline
- rotor thrust equal to weight.

2.3 The rotor

An important parameter in the theory of dynamic stability is the angle a between the centreline of the shaft and the thrust. This angle is generally assumed to be linearly dependent on the angular velocity of the rotor head and its linear velocity:

$$a = a_q \dot{\theta} + a_u (u + h \dot{\theta}) \tag{1}$$

See fig. 1a. The damping in pitch a_q and the speed stability a_u depend on geometric-, mass-, and operational characteristics of the rotor blades. The sign con-



Fig. 1 Comparison between helicopter and mathematical model

vention, as indicated in Eq. (1) and fig. 1a, leads to positive values for a_q , and a_u in the normal case.

In Eq. (1), two terms are proportional to $\dot{\theta}$. Taking these together, then

$$a = a'_q \theta + a_u u, \tag{2}$$

where

$$a'_q = a_q + ha_u \tag{3}$$

As a'_q depends on h, it is strictly speaking no longer a rotor quantity. However, ha_u is generally small in comparison with a_q , so that a'_q is still mainly determined by the rotor.

The excentricity of the blade flapping hinges is assumed to be zero. This leads to zero moments from the rotor on the shaft.

2.4 Helicopter dynamics

The equations of motion for the helicopter in two degrees of freedom are, according to fig. 1a:

translation:
$$M\dot{u} = Mg(\theta - a)$$
 (4)

rotation:
$$I\hat{\theta} = -hMga$$
 (5)

After substitution of Eq. (2), these become:

$$M\dot{u} = Mg(\theta - a'_{q}\dot{\theta} - a_{u}u) \tag{6}$$

$$I\ddot{\theta} = -hMg(a'_{q}\dot{\theta} + a_{u}u) \tag{7}$$

These equations, although based on several assumptions as mentioned before, give very reasonable approximations for the type of motion, the period and the damping time of a hovering helicopter.

Dynamic models, representing the helicopter, should also obey to these equations.

2.5 Mathematical models

In order to improve the physical understanding of



Fig. 2 Mechanical model

the helicopter dynamic behaviour, it has been attempted to devise a mathematical model which is still simpler than shown in fig. 1a. This model is indicated in fig. 1b. An important feature is, that it does not longer include the rotor. On the shaft centreline, a point has been indicated at a distance a'_q/a_u above the helicopter centre of gravity. This point is, for reasons which will soon become clear, called the *damper point*. It moves with the helicopter fuselage and because of the chosen distance, its horizontal velocity components due to rotation and translation are $a'_{a}\dot{\theta}/a_{u}$ and u respectively. The ratio between these velocities is exactly equal to the ratio of the rotor forces $Mga'_q \theta$ and $Mga_u u$, these being two of the three horizontal components of the thrust. If therefore a horizontal damper is assumed to act at the damper point, this will exert forces dependent on θ and u in the right proportion. It is important to note, that the distance between the damper point and the helicopter centre of gravity is by definition constant and almost fully determined by rotor quantities (if the small correction ha_u on a_q is disregarded).

The damper point should not be confused with the so-called neutral point above a rotor, which is sometimes used in rotor theory, particularly of German origin. The neutral point is related to the moment which a rotor exerts on the shaft, for instance due to excentric flapping hinges. Such moments are not considered here.

For normal helicopters, a'_q/a_u may be of the order of 100 ft, so the damper point is considerably above the rotor.

The damper constant, being the ratio of the damper force and velocity, should be equal to Mga_u in order to produce a force of the right magnitude.

The third horizontal component $M_g\theta$ of the rotor thrust in the mathematical model is assumed to act at the helicopter centre of gravity. This guarantees the absence of a moment dependent on θ , as in fact is required by Eq. (7).

The equation for the horizontal motion of the mathematical model appears to be:

$$M\dot{u} = Mg\theta - Mga_u \cdot \left(\frac{a'q}{a_u}\dot{\theta} + u\right) \tag{8}$$

and this is equivalent to Eq. (6). In order to fulfill the moment equation (7), it is necessary to scale the moment of inertia with the factor a'_q/ha_u . The reason for this scaling can also be explained in other terms. Consider the distance a'_q/a_u , which is larger then *h*. However the damper force is equal to the two horizontal components of the rotor thrust. Therefore, the moment will be larger and in order to obtain the same angular acceleration, the moment of inertia should be scaled up. The corresponding equation is

$$\frac{a'_{q}}{ha_{u}}I\cdot\ddot{\theta}=-\frac{a'_{q}}{a_{u}}\cdot Mga_{u}\cdot\left(\frac{a'_{q}}{a_{u}}\dot{\theta}+u\right] \qquad (9)$$

which is identical to Eq. (7). The mathematical model will therefore have the same dynamic characteristics as the helicopter. The time scale of the motions is equal to one.

These considerations show, that the action of a helicopter rotor is equivalent to the combined action of a physically more understandable damper at the damper point and a force $Mg\theta$ at the centre of gravity, both forces being horizontal.

Some general results may be obtained by paying closer attention to this model. It is however prefered first to proceed to the description of the mechanical model. The discussion may be found in Ch. 4.

3 Mechanical model

In developing the mathematical model, the question did arise whether it would be possible to materialize this model and for which purposes it could be used.



Fig. 3 Diagram of mechanical model

The first question is probably best answered by refering to figs. 2, 3 and 4, showing diagrams and a picture of the mechanical model. This is similar to the mathematical model, but constructed at a length scale λ and a mass scale μ . It consists of a horizontal rail, along which a car is allowed to move. This car represents the centre of gravity of the helicopter. A bar representing the moment of inertia of the helicopter is mounted on pivots in this car. The upper end of the bar, corresponding to the damper point, is equipped with an air damper. In order to exert the force $Mg\theta$ at the centre of gravity, the car is connected to a second car on a sloping rail. Its gradient is varied by a servomechanism. This receives an input from a potentiometer in the first car, measuring the attitude angle θ of the bar with respect to the vertical. The weights of cars and bar together correspond to the helicopter weight (scale μ). As only the second car is on the sloping rail, its gradient is somewhat larger than, but proportional to, the attitude angle of the bar.

With regard to the model scales, the following re-

marks apply (fig. 2). All lengths of the mathematical model are reduced by the linear scale λ . As a'_q/a_u is large, λ must be chosen rather small in order to obtain reasonable model dimensions.

For the time scale, one must remember that g, having the dimension $[l/t^2]$, is equal for the mathematical and mechanical models. So, a time scale of l/λ must be accepted. Independent of these scales is the mass scale μ . Scales for other quantities such as I, damping constant, etc. are combinations of λ and μ .



Fig. 4 Mechanical model

The model as shown in fig. 4 has been built from a universal construction system (Swedish FAC X-2), a servo component kit (English Feedback Ltd) and some model railway elements.

4 Discussion

4.1 General

Having described the mathematical and mechanical models, it is possible to go into some more detail. The discussion will be given in terms applying to the mathematical model (fig. 1b); for the mechanical model, the scales have to be taken into account (fig. 2). The length a'_{q}/a_{u} of the mathematical model may for a moment be considered as the length of a simple pendulum. Its period of oscillation would then be $T=2\pi V/(a'_{q}/ga_{u})$. This equation was first derived by Hohenemser in 1944 by starting from the equations of motion and neglecting the helicopter moment of inertia. This approximation is however seldom appropriate. The mathematical model shows two reasons. First, the helicopter moment of inertia had to be scaled up with the factor a'_{q}/ha_{u} and the presence of a moment of inertia (compound pendulum instead of simple pendulum) leads anyhow to larger periods. Secondly, the upper point of the length a'_{q}/a_{u} is not fixed to space but attached to the damper. This also leads to an increase in period.

4.3 Helicopter instability

The action of the rotor thrust and weight on the helicopter may, according to the models, be considered as being equivalent to that of two horizontal forces (fig. 1b). The upper force, depending on a'_q and a_u , is purely damping. Energy is permanently withdrawn from the system at the damper point. The only reason for instability must therefore be sought in the force $Mg\theta$. If a'_q and a_u would be zero, then the damper would exert no moment about the centre of gravity and the angular velocity $\hat{\theta}$ would be constant. The translational motion, which is then only influenced by the force $Mg\theta$ at the centre of gravity, will have a linearly increasing acceleration which is obviously unstable. The introduction of the forementioned damping leads to a statically stable motion, but the dynamic instability remains in the form of diverging oscillations.

4.4 Artificial stabilization

In its simplest form, artificial stabilization is obtained by cyclic control inputs to the rotor which make the thrust angle *a* dependent on fuselage attitude θ :

$$a = a'_q \dot{\theta} + a_u u + a_\theta \theta \tag{10}$$

Substitution in the equation of motion, (Eq. 5), shows, that artificial stabilization leads to an extra moment on the helicopter of $-hMga_{\theta}\theta$ about the centre of gravity. In terms of the mathematical model, this means that the force $Mg\theta$ (or more precisely $Mg\theta$ $(1-a_{\theta})$), should act at a point $(a'_q/a_u)a_{\theta}$ below the centre of gravity.

It is obvious that such a force tends to decrease the angle θ , thus having a stabilizing effect on the motion.

As the moment arm is proportional to a_{θ} , larger values of this coefficient will naturally be more favourable.

4.5 Sling load

Helicopters are often used for the transportation of external loads. In several cases these may have an unfavourable influence on the flying qualities, thus restricting the operational possibilities.



Fig. 5 Sling load on mechanical model

In order to avoid such restrictions, different methods of load suspension have been devised, their common idea being to avoid moments of the load about the helicopter centre of gravity.

Some preliminary tests on the mechanical model have been made to demonstrate the influence of the sling load on the motion. Fig. 5 shows this model with sling load attached. Because the bar represents the fuselage, the load may directly be suspended from this bar.

5 Conclusions

Theoretical considerations and experiments on models, representing the dynamic characteristics of a hovering helicopter with two degrees of freedom, with and without artificial stabilization and sling load, have led to the following conclusions.

1 The damper point on the rotor shaft centreline at a distance a'_a/a_u above the helicopter centre of gravity is a concept which may be used for improving the physical understanding of helicopter instability.

2 The complicated action of rotor thrust and weight on a helicopter may for stability considerations be replaced by that from *two horizontal forces*: one at the damper point and the other at the centre of gravity, proportional to the attitude angle.

3 It has appeared possible to construct a simple *me-chanical model* on the basis of the mathematical model which can be used for demonstrating dynamic characteristics and amongst others' the influence of artificial stabilization and/or sling load on the motion.

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Analysis of a symmetrical swept-back box beam with non-swept centre part.

bу

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Summary

The box beam has all its ribs parallel to the plane of symmetry. It is analysed for 3 symmetrical and 2 antimetrical loading cases without much idealization of the structure. In 3 cases rigid supports at the corners of the centre part are present. Approximate stress-distributions are obtained by aid of the minimum theorem of the complementary energy. As an auxiliary problem also the stress-distribution of a swept part extended to both sides is calculated along the same lines. The solutions are judged by the degree of compatibility of some strains.

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This investigation has been performed under contract with the Netherlands Aircraft Development Board (N.I.V.).

Notations

E	Elasticity modulus.
v	Poisson ratio.
$G=\frac{E}{2(1+\nu)}$	Shear modulus.
х, у	coordinates in an oblique
	coordinate system.
x', y'	coordinates in an orthogonal
	coordinate system.
θ	angle between x and y axes.
h	plate thickness.
σ_x, σ_y, τ	stress components in an ob-
-	lique coordinate system,
	τ is also a shear stress (in a
	rectangular system) in rib- or
	spar webs

stress are used, one being a well-known type, the other ones designed for the purpose at hand.

For the description of the stresses in the swept parts of the structure the oblique coordinates of Hemp (ref. 2) are used.

2 Description of the structure (fig. 1)

Both outer parts of the structure have an angle of sweep of 45°, and they are connected by two non-swept cells in the middle. The structure consists of only-shearcarrying spar and rib webs (their normal stress carrying capacity is added to the booms), only-normal-stress-



Fig. 1 Dimensions of beam. Thickness webs cm (Normal) Crosssectional area booms cm².

Fig. 2 shows how a cross section of the swept part is Fig. 2 shows how a cross section of the swept part is Fig. 2 shows how a cross section of the swept part is consequently in the middle part 0.139500 cm.

in some respects idealized. To the equivalent stringer plate the spar booms contribute to an amount equal to half of the (normal) cross sectional area of a stringer so that a concentrated cross sectional area of 1.25 cm²– $\frac{1}{2} \times 1.24$ cm²=0.63 cm remains. Since, however, the spar webs are supposed to carry only shear stresses, booms (By doing so the bending-moment of inertia of an I section remains the same) and the final crosssectional area of the idealized boom becomes 1.23 cm², as is illustrated in fig. 2. Likewise, the actual crosssectional area of the spar booms in the centre part of 1.25 cm² is transformed into 1.23 cm² of the idealized boom. com² is transformed into 1.23 cm² of the idealized boom.

swept part do not enter into the calculations mentioning of these stiffnesses of the cross-section of fig. 2 (in the usual sense of elementary beam theory) is worthwhile. At an elasticity modulus E=700.000 kg/cm² and a

stressflow components. In an oblique system, s_x and s_y are called "axial stressflows", t a "tangential stressflow" t may also "shear stressflow" t may also occur in rib or spar webs.

occur in rib or spar webs. displacement with its projections.

 $\frac{\delta u_x}{\delta x} = \varepsilon_1$ strain components in coordinate system x, y. The components ε_x and ε_y are called "axial strain". In spat booms ε_x is a $\frac{\delta u_y}{\delta x} = \varepsilon_3$ $\varepsilon_3 = \varepsilon_3$ strain". In spat booms ε_x is a normal strain and in rib booms

 ϵ_y is a normal strain. Also shear strain in a rib web

shear strain in a spar web

matrix of the stress-strain relations $s_{x} = a_{11} \varepsilon_{x} + a_{12} \varepsilon_{y} + a_{13} \gamma$

 $S_{w} = d_{31} S_{w} + d_{32} S_{w} + d_{33} \gamma$ $S_{w} = d_{31} S_{w} + d_{32} S_{w} + d_{33} \gamma$ $S_{w} = d_{31} S_{w} + d_{32} S_{w} + d_{33} \gamma$ $d_{y} = d_{y}$ $d_{y} = d_{y}$ $d_{y} = d_{y}$

matrix of the stress-strain relations tions $\varepsilon_x = A_{11}s_x + A_{12}s_y + A_{13}t$ $\varepsilon_y = A_{21}s_x + A_{22}s_y + A_{23}t$ $\gamma = A_{21}s_x + A_{22}s_y + A_{33}t$ $A_{19} = A_{31}$ $A_{19} = A_{31}$ $A_{19} = A_{31}$ Complementary energy, defined

I Introduction

results in the analysis of swept wings (ref. 1). parallelogram-shaped skin panel leads to incorrect known that the concept of the only-shear-carrying cedure like the present one is necessary, since it is for a far less simplified structure is obtained. A profor a simplified structure only an approximate solution statically indeterminate. Instead of an exact solution plates, the present structure remains infinite-fold structures consisting of booms and only-shear-carrying are continuously distributed. In contradistinction to stringers in spar direction, connected with these skins the spar booms, but remains preserved. Only the stress carrying capacity of these skins is not added to skins of this beam are not idealized, i.e. the normal non-swept centre part is analysed. The upper and lower A symmetrical swept back box beam (fig. 1) with a

Py (6.1).

To obtain an approximate solution the application of the minimum theorem of the complementary energy is the most suitable. Several types of internal systems of

 $\begin{cases} s = 2y = 1\\ s = h_{D}y = h_{S}\\ T_{S} = x_{D}y = x_{S} \end{cases}$

 (n^x, n^n)

$$\frac{x\varrho}{z_{n\varrho}} + \frac{z\varrho}{z_{n\varrho}} = \Lambda$$
pute
$$\frac{\lambda\varrho}{z_{n\varrho}} + \frac{z\varrho}{\bar{\kappa}_{n\varrho}} = \Lambda$$

$$\begin{cases}
\varepsilon_3 = \left(\frac{x\varrho}{\bar{\kappa}_{n\varrho}} + \frac{\Lambda\varrho}{\bar{\kappa}_{n\varrho}}\right) = \Lambda \\
\varepsilon_3 = \frac{\Lambda\varrho}{\bar{\kappa}_{n\varrho}} = \bar{\kappa}_3 \\
\varepsilon_4 = \frac{\chi\varrho}{\bar{\kappa}_{n\varrho}} = \bar{\kappa}_3
\end{cases}$$

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 $*\Lambda$


Fig. 2 Idealization of the (normal) cross section in the swept part.

contraction coefficient $\nu = 0.3$, the bending stiffness is 28.275×10^8 kg cm² and the torsional stiffness is 14.973×10^8 kg cm². The ratio bending stiffness: torsional stiffness=1.89:1 is such as may occur in modern practice.

The cross sectional area of the rib boom at the bend is like that of the spar booms 1.25 cm^2 , that of the idealized one 1.85 cm^2 , $\frac{1}{6}$ of the cross-sectional area of the rib web at the bend being added. The (normal) crosssectional area of the other idealized rib booms is taken as 0.30 cm^2 , although only $\frac{1}{6}$ of the normal crosssectional area of the corresponding rib webs would just contribute this amount. A reduced amount for the total cross sectional area of the idealized boom is taken because the ribs are considered not to be fully effectively connected to the skin, the connection being interrupted by all the stringers.

3 Loading cases

Fig. 3 shows the loading cases to be considered. In the cases a.1 and a.2 (constant moments) only a swept part of the box beam is investigated as if there were to both sides an infinite sequence of equal cells, and thus the stress distribution in all cells is the same. Also in case a.3 (constant shear force) the box beam is infinite to both sides. In that case the stress distribution in the cells is to be separated into a part equal in all cells and a part the stresses of which are proportional to the cell number, the cells being numbered starting from the cell for which the right hand rib is loaded by the external shear force.

The other loading cases refer to the symmetrical swept back box beam discussed in section 2. In cases b.1, b.3 and c.1.2 the four corners A, B, C, D (fig. 1) of the centre part are connected to supports that can carry only vertical forces and do so without any displacements (i.e. these corners remain in one plane).

In the symmetrical swept back box beam only the stress distribution in the middle cells and the first three swept cells from the middle part is considered to be unknown. Outside this region the stress distribution is supposed to be that which is obtained from the corresponding cases of the infinite swept back box beam.

4 Oblique coordinates

4.1 Stress flows

Fig. 4 shows the coordinate directions used in the middle part and in the swept part at the right-hand side. Both coordinate systems are right-handed. A more detailed discussion of the use of oblique coordinates, as used for the swept part was given in refs. 2 and 1. The features needed here will be briefly repeated.

The states of stress in a plate, which is parallel to the xy plane is described by the oblique stress components σ_x, σ_y, τ or mostly by the stressflow components





Fig. 4 Coordinate systems used.

$$\begin{array}{c} s_x = h\sigma_x \\ s_y = h\sigma_y \\ t = h\tau \end{array}$$

$$(4.1)$$

where h = plate thickness. Fig. 5 shows an elementary parallelogram with sides dx and dy. On the right side dy of this parallelogram act forces $s_x dy$ and t dy, on the lower side dx the forces $s_y dx$ and t dx. From equilibrium considerations follows that both stressflows t are equal and further that, in the absence of mass forces, the differential equations

hold, which equations have the same form as in the case
of orthogonal coordinates. The stressflows
$$s_x$$
 and s_y will
be called axial stressflows, the stressflows t tangentia
stressflows. The corresponding quantities in the middle
part are normal stressflows s_x and s_y and shear stress
flows t.

.0 s

Also the notation

$$\begin{array}{c} 1 = s_x \\ 2 = s_y \\ 3 = t \end{array} \right\}$$

$$(4.3)$$

will be used.

The shear stressflows t in spar and rib webs in the middle as well as in the swept part are, of course, expressed in orthogonal coordinates x, z and y, z respectively.

4.2 Strains

A displacement vector u in the x, y plane is indicated by its projections u_x and u_y on the coordinate axes (fig. 6). The state of strain is indicated in the same way as in orthogonal coordinates, by the oblique strain components

 $\varepsilon_1 = \varepsilon_x$

 $\varepsilon_2 = \varepsilon_y$

 $\varepsilon_3 = \gamma$

$$\left. \begin{array}{c} \varepsilon_{x} = \partial u_{x} / \partial x \\ \varepsilon_{y} = \partial u_{y} / \partial y \\ \gamma = \partial u_{x} / \partial y + \partial u_{y} / \partial x \end{array} \right\}.$$

$$(4.4)$$

(4.5)

(4.6)

Also the notation

will be used.

$$\frac{\partial s_x}{\partial x} + \frac{\partial t}{\partial y} = 0$$

$$\frac{\partial s_y}{\partial y} + \frac{\partial t}{\partial x} = 0$$
(4.2)

4.3 Stress-strain relations

The stress-strain relations for an isotropic plate parallel to the x, y plane are* $\varepsilon_i = A_{ij}s_j, \quad i = 1, 2, 3, \quad j = 1, 2, 3$

where

$$A_{ij} = \frac{1}{Eh} \begin{vmatrix} \frac{1}{\sin\theta} & \frac{\cos^2\theta - \nu \sin^2\theta}{\sin\theta} & 2 \operatorname{ctn} \theta \\ \frac{\cos^2\theta - \nu \sin^2\theta}{\sin\theta} & \frac{1}{\sin\theta} & 2 \operatorname{ctn} \theta \\ 2 \operatorname{ctn} \theta & 2 \operatorname{ctn} \theta & \frac{2(1 + \cos^2\theta + \nu \sin^2\theta)}{\sin\theta} \end{vmatrix}$$
(4.7)

The matrix of the inverse relations

$$a_{ij}\varepsilon_j$$
 (4.8)

$$a_{ij} = \frac{Eh}{(1-\nu^2)\sin^3\theta} \begin{vmatrix} 1 & \cos^2\theta + \nu\sin^2\theta & -\cos\theta \\ \cos^2\theta + \nu\sin^2\theta & 1 & -\cos\theta \\ -\cos\theta & -\cos\theta & \frac{1+\cos^2\theta - \nu\sin^2\theta}{2} \end{vmatrix}$$
(4.9)

 $S_i =$

The matrix a_{ij} of the equivalent stringer plate is

$$a_{ij} = \begin{vmatrix} EA_s & 0 & 0 \\ a_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$
(4.10)

• The summation signs for *i* and *j* are omitted everywhere





Fig. 6 Displacement projections in a plate parallel to the xy plane of the oblique coordinate system in fig. 4.



Fig. 5 Stress flow components in a plate parallel to the xy plane of the oblique coordinate system in fig. 4.

 A_s is the (normal) cross sectional area of a stringer, a_s the stringer spacing, measured in y-direction.

To obtain the matrix a_{ij} of the combination of the isotropic plate and the equivalent stringer plate, the matrices (4.9) and (4.10) must be added. By inversion of the resulting matrix the stressflow-strain relations for the combination arise in the shape

$$\varepsilon_i = A_{ij} s_j \tag{4.11}$$

and the elements of the matrix A_{ij} are

$$A_{11} = \frac{1}{E\left(\frac{A_s}{a_s} + h\sin\theta\right)}$$

$$A_{12} = A_{21} = A_{11}(\cos^2\theta - \nu\sin^2\theta)$$

$$A_{13} = A_{31} = 2A_{11}\cos\theta$$

$$A_{22} = A_{11}\left\{1 + \frac{A_s(1+\nu)\sin\theta}{a_sh}\left(1 + \cos^2\theta - \nu\sin^2\theta\right)\right\}$$

$$A_{23} = A_{32} = 2A_{11}\left\{\frac{A_s(1+\nu)\sin\theta}{a_sh} + 1\right\}\cos\theta$$

$$A_{33} = A_{11}\left\{\frac{2A_s(1+\nu)\sin\theta}{a_sh} + 2(1 + \cos^2\theta + \nu\sin^2\theta)\right\}.$$
(4.12)

The elastic energy per unit of surface (square cm) is given by the expressions

$$U = \frac{1}{2} s_i \, \varepsilon_i / \sin \theta \tag{4.13}$$

$$U = \frac{1}{2} A_{ij} s_i s_j / \sin \theta \tag{4.14}$$

$$U = \frac{1}{2} a_{ij} \varepsilon_i \varepsilon_j / \sin \theta \tag{4.15}$$

from which only (4.14) will be used in the calculations. This expression reduces to

$$U = \frac{1}{2} A_{ij} s_i s_j \tag{4.16}$$

if the strain energy is taken per unit rhomb of the coordinate system.

5 Flexibilities of the structure

The strain energy of an element dl of a boom is (N normal force, A normal cross sectional area)

$$U = \frac{1}{2} \frac{N^2}{EA} dl.$$
 (5.1)

In the structure occur four classes of booms with different values of

$$\frac{l}{EA}$$
.

They are indicated in fig. 7a.*

Fig. 7b shows the two sorts of combinations of stringers with isotropic skin which occur. Their values $A_{ij}xy$ (A_{ij} of formulas (4.12) multiplied by the lengths of both sides of the parallelograms) are also given.

^{*} The unit of force is the kg, the unit of length the cm everywhere.



Fig. 7 Flexibilities of structural components, multiplied with 10⁵. (unit of force kg, unit of length cm)

In fig. 7c the different only-shear-carrying webs are shown, together with their values F/Gh (F=surface of one web). The strain energy of a web is

$$U = \frac{1}{2}t^2 F/Gh \,. \tag{5.2}$$

6 Application of the theorem of the minimum of the complementary energy

6.1 The theorem

The theorem of the minimum of the complementary energy can be stated in the following way (ref. 3, page 286).

Of all states of stress satisfying the conditions of equilibrium in the interior and on that portion of the surface where the surface forces are prescribed, the actual state of stress is such as to minimize the expression for the complementary energy.

$$V^* = \frac{1}{2} \int S \cdot R dv - \int_{u} k \cdot u df.$$
 (6.1)

The scalar product $S \cdot R$ is the sum of the scalar products of the stress components acting on the volume

element dv and the infinitesimal displacements according to the strains of the volume element; shorter: the scalar product of stresses and strains of volume element dv. The product $\frac{1}{2} S \cdot R dv$ is called the strain energy of the volume element dv and the first integral of (6.1), which extends over the whole volume of the body, the strain energy of the body.

Further, k are the forces (per unit area), acting at the surface f of the body and u the displacements through which these forces act. The symbol u at the integral sign means that the integral extends only over that portion of the surface where the displacements are prescribed.

It must be kept in mind, that when applying the minimum principle (6.1), in the interior of the body primarily only the stress state is varied and that the strains R in (6.1) follow these varying stresses by means of the stress-strain relations. Unless V^+ is indeed the minimum, these strains do not satisfy compatibility conditions. Thus, for a unit rhomb(of the oblique coordinate system) of the anisotropic combination of skin and equivalent stringer plate the expression (4.16) will be used.

The state of stress S, together with its external forces k, is now considered to be the sum of a number of states of stress, each with its external forces:

$$S = S_0 + X_i S_i$$

$$k = k_0 + X_i k_i .$$
(6.2)

The state of stress S_0 , together with its external load k_0 , the so-called "zero system of stress", everywhere satisfies the equilibrium conditions and where the external load is prescribed k_0 equals this prescribed load. A state of stress S_i , together with its external loads k_i also satisfies equilibrium conditions, but where the external loads are prescribed the forces k_i are zero and where the displacements are prescribed the resultant of these forces k_i (of every system S_i) must be zero. The stress systems S_i , which must be linearly independent from each other, are called "internal systems of stress"*. In an *n*-fold statically indeterminate structure *n* such systems may be constructed, and in (6.2) the unknowns X_i are the statically indeterminate quantities, which have to be determined with the aid of the minimum theorem (6.1). The unknowns X_i will be called the participation factors of the internal systems of stress.

The equations (6.2) are substituted in (6.1).

$$V^* = \frac{1}{2} \int (S_0 + X_i S_i) \cdot (R_0 + X_j R_j) dv - \int (k_0 + X_i k_i) \cdot u df$$
(6.3)

$$V^{*} = \frac{1}{2}\lambda_{00} + X_{i}\lambda_{0i} + \frac{1}{2}X_{i}X_{j}\lambda_{ij} - \int_{u}^{u} (k_{0} + X_{i}k_{i}) \cdot u \, df$$
(6.4)

• In ref. 1 the "zero system of stress" is called "basic stress system" and the "internal systems of stress" are called "supplementary stress systems". It is believed now that the nomenclature of ref. 1 is to be recommended.

where

$$\lambda_{00} = \int S_0 \cdot R_0 \, \mathrm{d}v$$

$$\lambda_{0i} = \int S_0 \cdot R_i \, \mathrm{d}v = \int S_i \cdot R_0 \, \mathrm{d}v$$

$$\lambda_{ii} = \lambda_{ii} = \int S_i \cdot R_i \, \mathrm{d}v = \int S_i \cdot R_i \, \mathrm{d}v$$

$$\lambda_{ii} = \lambda_{ii} = \int S_i \cdot R_i \, \mathrm{d}v = \int S_i \cdot R_i \, \mathrm{d}v$$

(6.5)

In order that V^* be a minimum the equations

$$\frac{\partial V^*}{\partial X_i} = \lambda_{0i} + X_j \lambda_{ij} - \int_u^j k_i \cdot u \, df = 0 \qquad (6.6)$$

must be satisfied.

The solution for the unknowns X_f is

$$X_{j} = (\lambda_{ij})^{-1} (-\lambda_{0i} + \int_{u}^{j} k_{i} \cdot u \, \mathrm{d}f) = 0 \qquad (6.7)$$

where $(\lambda_{ij})^{-1}$ is the inverse matrix of the matrix λ_{ij} .

6.2 The "zero systems of stress" for the 8 loading cases.

To the 8 loading cases described in section 3 (fig. 3) belong 8 zero systems of stress, which are shown in the figures 8 to 14. Note that the loading cases c.1.1. and



Fig. 8 Zero system of stress in one cell of swept part for loading case a.1. Forces in kg. Stressflows in kg/cm. Dimensions in cm.



Fig. 9 Zero system of stress in one cell of swept part for loading case a.2. Forces in kg. Stressflows in kg/cm. Dimensions in cm.

c.1.2 may have, of course, the same zero system of stress (fig. 14). There are reasons to choose the zero systems of stress as simple as possible, other reasons to choose them as good as possible in accordance with the expected stress distribution. The latter has been done in the present calculations. In the swept parts the stresses of the zero systems of stress are as they follow from the elementary theory for the hollow box beam . (i.e. the beam without ribs) loaded by a moment or shear force. This can, however, not be seen immediately by inspection, because the stressflows in the skins with distributed stringers are expressed in oblique coordinates and because both constant moments which are used, are each a combination of a bending and a torsional moment.

The figures 10-14 show the zero systems of stress in half of the middle part and the swept part at the right-hand side only.



Fig. 10 Zero system of stress in one cell of swept part for loading case a.3. Forces in kg. Stressflows in kg/cm. Dimensions in cm.

Fig. 11 Zero system of stress at root of swept part for loading case b.1. Forces in kg. Stressflows in kg/cm. Dimensions in cm.

Fig. 12 Zero system of stress at root of swept part for loading case b.2. Forces in kg. Stressflows in kg/cm. Dimensions in cm.



Fig. 13 Zero system of stress at root of swept part for loading case b.3. Forces in kg. Stressflows in kg/cm. Dimensions in cm.



Fig. 14 Zero system of stress at root of swept part for loading case c.1, Forces in kg. Stressflows in kg/cm. Dimensions in cm.

Fig. 15 Situation of internal systems of stress.

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FIG. 3

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6.3 The internal systems of stress

The figures 16–32 show the internal systems of stress which are to be used. Fig. 15 shows their location in the structure.

System nr. 1 (fig. 16) is a system with 4 external forces (with zero resultant) acting through the rigid support. For the other, pure internal systems of stress six types are to be distinguished. Type 1 (systems nrs. 2, 6, 12, 18, 24, figs. 17, 21, 27) is convential for unswept wings and the only one to be used if it is supposed that all plates carry only shear stresses along their edges. The systems of type 2 (nrs. 3, 7, 13, 19, 25, figs. 18, 22, 28) leave the stressflows s_x constant in y-direction, those of type 3 (nrs. 4, 8, 14, 20 and 26, figs. 19, 23, 29) allow them to vary linearly in y-direction and those of type 4 (nrs. 5, 9, 15, 21, 27, figs. 20, 24, 30) to vary parabolically in y-direction. For all the types 2, 3 and 4, the stressflows s_y remain zero and, consequently, they have the important restriction that the stress flows s_x vary linearly in x-direction (as do the normal forces in the spar booms) and the stressflows t are constant in xdirection.

Use of the afore mentioned systems only, applied to the box beam infinite in length to both sides, loaded by a moment or a shear force, leave the rib booms unstrained and as approximate solution the exact solution for the hollow box beam is obtained (In this case, where the zero systems of stress are already this solution the unknowns all become zero). Type 5 (systems nrs. 10, 16, 22, 28, figs. 21, 25, 31) and type 6 (systems nrs. 11, 17, 23, 29, figs. 22, 26, 32) contain non-zero stressflows s_y , thereby allowing the tangential stressflows t to vary in x-direction within a panel. Of course, the systems 6 to 29 incl. have their counter part in the left-hand swept part of the box beam (nr. 6*...29*), which are pure reflections with respect to the (vertical) plane of symmetry of the structure of the corresponding systems nr. 6...29.

6.4 Matrix of the coefficients of the unknowns (λ_{ij}) and of the known terms (λ_{0i}) .

Table 1 gives the values λ_{ij} , i = 1 ... 29; j = 1 ... 29, of (6.5), table 2 the values λ_{ij} , i = 6 ... 11, $j = 6^* ... 11^*$

and table 3 the values λ_{0t} of (6.5) for the loading cases of the symmetrical swept back box beam.

The values λ_{ij} and λ_{0j} were calculated according to systematic matrix procedures applied before (ref. 4), but these procedures had to be adapted to this structure in which skin elements with more than one flexibility (the coefficients A_{ij}) occur; besides, these structural elements introduce surface integrals next to line integrals. In all cases however these surface integrals were of the shape

$$\int \int f_1(x) f_2(y) \, \mathrm{d}x \, \mathrm{d}y = \int f_1(x) \, \mathrm{d}x \int f_2(y) \, \mathrm{d}y$$

or in a few cases the sum of two or three terms of this form. The values of all integrals were not determined by actual integration, but all line and surface integrals were determined with the aid of table 4, which in fact is an extension of a table given in ref. 5.

Table 5 gives a scheme of all occurring values λ_{ij} , extended for all internal systems of stress 30...35, 36...41, 30*...*35*, 36*...41*, which easily are imagined as a continuation of the systems given in fig. 15.

The matrix is divided in matrices M_{ij} , where $M_{ij} = M'_{ji}$. Each submatrix again is subdivided by horizontal and vertical lines, and as indicated, several submatrices have zero elements. The values of the other elements are to be derived from tables 1 and 2 if it is further noticed that

$$\lambda_{ij} \ (i=6...17, j=18...29) = \lambda_{ij} \ (i=18...29, j=30...41).$$

Table 5 shows also how the matrix of elements λ_{0i} is divided into sub-matrices B_i .

7 Solutions of the unknowns for the infinite swept back box beam

7.1 First loading case constant moment (Case a.1, fig. 3).

Consider the swept part at the right-hand side in fig. 15 extended to both sides. In that structure an infinite series of sets of internal systems of stress like the set consisting of the system $18 \dots 23$ is possible, giving rise to an infinite number of equations (6.6) with an infinite number of unknowns.

The shape of this set is

$$DZ_{n-4} + CZ_{n-3} + BZ_{n-2} + C'Z_{n-1} + D'Z_n = -H$$

$$DZ_{n-3} + CZ_{n-2} + BZ_{n-1} + C'Z_n + D'Z_{n+1} = -H$$

$$DZ_{n-2} + CZ_{n-1} + BZ_n + C'Z_{n+1} + D'Z_{n+2} = -H$$

$$DZ_{n-1} + CZ_n + BZ_{n+1} + C'Z_{n+2} + D'Z_{n+3} = -H$$

$$DZ_n + CZ_{n+1} + BZ_{n+2} + C'Z_{n+3} + D'Z_{n+4} = -H$$
(7.1)



in cm.



Fig. 19 Internal system of stress in the two cells of middle part. Type 3. Forces in kg. Stressflows in kg/cm. Dimensions in cm.

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Fig. 22 Internal system of stress at root of swept part. Type 2. Forces in kg. Stressflows in kg/cm. Dimensions in cm.



Fig. 23 Internal system of stress at root of swept part. Type 3. Forces in kg. Stressflows in kg/cm. Dimensions in cm.



Fig. 24 Internal system of stress at root of swept part. Type 4. Forces in kg. Stressflows in kg/cm. Dimensions in cm.







Fig. 32 Internal system of stress in one cell of swept part. Type 6. Forces in kg. Stressflows in kg/cm. Dimensions in cm.

In (7.1) each equation is a matrix equation where Z_n is a column matrix consisting of the unknowns X_{18} ... X_{23} and Z_{n+1} a column matrix consisting of the unknowns $X_{24} ldots X_{29}$ etc. The column matrices *H* together form the matrix λ_{0i} of (6.6), the elements of each matrix *H* being given by (6.5). The matrix λ_{ij} in (6.6) now obviously has the form



Fig. 31 Internal system of stress in one cell of swept part. Type 5. Forces in kg. Stressflows in kg/cm. Dimensions in cm.

in which D, C, B, C', D' are submatrices from which the numerical values of the elements can easily be taken from the matrix table 1.

Table 6 gives these numerical values. The numbering of the rows and columns corresponds with that of matrix table 1.

In the solution of the unknowns of course the participation factors of all systems of the same type are equal, so that

$$\ldots = Z_{n-1} = Z_n = Z_{n+1} = \ldots$$
 (7.3)

and therefore of all the equations (7.1) only one remains $[D+C+B+C'+D']Z_n = FZ_n = -H. \quad (7.4)$

Also the matrix F is given in table 6. The matrix H becomes

$$H = 10^{-5} \begin{vmatrix} 0 \\ 0 \\ -570590 \\ 0 \end{vmatrix}$$

and the solution is

$$Z_{n} = \begin{vmatrix} X_{18} \\ X_{19} \\ X_{20} \\ X_{21} \\ X_{22} \\ X_{23} \end{vmatrix} = \begin{vmatrix} 0 \\ +0.268085 \\ 0 \\ +0.079286 \\ +0.733871 \\ 0 \end{vmatrix}.$$
(7.5)

7.2 Second loading case constant moment (case a.2, fig.3). For this case the matrix H becomes

$$H = 10^{-5} \begin{vmatrix} 0 \\ 0 \\ 0 \\ -346516 \\ 0 \end{vmatrix}$$

and the solution is

$$Z_{n} = \begin{vmatrix} X_{18} \\ X_{19} \\ X_{20} \\ X_{21} \\ X_{22} \\ X_{23} \end{vmatrix} = \begin{vmatrix} 0 \\ +0.162806 \\ 0 \\ +0.048150 \\ +0.445676 \\ 0 \end{vmatrix}.$$
(7.6)

7.3 Constant shear force (case a.3, fig. 3).

Suppose the centre rib of the internal systems of stress nr. 18...21 is *n* rib spacings from the loaded rib. Again of the infinite number of equations only six are of importance and they are

$$DZ_{n-2} + CZ_{n-1} + BZ_n + C'Z_{n+1} + D'Z_{n+2} + nR + S = 0$$
(7.7)

where the meaning of D, C, B, C', D', Z_n is that of section 7.2, and also the column matrix nR + S consists again of the elements λ_{0t} , i=18...23. The numerical values of R and S are

$$R = 10^{-5} \begin{vmatrix} 0 \\ 0 \\ 0 \\ -76053.34 \\ 0 \end{vmatrix}, S = 10^{-5} \begin{vmatrix} 0 \\ 0 \\ -38026.67 \\ -23819.30 \end{vmatrix}.$$

Suppose Z_n is of the form

$$Z_{n+i} = (n-i)U + Y \tag{7.8}$$

where U and Y are unknown column matrices. Equation (7.8) substituted into (7.7) gives

$$D\{(n+2)U+Y\}+C\{(n+1)U+Y\}+B\{nU+Y\} + C'\{(n-1)U+Y\}+D'\{(n-2)U+Y\}+nR+S=0$$

or

$$nFU + GU + FY + nR + S = 0 \tag{7.9}$$

where F is according to (7.4) and

$$G = \{2D + C - C' - 2D'\}.$$
(7.10)
Of course (7.9) separates into

$$FU + R = 0 \tag{7.11}$$

$$GU + FY + S = 0 \tag{7.12}$$

and from the two matrix equations (7.11) and (7.12) with two unknown column matrices U and Y the solutions for U and Y are obtained.

They are

$$U = \begin{vmatrix} 0 \\ +0.0357328 \\ 0 \\ +0.0105679 \\ +0.0978169 \\ 0 \end{vmatrix}$$
$$Y = \begin{vmatrix} +0.06098531 \\ 0 \\ +0.2599130 \\ 0 \\ +0.0489085 \\ +1.100151 \end{vmatrix}$$

8 Solution of the unknowns for the symmetrical swept back box beam (loading cases b and c, fig. 3)

8.1 General

In these cases the participation factors of the internal systems of stress nr. 1...29, 6^* ...29*, (fig. 14) are the unknowns.

Both swept parts of the wing are still considered to be infinite in length, but systematic continuation of the sets of internal systems of stress, nr. $30 \dots 36$, $37 \dots 42$ etc at the right-hand side and nr. $30^* \dots 36^*$, $37^* \dots$ 42^* etc does not yield new unknowns and new equations because the participation of these systems is supposed to be as they follow from the solutions of section 7. In an earlier investigation (ref. 1) it was proved for a clamped swept box beam that this approximation is allowable for both loading cases constant moment.

The equations (6.6) for all the loading cases in discussion take the form of table 5. The division of the matrix λ_{ij} into submatrices M_{ij} and of the matrix λ_{0i} into sub-matrices B_i has already been discussed in section 6.4.

8.2 Symmetrical loading cases (cases b, fig. 3).

In these cases the participation of the antimetrical internal system of stress nr. 1 (fig. 16) is zero.

In terms of the sub matrices of table 5 the equations to be solved read

 $M_{33} V_3^* + M_{32} V_2 + M_{ab} V_3 = -M_{34} V_4^* - B_3^* \qquad (8.1)$

$$M_{23}V_3^* + M_{22}V_2 + M_{23}V_3 = -B_2 \qquad (8.2)$$

$$M^{p}{}_{b}V_{3}^{*} + M_{32}V_{2} + M_{33}V_{3} = -M_{34}V_{4} - B_{3}$$
(8.3)

in which thus column matrices V_3^* , V_2 and V_3 are unknown and the column matrices V_4^* and V_4 are known from section 7.

Since
$$V_3^* = V_3$$
 $V_4^* = V_4$ $B_3^* = B_3$

equations (8.1) and (8.3) are identical and from (8.1)... (8.3) results the matrix equation containing 28 equations with 28 unknowns

$$\begin{vmatrix} \frac{1}{2}M_{22} & M_{23} \\ M_{32} & M_{33} + M_{ab} \end{vmatrix} \begin{vmatrix} V_2 \\ V_3 \end{vmatrix} = \begin{vmatrix} -\frac{1}{2}B_2 \\ -B_3 - M_{34}V_4 \end{vmatrix} .$$
(8.4)

The solution of this matrix-equation is given in table 7.

8.3 Asymmetrical loading cases.

8.3.1 Case c.1.1, fig. 3.

In the absence of supports at the corners of the middle part, again the participation of the internal system of stress no. 1 is zero. Also the participation of the symmetrical internal systems of stress nr. 2...5 is zero. In the equations (8.1)...(8.3) are now

$$V_3^* = -V_3, \quad V_4^* = -V_4, \quad Z_2 = 0$$
 (8.5)
 $B_3^* = -B_3, \quad B_2 = 0.$

The equations reduce to the 24 equations with 24 unknowns

$$(M_{33} - M_{ab})V_3 = -B_3 - M_{34}V_4.$$
(8.6)

The solution of this matrix-equation is given in table 8.

Again the participation of the symmetrical internal systems of stress nr. 2...5 is zero, but since rigid supports at the corners of the middle part are present, internal system of stress nr. 1 plays a role. In terms of the submatrices of table 5 the equations to be solved read

$$M_{33} V_3^* - M_{31} V_1 + M_{ab} V_3 = -M_{34} V_4^* - B_3^* \quad (8.7)$$

$$-M_{13}V_3^* + M_{11}V_1 + M_{13}V_3 = -B_1 \quad (8.8)$$

$$M_{ab} V_3^* + M_{31} V_1 + M_{33} V_3 = -M_{34} V_4 - B_3 .$$
(8.9)

Equations (8.5) substituted into equations (8.7) and (8.9) gives for the latter equations the same resulting equation.

So from (8.7)... (8.9) remain 25 equations with 25 unknowns

$$\begin{vmatrix} \frac{1}{2}M_{11} & M_{13} \\ M_{31} & M_{33} - M_{ab} \end{vmatrix} \begin{vmatrix} V_1 \\ V_3 \end{vmatrix} = \begin{vmatrix} -(\frac{1}{2})B_1 \\ -B_3 - M_{34}Z_4 \end{vmatrix} .$$
(8.10)

The solution of this matrix-equation is given in table 9.

9 Determination of stresses

Figs. 33... 37 show which 168 stresses are calculated for the 5 loading cases of the symmetrical swept back box with non-swept middle part. A part of these stresses, only occurring in a swept cell, were also calculated for the loading cases constant moment or constant shear force of the swept back box beam infinite to both sides.

The stresses were composed from the zero systems of stress and the internal systems of stress by the matrix computation

$S = S_0 + X_i S_i$

where S is the column matrix of stresses in all points shown in the figs. $33 \ldots 37$, and S_0 and S_i are the matrices of stresses in these points due to the zero system of stress and the internal systems of stress respectively.

10 Discussion of results

The figures 38...77 give calculated stresses for the three loading cases of the swept back box beam infinite to both sides and the five loading cases of the symmetrical swept back box beam (fig. 3). On every figure in the upper right-hand corner a picture of the loading case in question is given. The sequence for every loading case is:

Normal forces in upper booms, stressflows s_x , stressflows s_y , stressflows t (see the small squares and rhombs, which define the stressflows) and the stressflows in the webs.

In parenthesis stresses as they follow from the "zero systems of stress" are given. These are stresses only satisfying equilibrium conditions and chosen as good as possible (they are the exact solution for the hollow box beam in the swept part). In this way clearly the influence of the use of the internal systems of stress is demonstrated.

The influence of the systems of type 1 (figs. 17, 21, 27) can be traced in figures 42, 47, 52, 57, 62, 67, 72, 77. For the swept back box beam, infinite to both sides, and loaded by a constant moment, the participation of systems of this type is zero (figs. 42, 47), which is to be seen from formulas (7.3) and (7.4). In the loading cases of figs. 57, 62, 67 there is a lowering in shear stress of some importance in the rib web at the bend and a change in spar web stresses near this bend. In fig. 77 part of the change of the shear stress 0 into -23.22 of the rib web at the bend is due to the internal system nr. 1, fig. 16, which is not of type 1.

The influence of the systems type 2, 3, 4 (type 2, figs. 18, 22, 28, type 3, figs. 19, 23, 29, type 4, figs. 20, 24, 30) is not detected separately at first sight, but their collective influence is seen by considering values of stressflow s_x in the skin-continuised stringer combination along rib booms (figs. 39, 44, 49, 54, 59, 64, 69) and by considering values of stressflows t in the skin-stringer combination midway between two ribs (the afore mentioned results are not affected by types 5 and 6). From formulas (7.3) and (7.4) it is to be seen that

participation of type 3 is zero for the swept back box beam infinite to both sides which is loaded by a constant moment, and in view of the results also the participation of the two other types in question seems not to be strong. For the loading cases of the symmetrical swept back box beam, however, in the neighbourhood of the bend, considerable differences may arise, e.g. in figs. 64, 66, 74.

The influence of types 5 and 6 (type 5, figs. 25, 31, type 6, figs. 26, 32) is revealed by stressflows s_y being present, nonlinear variation of normal forces in sparbooms between two intersections with ribs, non-linear variation in x-direction of stressflows s_x , between two ribs and the not-being constant in x-direction of stressflows t. Further, for the swept back box, loaded by a constant moment, infinite to both sides, type 6 does not participate (formulas (7.3) and (7.4)), but the presence of type 5 is very clearly demonstrated by non-zero rib boom forces being present (figs. 38, 43).

In fig. 55....62 and fig. 68....77 (loading cases of the symmetrical swept back box beam) is also indicated the figure showing the stresses at infinity. Comparison learns that in general the stresses in the third cell from the root tend, as expected, rather well to the values which they must have at infinity. Besides, as mentioned in section 8.1, it was assumed in the calculations that the stresses in the fourth and further cells would practically be so, and the participation factors of the internal stresses in these cells were taken equal to those of the swept beam infinite to both sides (that alone, of course, is not a reason why the stresses in the third cell should tend to the values of the fourth and further cells).

The investigations of ref. 6 have learned that the participation factors or the stresses in the respective cells do not tend to their values at infinity according to a simple geometrical series, but according to the sum of many convergent and sometimes oscillating series. So not always corresponding stresses change monotonously from cell to cell.

11 Degree of compatibility

The figures 78...85 show for the 8 loading cases normal strains in booms and in the adjacent skins, as derived from the stresses of figures 38...77 and the flexibilities of fig. 7. If the solutions for the stresses were exact in a point of a boom the strain in that boom and in the adjacent skin had to be the same. Only the compatibility of the strains mentioned is investigated, but of course there are numerous other tests, for example compatibility of displacements u_x along rib booms and the degree to which the compatibility equations $\partial^2 \varepsilon_x / \partial^2 y + \partial^2 \varepsilon_y / \partial x^2 = 2\partial^2 \gamma / \partial x \partial y = 0$ in a skin plate are satisfied.

In the figures 78, 79, 80 for the swept back box beam infinite to both sides the agreement is rather good. The incompatibility between the rib boom and its left

and right skins near the spar booms is not astonishing, for the limiting case of a rib boom with vanishing stiffness gives a constant state of strain in the skin (with non-zero strain ε_y) and at the end point of a rib boom the strain in the rib boom is always zero for the present idealizations).

In the figures 81 . . . 85 for the 5 loading cases of the symmetrical swept back box beam with middle part, strong incompatibilities still occur near the bend at the trailing edge. Obviously the chosen degrees of freedom are not sufficient to describe the stress singularities at this corner adequately. It is questionable if further extensions with similar types of internal systems as used will give much improvement, since in the corner in question probably infinite tangential stressflows t occur. It is to be expected that the stress-singularities in this corner have to be attacked with the aid of the differential equations in question (ref. 6, 7). Where incompatibility exists between booms and adjacent skins it is to be expected that the values of the skins are the better ones. In general the booms are rather light with respect to the skins and their strain energy is a smaller part of the total strain energy. Along the rib boom at the bend it remains however unclear which of both strains in the skins right and left of this boom will yield the best approximation. But, again, other compatibility requirements, such as the compatibilitydifferential equations are perhaps rather well fulfilled. There are other reasons to take the defects in the corners at the bend not too seriously. In the vicinity of the corner the idealizations of the structure, such as booms which can carry only normal forces are longer valid. Moreover, in practice such a corner will be constructed with the aid of local reinforcements.

12 Conclusions

It is not possible to describe the stresses near the root of the swept part and in the middle part with the aid of the elementary beam theory of torsion and bending, nor the stresses in all the rib booms.

The minimum theorem of the complementary energy with the use of 6 degrees of freedom for each cell yields a stress distribution which seems to be a good approximation in view of the degree of compatibility of the strains. However, the stress-singularities which are probably present at the trailing edge of the rib at the bend are not described adequately. It is to be expected that these singularities have to be attacked along other lines.

13 References

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Values of λ_{ij} (multiplied by 10⁵) $i = 1 \dots 29$, $j = 1 \dots 29$.

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	1	2	3	4	5	6	7	8	9	10	11	12	13		14	15
1	+30058	0	0	0	0	+ 13165	0	0	0	.0	0	- 2674.3	0		0	0
2	0	+99892	0	-33639	0	- 16290	0	- 8409.7	0	0	+4204.9	+ 2674.3	0		0	0
3	0	0	+250610	0	+136720	- 18638	-105110	- 2588.6	- 73066	+158054	0	0	+ 7321.6		0	+ 4706.8
4	• 0	-33639	0	+ 56859	0	- 8409.7	+ 2588.6	- 6875.0	+ 2218.8	0.	+4783.0	0	0	+	726.35	i 0
5	0	0	+136720	0	+ 98765	11183	- 73066	- 2218.7	- 48407	+107448	0	0	+ 4706.8		0	+ 3137.8
6	+13165	- 16290	- 18638	- 8409.7	- 11183	+204510	+ 31752	-64392	+ 22365	+ 37276	+4204.9	- 2899.0	- 24161	_	23786	11183
7	0	0	-105110	+ 2588.6	— 73066	+ 31752	+ 229800	+ 7671.2	+100330	- 68915	2588.6	+ 5523.2	- 95124	+	10259.6	- 77406
8	0	- 8409.7	2588.6	- 6875.0	- 2218.7	- 64392	+ 7671.2	+86889	0	+ 5177.2	-6475.5	- 23786	- 10260	+	4627.1	- 11424.2
9	0	0	- 73066	+ 2218.8	- 48407	+ 22365	+100330	0	+ 83469	- 38461	-2218.8	0	- 77406	+	11424.2	- 48811
10	0	0	+158054	0	+ 107448	+ 37276	- 68915	+ 5177.2	- 38461	+258726	0	0	- 14643		0	- 9413.5
11	0	+ 4204.9	0	+ 4783.0	0	+ 4204.9	- 2588.6	- 6475.5	- 2218.8	0	+8654	0	0		726.35	5 0
12	- 2674.3	+ 2674.3	0	0	0	- 2899.0	+ 5523.2	-23786	0	0	0	+208640	0		95145	0
13	0	0	+ 7321.6	0	+ 4706.8	- 24161	- 95124	-10260	- 77406	- 14643	0	0	+371770		0	+169720
14	0	0	0	+ 726.35	0	23786	+ 10259.6	+ 4627.1	+ 11424.2	0	- 726.35	- 95145	0	+:	124700	0
15	0	0	+ 4706.8	0	+ 3137.8	- 11183	- 77406	-11424.2	- 48811	- 9413.5	0	0	+169720		0	+135680
16	0	0	- 14643.0	0	- 9413.5	+ 57744	- 89935	+14954	- 40742	+ 29286	0	- 20468	+137040		9776.9	+105170
17	0	0	0	- 726.35	0	+ 11893	- 7477.1	-14525	- 7285.6	0	+ 726.35	+ 11893	+ 4888.5		3266.7	+ 5066.8
18	0	0	0	0	0	+ 5348.6	0	0	0	0	0	- 8247.6	+ 5523.2		23786	0
19	0	0	0	0	0	0	+ 45150	0	+ 29025	0	0	- 5523.2	-170780		7671.2	126040
20	0	0	0	0	0	0	0	+4479.2	0	0	0	- 23786	+ 7671.2		2878.5	+ 9205.4
21	0	0	0	0	0	0	+ 29025	0	+ 19350	0	0	0	-126040	_	9205.4	- 81235
22	0	0	. 0	0	0	0	- 90300	0	- 58050	0	0	+ 20468	+ 61379	+	9776.9	+ 56531
23	0	0	0	0	0	0	0	- 4479.2	0	0	0	+ 11893	- 4888.5	_	7019.5	- 5066.8
24	0	0	0	0	0	0	0	0	0	0	0	+ 5348.6	0		0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	+ 45150		0	+ 29025
26	0	0	0	0	0	0	0	0	0	0	0	0	0	+	4479.2	· 0
27	0	0	0	0	0	0	0	0	0	0	0	0	+ 29025		0	+ 19350
28	0	0	0	0	0	0	0	0	0	0	0	0	- 90300		0	- 58050
29	0	0	0	0	0	0	0	0	0	0	0	0	0		4479.2	0

	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	- 14643	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	- 726.35	0	0	0	0	0	0	0	0	0	0	0	0
5	- 9413.5	0	0	0	0	0	0	0	0	0	0	· 0	0	0
6	+ 57744	+11893	+ 5348.6	0	0	0	0	0	0	0	0	0	0	0
7	- 89935	- 7477.1	0	+ 45150	0	+ 29025	- 90300	0	0	0	0	0	0	0
8	+ 14954	-14525	0	0	+ 4479.2	0	0	- 4479.2	0	0	0	0	0	0
9	- 40742	- 7285.6	0	+ 29025	0	+ 19350	- 58050	0	0	0	0	0	0	0
10	+ 29286	0	0	0	0	0	0	0	- 0	0	0	. 0	0	0
11	0	+ 726.35	0	0	0	0	0	0	0	0	0	0	0	0
12	- 20468	+11893	- 8247.6	- 5523.2	- 23786	0	+ 20468	+11893	+ 5348.6	0	0	0	0	0
13	+137040	+ 4888.5	+ 5523.2	-170780	+ 7671.2	-126040	+61379	- 4888.5	0	+ 45150	0	+ 29025	- 90300	0
14	- 9776.9	- 3266.7	- 23786	- 7671.2	- 2878.5	- 9205.4	+ 9776.9	- 7019.5	0	0	+ 4479.2	0	0	- 4479.2
15	+105170	+ 5066.8	0	-126040	+ 9205.4	- 81235	+ 56531	- 5066.8	0	+ 29025	0	+ 19350	- 58050	0
16	+286445	0	0	- 90300	0	- 58050	+180600	0	0	0	0	0	0	0
17	0	+13262	0	0	- 4479.2	0	0	+ 4479.2	0	0	0	0	0	0
18	0	0	+211310	0	- 95145	0	_ 20468	+11893	- 8247.6	- 5523.2	- 23786	0	+ 20468	+11893
19	- 90300	0	0	+409600	0	+194040	+ 61379	+ 4888.5	+ 5523.2	- 170780	+ 7671.2	-126040	+ 61379	- 4888.5
20	0	- 4479.2	- 95145	0	+128450	0	- 9776.9	- 7019.5	- 23786	- 7671.2	- 2878.5	- 9205.4	+ 9776.9	- 7019.5
21	- 58050	0	0	+194040	0	+151890	+ 56531	+ 5066.8	0	-126040	+ 9205.4	- 81235	+ 56531	- 5066.8
22	+180600	0	- 20468	+ 61379	- 9776.9	+ 56531	+437765	0	. 0	- 90300	0	- 58050	+180600	0
23	0	+ 4479.2	+ 11893	+ 4888.5	- 7019.5	+ 5066.8	0	+17015	0	0	- 4479.2	0	0	+ 4479.2
24	0	0	- 8247.6	+ 5523.2	- 23786	0	0	0	+211310	0	- 95145	0	- 20468	+11893
25	0	0	- 5523.2	-170780	- 7671.2	-126040	- 90300	0	0	+409600	0	+194040	+ 61379	+ 4888.5
26	0	0	- 23786	+ 7671.2	- 2878.5	+ 9205.4	0	- 4479.2	- 95145	0	+128450	0	- 9776.9	- 7019.5
27	0	0	0	-126040	- 9205.4	- 81235	58050	0	0	+194040	0	+151890	+ 56531	+ 5066.8
28	0	0	+ 20468	+ 61379	+ 9776.9	+ 56531	+180600	0	- 20468	+ 61379	- 9776.9	+ 56531	+437765	0
29	0	0	+ 11893	- 4888.5	- 7019.5	- 5066.8	0	+ 4479.2	+ 11893	+ 4888.5	- 7019.5	+ 5066.8	0	+17015

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TABLE 2

Values λ_{ij} , $i = 6 \dots 11$, $6^* \dots 11^*$ multiplied by 10^5 .

	6*	7*	8*	9*	10*	11*
6	+5348.6	0	0	0	0	0
7	0	+45150	0	+29025	-90300	0
8	0	0	+4479.2	0	0	-4479.2
9	0	+29025	0	+19350	58050	0
10	0	-90300	0	- 58050	+180600	0
11	0	0	- 4479.2	0	0	+4479.2
-						

TA	DT	E.	2
- 1 /4	n 1	л г .	

Values λ_{0i} (multiplied by 10⁵) for the loading cases:

	b.1	b.2	b.3	c.1.1	c.1.2	-
1	0	0	0	0	+ 857955.2	 B1
2	+185714.2		+ 571665.4	0	0	
3	-133299.6	- 91119.4	+ 68503.8	0	0	
4	- 12541.0	+ 12541.0	- 102961.8	0	0	B ₂
5	0	0	+ 88765.0	0	0	
6	668929.1	+ 191415.0	-1675392	- 256968.5	-256968.5	
7	- 98573.6	- 13634.6	-198270	-134449.7	-134449.7	
8	+ 12540.9	- 12540.9	- 7075,8	+ 6270.5	+ 6270.5	
9	- 19154.4	+ 19154.4	-102202	- 55530.1	- 55530.1	
10	+ 38282.8	- 88154	+ 74484.9	- 16329.9	- 16329,9	
11	+ 6270.5	- 6270.5	+ 25730.1	+ 6270.5	+ 6270.5	<u> </u>
12	+ 92857.1	- 92857.1	+315167.5	. 0	0	
13	0	0	+ 70042.6	+ 17735.6	+ 17735.6	
14	6270.5	+ 6270.5	- 21282.7	- 6270.5	- 6270.5	
15	0	0	+ 44382.5	+ 11401.5	+ 11401.5	
16	- 570590.1	- 346516	-482325	- 606062.2	- 606062.2	
17	+ 6270.5	- 6270.5	- 2536.6	+ 6270.5	+ 6270.5	
18	0	0	0	0	0	B3
19	0	0	0	0	0	
20	0	0	0	0	0	
21	0	0	0	0	. 0	
22	- 570590.1	346516		- 570590.1	570590.1	
23	0	0	- 23819.3	0	0	
_24	. 0	0	0	0	0	
	0	0	0	0	0	
26	0	0	0	0	0	
	0	0	. 0	0	0	
28	- 570590.1	- 346516	190133.3	- 570590.1	- 570590.1	
29	00	0	- 23819.3	0	0	
	-					

B₁, B₂, B₃ matrices of table 5.

TABLE 4

· ·							
	1	0	² /3	0	⁸ /15	1/2	0
	D	1/3	0	2/15	0	1/6	0
	2/3	D	8/15	0	16/35	1/3	4/15
	10	2/15	0	8/105	0	1/15	0
	8/15	0	16/35	0	128/315	4/15	³² /105
	1/2	1/6	1/3	1/15	4/15	1/3	0
1-2	0	0	4/15	0	³² /105	0	4/5
	(f(x) =1	ţ		i 1 f(x) =- 8x ³ +	2×	 f(x) = x +½	<u>%</u> +½ ccccccc111111111111111111111111111111
- ½ +½	χ f(x)≑2x	,	-%	² _f(x) = 16 x ⁴ -	8x ² +1	-%	-2 +)
	$f(x) = -4x^2 + 1$		-			f {x} = - 12	"2 .

Values of $C = \int_{-1}^{+1} f_1(x) f_2(x) dx.$

TABLE 5

General scheme of the matrix equation (6.6) which is discussed in section 8.1.



Matrix	D	6	7	8	9	10		11
	18 -	+ 5348.6	0	0	0	0		0
	19	0	+ 45150	0	+ 29025	0		0
	20	0	0	+ 4479.2	0	0		0
	21	0	+ 29025	0	+ 19350	0		0
	22	0	- 90300	0	- 58050	0		0
	23	0	<u>`0</u>	4479.2	0	0		0
Matrix	С	12	13	14	15	16		17
	18	- 8247.6	+ 5523.2	- 23786	0	0		0
	19 ·	- 5523.2	. — 170780	— 7671,2 °	-126040	- 90300		0
	20	- 23786	+ 7671.2	— 2878.5	+ 9205.4	0		4479.2
	21	0	- 126040	- 9205.4	- 81235	- 58050		0
	22	+ 20468	+ 61379	+ 9776.9	+ 56531	+180600		0
	23	+ 11893	4888.5	- 7019.5	- 5066.8	0	+	4479.2
Matrix	B	18	19	20	21	22		23
	18	+211210	0	— 95145	0	- 20468	+	11893
	19	0	+409600	0	+ 194040	+ 61379	+	4888.5
	20	— 95145	0	+128450	0	- 9776.9	_	7019.5
	21	0	+ 194040	0	+151890	+ 56531	+	5066.8
	22	- 20468	+ 61379	— 9776.9	+ 56531	+437765		0
	23	+ 11893	+ 4888.5	- 7019.5	+ 5066.8	0	+	17015
Matrix	C'	24	25	26	27	28		29
	18	- 8247.6	- 5523.2	- 23786	0	+ 20468	+	11893
	19	+ 5523.2	170780	+ 7671.2		+ 61379	_	4888.5
	20	- 23786	— 7671.2	- 2878.5	9205.4	+ 9776.9		7019.5
	21	0		+ 9205.4	- 81235	+ 56531	_	5066.8
	22	0	- 90300	0	58050	+180600		0
	23	0	0	- 4479.2	0	0	+	4479.2
Matrix	D'	24	25	26	27	28		29
	12	+ 5348.6	0	0	0	0		0
	13	0	+ 45150	0	+ 29025	- 90300		0
	14	0	0	+ 4479.2	0	0		4479.2
	15	0	+ 29025	0	+ 19350	- 58050		0
	16	0	0	0	0	0		0
	17	0	0	0	0	0		0
Matrix	F			,				
		+205512	0	-142717	0	0	+	23786
		0	+158340	0	0	- 57842		0
		-142717	0	+131651	0	0		22997.4
		0	0	0	+ 28120	- 3038		0
		0	— 57842	0	- 3038	+798965		0
		+ 23786	0	- 22997.4	0 .	0	+	25973

TABLE	6

- - -

Matrices D, C, B, C', D' and F (multiplied by 10⁵) occurring in formula (7.2).

TABLE 7

Participation factors symmetrical loading cases

.

TABLE 8

Participation factors antimetrical loading case without rigid supports Participation factors antimetrical loading case with rigid supports

TABLE 9

loa	ading case b.1	b.2	b.3	loading case c.1.1	loading case c.1.2
2	+2 292343		+10.8020776	6206579	1 -33,132315
3	+ 3.834579	+1.447124	+ 3.5791863	7 + 1.982464	6 + 5.433665
4	+4.307852	- 3.284639	+ 19.3521490	8 -1.041746	7 + .637191
5	+4.110650	524756	+ 5.3746518	9 + .425731	8 + 4.143080
6	+ 5.851091	-4.079304	+20,3096442	10 -1.868359	9 + .163858
7	+ .697233	+1.969909	- 3.0631326	11382547	10 - 3.156394
8	+4.936828	-3.851353	+19.2625702	12 + .557996	11 - 3.971444
9	+1.208897	+.0115231	+ .7893845	13 — .0202257	12 + .959591
10		+ .591134	- 5.1070861	14 + .614178	13 + .312512
11	0854942	+ .398591	- 2.8952024	15 -1.636491	14 + 1.725572
12	+ .857906	+ .356053	+ 1.9224102	16 +3.720383	15 + .502524
13	+ .417165	259696	+ 1.1141007	17 + .878594	16 + 1.072252
14	+1.597541	184259	+ 4.6839652	18 + .369010	17 — 1.377486
15	+ .720154	- 2.498956	+ 5.8864388	19 + .176913	180587047
16	+1.033200	+3.945093	5.8242580	20 + .536691	19 + .331333
17	- ,365512	+2.857978	- 3.9241115	21 + .216437	, 200058767
18	113771	+ .642627	- 1.1255474	22 + .216421	21 + .455991
19	+ .340718	+ .0039861	+ .5558026	23 -1.390782	22 + .425514
20	074807	+ .864025	- 1.1670058	24 + .0533054	23 + 1.455721
21	+ .415339	+ .0331486	+ .9554938	25 + .285282	24 — .0639610
22	+ .551032	0423101	+ .6541282	26 + .0612044	25 + .288944
23	+1.540812	3.115509	+ 9.7719453	27 + .193275	26 — .0748243
24	072694	+ .129805	3057630	28 + .612914	27 + .0685889
25	+ .285385	+ .178887	+ .0967015	29 + .455042	28 + .865529
26	0852378	+ .152045	1664379		29 + .251026
27	+ .0536983	+ .240000	3170404		
28	+ .874022	+ .179379	+ 1.0098333		
29	+ .202830	+ .539074	+ .8004748		



Points where forces and stress flows are calculated for all 8 loading cases





AT FRONT AND REAR SPAR THE RIB BOOM FORCES ARE ZERO, THE RIB AT THE BEND EXCLUDED. Fig. 38 Normal forces in upper booms (kg).



Fig. 39 Stressflows s_x in upper skin-stringer combination (kg/cm).



AT FRONT AND REAR SPAR THE STRESSFLOWS SY ARE ZERO. IN THE APPLICATION OF THE MINIMUM THEOREM FOR THE STRESSES THE STRESSFLOWS SY ARE NOT ALLOWED TO VARY IN SPAR DIRECTION WITHIN A SKIN FIELD.

Fig. 40 Stressflows s_y in upper skin-stringer combination (kg/cm)





Fig. 41 Stress flows t in upper skin-stringer combination (kg/cm).



Fig. 42 Shearstressflows in spar and rib webs (kg/cm).



AT FRONT AND REAR SPAR THE RIB BOOM FORCES ARE ZERO, THE RIB BOOM AT THE BEND EXCLUDED.

Fig. 43 Normal forces in upper booms (kg).



























Fig. 49 Stressflows s_x in upper skin-stringer combination (kg/cm).



AT FRONT AND REAR SPAR THE STRESSFLOWS SY ARE ZERO. IN THE APPLICATION OF THE MINIMUM THEOREM FOR THE STRESSES THE STRESSFLOWS SY ARE NOT ALLOWED TO VARY IN SPAR DIRECTION WITHIN A SKIN FIELD.

Fig. 50 Stressflows s_y in upper skin-stringer combination (kg/cm).



Fig. 51 Stress flows t in upper skin-stringer combination (kg/cm).







Fig. 53 Normal forces in upper booms (kg).



Fig. 54 Stressflows s_x in upper skin-stringer combination (kg/cm).







Fig. 56 Stress flow t in upper skin-stringer combination (kg/cm).



Fig. 57 Shearstressflows in spar and rib webs. (kg/cm).



Fig. 58 Normal upper forces in booms (kg),



Fig. 59 Stressflows s_x in upper skin-stringer combination (kg/cm).



Fig. 60 Stressflows s_y in upper skin-stringer combination (kg/cm).



Fig. 61 Stressflow t in upper skin-stringer combination (kg/cm).



Fig. 62 Shearstressflows in spar and rib webs. (kg/cm).



Fig. 63 Normal froces in upper booms (kg).



Fig. 64 Stressflows s_x in upper skin-stringer combination (kg/cm).



Fig. 65 Stressflows s_y in upper skin-stringer combination (kg/cm).



Fig. 66 Stress flow t in upper skin-stringer combination (kg/cm).



Fig. 67 Shearstressflows in spar and rib webs. (kg/cm).





Fig. 73 Normal forces in upper booms (kg).



Fig. 74 Stressflows sx in upper skin-stringer combination (kg/cm).



Fig. 75 Stressflows s_y in upper skin-stringer combination (kg/cm).



Fig. 76 Stress flow t in upper skin-stringer combination (kg/cm).



Fig. 77 Shearstressflows in spar and rib webs. (kg/cm).



Fig. 68 Normal forces in upper booms (kg).



Fig. 69 Stressflows sx in upper skin-stringer combination (kg/cm).



Fig. 70 Stressflows sy in upper skin-stringer combination (kg/cm).



Fig. 71 Stress flow t in upper skin-stringer combination (kg/cm).



Fig. 72 Shearstressflows in spar and rib webs. (kg/cm).



Fig. 78 ε_x in upper spar booms and adjacent skin. ε_y in upper rib boom and adjacent skin. Multiplied by 10⁵.



Fig. 79 ε_x in upper spar booms and adjacent skin. ε_y in upper rib boom and adjacent skin. Multiplied by 10⁵.



Fig. 80 ε_x in upper spar booms and adjacent skin. ε_y in upper rib boom and adjacent skin. Multiplied by 10⁵.



Fig. 81 ε_x in upper spar booms and adjacent skin. ε_y in upper rib boom and adjacent skin. Multipled by 10⁵.



Fig. 82 e_x in upper spar booms and adjacent skin. e_y in upper rib boom and adjacent skin. Multiplied by 10⁵.







Fig. 84 ε_x in upper spar booms and adjacent skin. ε_y in upper rib boom and adjacent skin. Multiplied by 10⁵.



Fig. 85 ε_x in upper spar booms and adjacent skin. ε_y in upper rib boom and adjacent skin. Multiplied by 10⁵.

REPORT NLR-TR S. 597

Results of strain measurements performed on a polystyrene swept back box beam with a non swept-centre part, and their comparison with theoretical results

by

R. M. Lichtveld and P. J. Sevenhuysen

Summary

Strain gauge measurements were made on a polystyrene swept-back box beam. Three symmetrical and one antisymmetrical loading case were investigated. In order to minimize the influence of visco-elastic behaviour use was made of a constant-displacement method in loading the test-specimen.

The results are compared with theoretical results derived previously by means of the principle of minimum complementary energy. Good agreement between test and theoretical results is found. For 2 of the 4 loading cases investigated the comparison showed some differences in the region of high stress concentration. Indications could be found how the theoretical results might still be improved.

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Q — Force (see fig. 11)	
R — Electrical resistance (Ohms)	
10 tables s — Direct stress flow related to carte	sian co-
55 figures ordinates (kg/cm)	

- Direct stress flow related to oblique co-Ī ordinates (see ref. 1) (kg/cm) T- Torsional moment (kg cm) - Shear stress flow related to cartesian cot ordinates (kg/cm) Ī - Shear stress flow-related to oblique coordinates (ref. 1) (kg/cm) u - Displacement or deflection x and y =co-ordinates - Shear strain γ_{xy} - Half of the thickness of the boom of the Δ box beam (see fig. 5) - Direct strain (cm/cm) ε θ - Sweep angle of swept back box beam - Poisson's ratio v - Rotation of cross-section around the elastic Ø axis of a beam (fig. 23)

Indices

m	refers to measured strains
с	- refers to the measured strains corrected for
	strain gauge stiffness
x .	- refers to x-direction
у	refers to y-direction
45°	- réfers to a direction making an angle of 45°
	degrees with the x-axis

1 Introduction

In refs. 2 and 3 BENTHEM discusses a method for the analysis of swept wing structures with discrete ribs in flight direction. The analysis allows for direct stresses in the skins; thus the commonly used assumption of only shear-carrying skins has been dropped. The method of analysis is based on the principle of minimum complementary energy.

In ref. 1 this method is applied in the analysis of a symmetrical swept-back box beam with a non-swept centre part. 28 internal systems of stress of 6 different types are used in these computations.

In order to check the theoretical results of ref. 1 strain measurements were made on a plastic scale model of the box beam analysed in ref. 1. The results and a comparison of the results with the computed values of ref. 1 are reported in this paper.

2 Test programme

In conformity with ref. 1 the following four loading cases were applied to the model box-beam (see fig. 1):

case B 1	- symmetrical torsional moments
case B 2*	- symmetrical bending moments
case B 3	symmetrical downward shear loads
case C 1.2*	- asymmetrical torsional moments



The values of the loads given in this figure refer to the computations of ref. $\mathbf{1}$



The magnitude of the loads on the model was chosen so that the maximum measured strains were about 1200 μ strain. The test results were afterwards corrected for the difference in loads applied to the model and those used in ref. 1, as well as for the differences in scale.

The loads were applied in the centre of the end ribs of the model. To obtain a good load diffusion the endribs consisted of 10 mm thick polystyrene plates stiffened with dural stringers.

To reduce the creep effects associated with the use of plastics for structural models the method of application of constant deflections was employed. The deflections were applied at the tips, while the forces necessary to apply these deflections were measured. In order to retain symmetry the deflections of both tips were applied in such a way that the loads at both tips were equal at the moment directly after application of the deflections. The application of the deflections could be done in a very short time (< 1 sec.).

The strains in the model were measured on the upper skin and booms and on the rib and spar webs of the starboard half of the model. On the port half a few strain gauges were applied in order to gain some information about the symmetry of the model and the loads.

[•] For the loading cases B 2 and C 1.2 the direction of the load-vectors in the tests was opposite to that used in the calculations of ref. 1. In this report all numerical values refer to the loads applied in the tests.

In total 141 strain gauge bridges were used with 262 active gauges (the gauges at both sides of skin and . webs were combined in one Wheatstone bridge). The type of the gauges was Hottinger Impa 10/120 FA 1. In fig. 2 the locations of the strain gauges are shown.



Fig. 2 Location of the strain gauges.

Deflections of the model were measured at both wing tips at the front and rear spars.

A special test was performed to compare the creep effects that appear in the case of application of constant loads and in the case of application of constant displacements.

3 The test set-up

3.1 The model

The layout and the cross-section of the box beam of ref. 1 are shown in figs. 3 and 4. The layout and the



Fig. 3 The lay-out dimensions of the box beam of ref. 1 and those of the test model.

The dimensions of the test model are given in parentheses.

cross-section of the test-model are shown in figs. 3 and 5. The length-scale of the test model was 1:2. The thickness scale of the stress carrying members was 1:1, so that the scale for the cross-sectional area of these members was 1:2, and the scale for the cross





Fig. 4 Cross-section of the box beam of ref. 1.



Fig. 5 Cross-section of the model box beam. Dimensions in cm

sectional moment of inertia was 1:8 (the moments of inertia of the skin, stringers and spar booms with respect to their centroidal axes are neglected; see table 1).

The test-model was built up from polystyrene plates, stringers and booms* The stringers and booms were milled out of plates of 6 and 10 mm thickness. All plates used for the skin and the webs were milled to thickness. The deviation of the mean thickness of each plate with respect to the nominal thickness amounted



Fig. 6 The situation of the plates which form the upper skin of the model.

^{*} This material was supplied in the form of plates with thicknesses of 1, 2, 6 and 10 mm \pm 10%, trade mark "Trolitul". The manufacturer was Dynamit Aktien Gesellschaft, vormals Alfred Nobel and Co, Troisdorf Bez. Köhn.

after milling to ± 0.02 mm. In addition a tolerance of ± 0.015 mm with respect to the mean thickness was allowed. The situation of the various plates in the upper and lower skins of the model is shown in the figs. 6 and 7.



Fig. 7 The situation of the plates which form the lower skin of the model.

All bonded joints in the structure were made with a solution of polystyrene in carbitol-acetate. Fig. 8 shows a photograph of the model under construction.



Fig. 8 The model under construction.

The strain gauges at the inner side of the box beam are already fitted. The polystyrene strips carrying the dummies are seen to be placed in the direct neighbourhood of the active gauges. All lead wires were led along the stiffeners to the tip ribs (see left part of the photograph).

In table 1 the bending and torsional stiffnesses of the actual test-model, measured after milling and assembling, are compared with those of the ideal test-model. The actual bending stiffness is 3.06% too high and the actual torsional stiffness is 1.43% too high.

Reasons to choose polystyrene as the material to build the model were

- The mechanical properties of polystyrene show little sensitivity to changes in the relative humidity of the ambient air.
- Poisson's ratio for polystyrene shows very good agreement with that for aluminium.

- The creep properties are not more unfavourable than those of other plastics in use for model testing.

A disadvantage presented itself when during the building of the model unexpected difficulties arose with the bonded joints between the stiffeners and the skin. It appeared not possible to obtain satisfactory joints with benzol when large parts had to be bonded. After an investigation into the properties of some other solvents and cements it was found that carbitol acetate gave satisfactory results without the necessity to use complicated bonding procedures. In order to obtain a cement of higher viscosity some polystyrene was dissolved in this solvent.

The mechanical properties of polystyrene given in the literature are $E = 0.320 \cdot 10^5$ to $0.349 \cdot 10^5$ kg/cm², and $\nu = 0.31$. A large number of strain gauge measurements on polystyrene test-specimens gave $E = 0.336 \cdot 10^5$ kg/cm² (ref. 4). The latter value is used in this report.

3.2 The test-rig

Fig. 9 gives a general impression of the test rig with



Fig. 9a The test rig with the model.

the model. Fig. 10 shows the supports of the model. With these supports the boundary conditions, applied in ref. 1, are satisfied.

The application of the deflections to the model was done by means of a cable- and lever-system. Fig. 11 shows schematically the method of application of the torsional moments. The bending moment was applied in a similar manner. The use of a loading lever made application of the deflections possible within a very short time. The application of deflection increments was performed by unloading, adjusting the pin-lever



Fig. 9b The scanning apparatus with the recorder and the calibrating apparatus.



Fig. 10 The supports of the model.

attachment and retightening the loading lever. The time necessary for applying a deflection increment was about 5 seconds. The cable spanner was used as a means to adjust the deflections of one of the end ribs of the model in order to equalize the measured loads at both tips. This was done immediately after each increment of the deflection. The loads following from the applied deflections were measured by means of strain gauge dynamometers (measuring range 150 kg, linearity better than 1%) connected to a Peekel static strain indicator. The sensitivities of the dynamometers were determined from the calibration curves shown in fig. 12. This calibration was done before the tests were started. The highest load measured during the tests was 68 kg (loading case C 1.2).



Fig. 11 The cable and lever system for the application of the constant deflections.



Fig. 12 Calibration curves for the dynamometers.

3.3 The strain and deflection measurements

The positions of the strain gauges on the model are shown in fig. 2. At all rosette stations on the skin, spar webs and rib webs strain gauges were cemented at both sides of the plate. This was done in order to eliminate the effects of plate-bending due to the crushing effect and initial imperfections. The type of strain gauges used was Hottinger Impa 10/120 FA 1, $R = 120 \Omega$, k = 2, helical grid in plastic base, grid dimensions 2×10 mm. The dimensions of the gauges were cut down to 6×20 mm in order to reduce the stiffness. The gauges were bonded with Eastman 910 cement. Tests showed this strain gauge-cement combination to be usable up to strains of about 2000 μ -strain (2 $\cdot 10^{-3}$ cm/cm). The loading of the model box beam was therefore chosen at such a level that the highest strained gauges showed maximum strains of 1000 to 1200 μ -strain.



Fig. 13 Measuring circuit of strain gauges on skin and web panels.

The strain gauge measuring circuits were connected as follows. Each active gauge was provided with its own dummy gauge. The strain gauges at both sides of the skin and webs were combined to form, together with their two dummies, complete Wheatstone bridges (fig. 13). The advantages of such a combination are:

- elimination of bending effects without having the disadvantage of doubling the number of strain gauge bridges and measuring data.
- 2. doubling of the voltage output.

The strain gauges on the spar and rib booms, provided with their own dummy gauges, were connected to the reference resistances built into the scanning apparatus (fig. 14), to form a complete Wheatstone bridge. The influence of the lead wires from the half strain gauge



Fig. 14 Measuring circuit of strain gauges on spar and rib booms.

bridges to the scanner was negligible because these wires were short (< 5 m).

All the dummy strain gauges were cemented on

strips of polystyrene (six dummies per strip) and placed in the direct neighbourhood of the active gauges. The dummies belonging to the active gauges on the inner sides of the skin and webs were consequently placed inside the box-beam (see fig. 8).

Because polystyrene is a bad conductor for the heat developed in the strain gauges special precautions have to be taken to keep the zero drift of the balanced strain gauge bridges within acceptable limits.

As causes of zero drift the following differences, between active and dummy strain gauges can be mentioned:

- (1) difference in temperature coefficient.
- (2) difference in temperature and temperature history.
- (3) difference in gauge factor and the coefficient of expansion of the grid material.
- (4) differences in chemical composition and physical properties due to manufacturing.

The following precautions were taken against excessive zero drift:

- (1) a constant bridge supply of low voltage (1 V) was used.
- (2) this bridge supply was applied during at least two hours before starting any tests in order to reach a temperature equilibrium.
- (3) the active and dummy strain gauges were shielded with cotton wool against draught.

The strain gauge bridges were connected to a scanner with a capacity of 203 bridges (7 groups of 29, see Fig. 9b).

This semi-automatic scanner S.A.R.A. is provided with 203 pairs of reference resistances for the use of half strain gauge bridges, and with bridge balance facilities for each bridge. Provisions were also present to connect the strain gauge bridges in parallel in groups of 29, each of the seven groups having its own accumulator. With the scanner the strain gauge bridges were connected automatically or by hand to a millivolt recorder. A diagram of the complete electrical circuit is given in fig. 15.



Fig. 15 The electrical circuit of strain gauges and scanner.

The mV-recorder was a Philips Type PR 2210 A/21 Nr. D 3445. The response time (1 sec. full scale) of this recorder determined the scanning time. The total of
156 strain gauge bridges were scanned in about 8 minutes.

Calibration of the recorder deflections in μ -strain was done by shunting a resistance of known value over one of the arms of a calibration bridge, thus producing a recorder deflection corresponding to a fictitious strain of known value. For each of the seven groups of strain gauge bridges a calibration bridge was available. For this calibration use was made of an apparatus specially developed for the purpose of calibrating strain gauge apparatus (ref. 5). A circuit scheme of this calibration apparatus is shown in fig. 16. Fig. 9b shows the scanning apparatus with the recorder and the calibrating apparatus.



Fig. 16 Circuitry of calibration apparatus.

Deflection measurements were made with the use of dial gauges. With three gauges at each wingtip the bending deflections in two directions and the torsion deflection were measured. Fig. 17 shows the position of these dial gauges.



Fig. 17 The position of the dial gauges at the wing tips.

4 Method of testing and evaluation of test results

As has been mentioned in section 2 there are 4 loading cases. The test procedure, which was the same for each loading case, went as follows. After the box beam had been pre-loaded by prescribing small tip deflections (about 2% of the maximum load level) at each tip, zero adjustment of the strain gauge bridges was performed.

To take advantage of the full scale of the recorder (11'') all gauges were balanced at approximately 5% of the scale. Hence, gauge circuits showing a negative output had to be connected reversely to the supply connections. The sensitivity of the recorder was now adjusted in such a way as to give nearly full deflection

at the highest expected bridge-output. After being balanced the bridges were scanned; also the dial gauges at the tips were read and dynamometer readings were taken. These observations were used as a zero reference.

The maximum deflection was then applied in 5 steps. After application of each deflection step the cable spanner of the left wingtip was adjusted so as to equalize the applied loads at both wingtips, and the following observations were made:

the output of the strain gauge bridges was recorded.
 the dial gauges were read.

(3) at the same time a rough plot was made of the dynamometer readings as a function of time at the prescribed deflection.

The time necessary to perform such a series of observations was about 10 minutes.

The plot of dynamometer readings shows a decrease of load with time, due to relaxation of the material (Fig. 18).



Fig. 18 The decrease of the load due to relaxation.

The load curve for a complete test thus had the form shown in fig. 19. Since the relaxation effects were small the loads, $P_1, P_2, P_3 \ldots$, measured directly after application of a new deflection step were assumed to be equal to the loads $P_1', P_2', P_3' \ldots$. The resulting error per deflection step is smaller than 1%, which results in an error of the order of $1\frac{1}{2}$ % in the final test results. Because of the presence of an elastic aftereffect after unloading the box beam measurements could only be made during increasing load.



Fig. 19 The effect of relaxation on the load curve.

In table 2 the magnitudes of the applied maximum loads and deflections are given.

The evaluation of the test results as described in the following was carried out for the greater part on an electronic digital computer. For each strain gauge bridge the ratio $d\epsilon/dP$ was determined by the method of least squares. Corrections were applied for the gauge factor and for the influence of the strain gauge stiffness on the measured strains. The last mentioned correction is described in Appendix A. The numerical values of this correction are:

skin and the webs with h = 0.2 cm $\varepsilon_c = 1.119 \ \varepsilon_m$ for the webs with h = 0.1 cm $\varepsilon_c = 1.238 \ \varepsilon_m$ for the booms $\varepsilon_c = 1.014 \ \varepsilon_m$ were ε_c denotes the corrected strain, and ε_m denotes the measured strain.

The shear deformations at the rosette stations were calculated from the expression (ref. 12)

$$\frac{\mathrm{d}\gamma_{xy}}{\mathrm{d}P} = 2 \frac{\mathrm{d}\varepsilon_{45}^{\circ}}{\mathrm{d}P} - \left\{\frac{\mathrm{d}\varepsilon_x}{\mathrm{d}P} + \frac{\mathrm{d}\varepsilon_y}{\mathrm{d}P}\right\}$$
(4.1)

where ε_x and ε_y denote the strains indicated by the gauges which were orientated in the positive x- and y-directions respectively, and ε_{45}° denotes the strain indicated by the gauge which was orientated under an angle of 45° with the positive x- and y-directions.

Since the strain gauges on the booms were not situated in the neutral plane of the booms a correction was made to allow for the strains due to bending. The bending-stress distribution has been assumed to be linear over the depth of the spars, and the strains measured on the booms were consequently reduced with a factor (see fig. 5):

$$\frac{d}{d+2\Delta} = 0.9091$$
 . (4.2)

Lastly a conversion factor was applied to transform the evaluated test results into strains that would occur in the metal box beam of ref. 1 when loaded in the way as shown in fig. 1. The final test results as given in this report thus refer to the loading cases given in fig. 1, and the metal box beam of ref. 1. The applied conversion factors are determined in Appendix B. For the loading cases B 1, B 2 and C 1.2. the conversion formula reads

$$(\varepsilon/M)_p = 0.012(\varepsilon/M)_m$$

that for loading case B 3 reads

$$(\varepsilon/P)_p = 0.024(\varepsilon/P)_m$$

(4.3)

where m refers to the model box beam and p to the prototype of ref. 1.

For comparison with the test results, the results of ref. 1, which are given in the form of stresses related to oblique co-ordinates, were transformed into strains related to the cartesian co-ordinates used for the testmodel (fig. 3). The transformation formulas used are given in table 3.

From a check on the vertical equilibrium between internal stresses and external loads (see section 5.2) it was found that two more corrections had to be applied to the test results. The first one was a correction for a small vertical load at the tip of the model that appeared in all loading cases where bending or torsional moments were applied (the cases B 1, B 2 and C 1.2). This vertical load was due to the resulting vertical deflection of the wing tips. As a result of this deflection the force applied by the cables of the loading system had a small vertical component (see fig. 11):

$$P=2\,\frac{u}{a}\,Q\,.$$

The magnitudes of these "false" loads, computed from the measured vertical deflection u, were for all loading cases of the order of 2% of the vertical load in loading case B 3. The correction consisted of adding the relevant percentage of the strains of case B 3 to the strains to be corrected. For loading case C 1.2 this correction was somewhat more complicated because of the asymmetric loading in this case. In fig. 20 the



Fig. 20 Method of correcting for the "false" shear loads in loading case C 1.2.

method of this correction is shown. In the computation of the stresses due to the load system C, shown in fig. 20, the engineer's theory for bending has been applied to the wing centre-section only. This is not quite correct, but since only a correction is concerned the resulting error will be of minor importance.

The second correction of the test results was necessary for loading case C 1.2 only. It concerns a correction for the finite stiffness of the frontspar supports (see detail of fig. 10); the stiffness of this configuration proved to be too small in comparison with other stiffnesses. In loading case C 1.2 a deformation of one or more of the supports under its loading results in a decrease of the reaction forces on the supports. For the symmetrical loading cases B 1, B 2 and B3 a correction was not necessary because the displacements of the

centre section of the box beam due to finite stiffness of the supports only had the effect of a rigid body displacement. The discrepancies between the test results, corrected for false shear load, and the theoretical results in the wing centre section (see fig. 45) are qualitatively in accordance with the supposed lack of stiffness in the ball-stiffener configuration. The non-linearity in the plots of applied tip deflections against the measured loads (fig. 21) also points to deformation of the ball-stiffener combination.



Fig. 21 Tip deflection-load curves for loading case C 1.2.

If the magnitude of the reaction forces, P_{rigid} , in the case of infinitely rigid supports were known a correction to the test results could be applied in the following way (see fig. 22). The total force applied to the vertical suffener by the measured web shear stresses P_{ahear} is compared with the reaction force P_{rigid} . The difference between the two forces, $P = P_{ahear} - P_{rigid}$ (P < 0), is then applied as a load on the structure. If this is done for all four supporting points of the box beam a self equilibrating load-system of forces P will be obtained which increases the real reaction forces P will be obtained which increases the real reaction forces are the astructure after a stating tude of those in the case of rigid supports. The strains tude of those in the case of rigid supports. The strains



Fig. 22 Vertical equilibrating forces on the vertical support stiffener.

 $P_{\text{rigid}} = \frac{T}{b} = \frac{10^5}{80} = 1250 \text{ kg}.$ $P_{\text{rigid}} = \frac{T}{b} = \frac{10^5}{80} = 1250 \text{ kg}.$ $P_{\text{rigid}} = \frac{1}{b} = \frac{10^5}{60} = 1250 \text{ kg}.$

infinitely rigid supports at cross-section B,

loads of case C 1.2 will be estimated.

found (fig. 23) that for an unswept box beam with

tion in its own plane. From elementary theory it is

under these loads no prevention of warping will occur; in addition this plane of symmetry will show no rota-

at the tips. In the vertical plane of symmetry of a beam

beam is loaded by antisymmetrical torsional moments

finitely rigid, and continuously distributed. The box

The ribs of the box beam will be assumed to be in-

The magnitude of the reaction forces Prigid for the

simplify these calculations to a relatively high degree.

Since it concerns only a correction it is allowable to

culated and applied as a correction to the test results.

in the structure due to this load-system can be cal-

Fig. 23 The reaction forces P for infinitely rigid supports at the points B.

In order to judge the effect of the correction as described above the load $P_{rigid} = 1250 \, kg$ will be compared with the load $(P_{ahear})_{th}$ calculated from the theoretical stresses given in ref. I, and with the load $(P_{ahear})_{teat}$ calculated from the test results.

This renders

Since the first value differs very little from one in comparison with the second one it can be concluded that the correction as described above is certainly allowable. The calculation of the strains due to the correcting

load system was done by means of the energy method. In fig. 24 the zero stress system and the two internal systems of stress, S_1 and S_2 , used in these calculations, are shown. The participation factor of an internal system of stress equivalent to S_1 and S_2 for the two cells of the



Fig. 24a Zero system of stress, S1



Fig. 24b Internal systems of stress S_1 and S_2 (participation factor 0.88310 and -0.88310 respectively) S1 and S2 are symmetrical with respect to the axis of symmetry a.a.



centrepart is zero because of the antisymmetric loading. Therefore this system of stress has not been used. In the figs. 41, 44 and 45 the corrected strains can be compared with the uncorrected values. The latter ones are placed between brackets.

5 Accuracy of the test results

5.1 The possible error in the test results

In the following a general impression will be given of the most evident error sources. The approximate error is given and the total possible error in the final test results is calculated. Some of the given values for the errors are estimated, in which case the given value is conservative.

The error sources can be divided into:

the relative errors:

- (a) errors in the measurement of the applied loads.
- (b) errors in the measurement of the strains as far as these are due to the electronic equipment and the strain gauge itself.
- (c) errors in the measured strains as far as these are due to mechanical and other causes.

the absolute errors:

(d) zero drift of the strain gauge bridges due to bad heat transmission from the strain gauges to the structure.

(e) errors due to the imperfect reproducibility of the electronic equipment and the mechanical installations.

Approximate numerical values for the relative errors are:

ad (a) 1.	non-linearity of the dynamometers	0.3%
2.	error of the electronic strain meas-	-
	uring apparatus	< l %
3.	other error causes	1%
ad (b) 4.	error of strain gauges (gauge fac-	
	tor)	1.5%
5.	non-linearity of the recorder	1%
6.	error of the calibration signal	< 0.2%
ad (c) 7.	error in the corrections for the	
	stiffness of the strain gauges* (see	
•	remarks at the end of Appendix B):	,
	for gauges on skin and webs $(h =$	
	0.2 cm)	< 6%
	for gauges on rib webs ($h = 0.1$	
	cm)	< 12%
	for gauges on the booms	< 0.7%
8.	for the strain gauge rosettes on	
۰.	skins and webs misalignment of	
	one or two gauges of the rosette of	
	2° results in an error of maximum	
	2% (ref. 6 p. 426).	
. 9.	error due to thickness tolerances of	
	the structural parts (see sect. 3):	
	for skins and webs $(h = 0.2 \text{ cm})$	1.5%
	for rib webs $(h = 0.1 \text{ cm})$	3%
	for booms	negligible
10.	accuracy with which Young's mo-	
	dulus was determined	3%
11.	error due to creep effects (see sec-	
	tion 5.3)	< 2.5%
The to	otal relative errors expressed as e	-

 $\sqrt{e_1^2 + e_2^2 + \dots}$, where e_1, e_2 are the above mentioned relative errors, amount to .

- 7% for skins and sparwebs with h = 0.2 cm
- 12% for rib webs with h = 0.1 cm
- 5% for spar- and rib booms

Measurements of the zero drift of a complete strain gauge bridge during several hours indicated a mean zero drift of 18 · 10⁻⁶ cm/cm/hr. If it is assumed that the zero drift increases linearly with time the approximate value of 10⁻⁵ cm/cm for the possible absolute error in the final test results is obtained.

In the given values for the possible relative errors the errors due to stiffness of the strain gauges dominate for the skin and web's. The equilibrium checks discussed in section 5.2 will show that these values are rather conservative.

The given values are conservative.

5.2 Checks on the reliability of the test results

In order to have a check on the reliability of the test results the equilibrium between the resulting internal forces (after application of all previously described corrections to the test results) and the externally applied loads was investigated. The following equilibrium checks have been made.



Fig. 25 The cross sections where eguilibrium-checks were made.

- (a) At cross-sections B and C (see fig. 25) the bending couples corresponding to the normal forces in the skins, stiffeners and booms determined from the strain gauge measurements were compared with the externally applied bending moments for loading cases B 2 and B 3. The deviation from equilibrium amounted to only 4%.
- (b) At the cross-sections B and C the torsional couples of the shearing stresses in the skins and spar webs and the torsion-producing components of the stringer and boom normal forces were compared with the externally applied torsional moments for the loading cases B 1 and C 1.2. For these loading cases the deviation from equilibrium amounted to 1 and 6% respectively.
- (c) At all cross-sections of the box beam which cut 2 or 3 web-rosettes (fig. 25, A to G inclusive), the vertical forces resulting from the shear strains in the cut spar and rib webs were compared with the externally applied shear loads including the reaction forces at the points of support of the model.

The maximum absolute value of the deviations from equilibrium for all cross-sections, except section A, amounted for loading case B 3 to 8%(in cross section B). For loading cases B 1, B 2 and C 1.2 the sum of the shear forces has to be zero (except for cross-sections D and E). For these loading cases the mean of the sum of all upward web shear forces and the sum of all downward web shear forces in the considered cross-section was taken as a basis of comparison. Compared to this basis the maximum absolute value of the deviations from zero for all crosssections, except section A, amounted for loading case B 1 to $2\frac{1}{2}$ % (in cross section G). For loading cases B 2 and C 1.2 these values were 7% and 7% (in the cross-sections B and F respectively).

For all loading cases the deviations in crosssection A were proportionally larger than the above given values, but the absolute values of these deviations were about as large as those in the other cross-sections. The values given above for loading case C 1.2 refer to the corrected test results as given in fig. 45.

- (d) At the points of support of the model the vertical forces resulting from the shear strains in the webs connected to the vertical supporting stiffener (see fig. 22) were compared with the external reaction force. This force was calculated from the external loads applied to the model. For loading case C 1.2 the reaction forces were, according to the discussion in sec. 4, assumed to amount to 1250 kg, and the web shear forces were taken as the corrected test results of fig. 45. For loading case B 1 the deviation from equilibrium amounted to 0.25%, both for the front and rear supporting points and for loading case B 3 to 0.8% and 6% respectively. For loading case C 1.2 these values were 3% and 3%. The reaction forces for loading case B 2 are zero. The deviations from zero of the resultant web shear force amounted for this loading case to 12% and 3% respectively of the mean of the sum of all upward web shear forces and the sum of all downward web shear forces in the point of support.
- (e) Finally the equilibrium of the junction of the spar boom of the wing centre section, the spar boom of the swept part of the wing and the boom of the root rib was checked. The measured boom stresses were extrapolated to find the stresses at the junction. The deviation from equilibrium at the junction of the front spar booms was in accordance with the order of the possible absolute error in the measured strains for all loading cases.

For the junction of the rear spar booms the equilibrium was, however, rather poor. When the direct stresses in spanwise direction of the skin, stringers and spar booms at both sides of the root rib were compared with each other it appeared that also the measured strains in the skin-stringer combination aft of stringer number 7 (see fig. 2) showed a considerable lack of equilibrium.

It is concluded that the measured strains in the area around the intersection of the root rib and the rear spar are not sufficiently accurate. This could be due to the very high stress gradients in this region.

5.3 Results of a comparison between a constant load test and a constant deflection test

The model was loaded with a constant shear load of 25,5 kg at both tips (loading case B 3) by means of dead weight, while during 2 hours the indicated strain of one complete strain gauge bridge was plotted against time. For this purpose one of the strain gauges on the skin, bonded in x-direction and situated in the swept part of the box-beam in the region of the intersection of rear spar and root rib was selected. Here the strains and the strain gradients are largest and consequently this region can be expected to show the strongest visco-elastic effects.

For the strain measurements use was made of a Peekel static strain indicator; the accuracy of the strain reading was $\pm 2 \times 10^{-6}$ cm/cm.

The same test was made with constant deflections applied to the tips of the model. The load level corresponding with these deflections was 26 kg.

The results of these tests were corrected for zero drift due to heat development in the strain gauges. Based on the results of special tests this zero drift was found to be 18×10^{-6} cm/cm per hour approximately (see sec. 5.1). 30 minutes after load application the following values for the strain increase $\Delta \varepsilon$, expressed as a percentage of the strain ε_0 indicated immediately after load application, were obtained:

case of constant load: $\Delta \varepsilon = +3.3\%$ of ε_0

case of constant deflection: $\Delta \varepsilon = -0.4\%$ of ε_0

60 minutes after load application these values amounted to:

 $\Delta \varepsilon = +4.5\%$ of ε_0 , and $\Delta \varepsilon = -1.1\%$ of ε_0

respectively (the time necessary to complete a test was about 1 hour for each of the loading cases). The negative values of $\Delta \epsilon$ are most probably due to non-linearity effects (i.e. the rate of strain increases more than in the case of a linear relation between stress and rate of strain). Since the strain gauge with which this value was obtained was lying in the region of highest strain it is to be expected that in other points of the structure smaller absolute values for $\Delta \epsilon$ occur and that in the points of lowest strain $\Delta \epsilon$ is of positive sign.

In sec. 4 an error in the measured load due to relaxation is mentioned, which resulted in a deviation of the values for $d\varepsilon/dP$ of about +1.5%. This error is of opposite sign compared with the value of $\Delta\varepsilon$ for the case of constant deflection. Based on the foregoing the total error in the test results due to plasticity effects can thus be estimated to vary between zero and about +2.5% dependent on the strain level. The smallest values are valid in the region of highest strain, the largest values in the regions of lowest strain. This accuracy is still favourable compared with the case of a constant load applied to the model.

6 Test results and discussion

6.1 Comparison between the test results and the theoretical results

In the figures 26 up to 45 inclusive the test results at all strain gauge locations on the upper skin and on the webs of the right winghalf are given together with the strains obtained from ref. 1. In these figures the numbers placed in a box give the strains for an infinitely long hollow box beam. These values were also obtained from ref. 1.

Because in ref. 1 the stresses were not always calculated at the points of location of the strain gauges the theoretical values given in the figures were obtained from graphical interpolation.

In the figures 46 up to 55 inclusive the comparison between test results and theoretical results is given in graphical form. The solid lines in these figures are drawn through the theoretical values, whereas the test results are plotted as points.

The tables 4, 5, 6 and 7 give a comparison between the strains measured on the right and on the left half of the model. The agreement between the strains on both halves is quite satisfactory.

6.2 Discussion

In tables 8, 9 and 10 a survey is given of the maximum deviations between test and theoretical results, as well as of the overall quality of agreement. From a study of this table and the figs. 46 to 55 inclusive it is to be concluded that the best agreement between test and theoretical results is found for loading case B 1. Also it can be seen that the strain distributions of the loading cases B 1 and C 1.2, which both involve a torsional moment, show a strong similarity. The same is the case with the loading cases B 2 and B 3 which both contain a bending moment.

On the whole, the agreement between test and theoretical values is somewhat less satisfactory for the bending moment-load cases than for the loading cases where a torsional moment is applied. From the latter, two load cases (B 1 and C 1.2) case C 1.2 shows less agreement than load case B 1; this can partly be attributed to the relatively large corrections applied to the test results to correct for the finite stiffness of the supports (see sect. 4).

In the following the strains in all structural parts of the box beam will be discussed in more detail.

The direct strains in the booms

The agreement between test and theoretical results

is satisfactory for loading cases B 1 and C 1.2. For case B 1 the gradients of the strain in the centre-section and in the first bay of the swept part are somewhat overestimated in the results of ref. 1. This discrepancy must be due to deviations in the participation factors of the internal systems of stress of the types 2 and 3. (see ref. 1, figs. 22, 23, 28 and 29; the internal systems of stress of type 1 cannot be the cause because the shear strains in the webs show good agreement between test and theoretical values).

The agreement between test and theoretical results for loading cases B 2 and B 3 (figs. 50 and 51) is rather poor.

For cases B 3 and B 2 the calculated gradients of strain are considerably overestimated. This could be caused by deviations in the participation factors of the internal systems of stress of the types 1 and 3 (ref. 1, fig. 21 and 23).

The discrepancies in the region of the root rib for all loading cases could be expected in view of the incompatibilities between the calculated strains in the booms and the adjacent skin, shown in fig. 81 to 85 inclusive of ref. 1.

On p. 20 of ref. 1 it is conjectured that the calculated skin stresses adjacent to the spar booms are better approximations than the calculated boom stresses, because of the fact that the booms give a smaller contribution to the total strain energy than the skin stiffener combination does. Comparison of figs. 81, 82, 83 and 85 of ref. 1 with figs. 26, 31, 36 and 41*) respectively shows this expectation to be fully confirmed for .the loading cases B 1 and B 2; for the cases B 2 and C 1.2 it is only true for the strains near the rear spar boom.

For the loading cases B 1 and C 1.2 the computed strains in the boom of the root rib are better approximations than the strains ε_y in the adjacent skin. This can be attributed to the fact that in the calculations the freedom of variation of the stresses in y-direction in the skin is very limited so that the strains ε_y in the skin are not very accurate.

The direct strains in x-direction in the skin

For loading cases B 1 and C 1.2 the agreement between test and theoretical values is quite good. In most points the difference is smaller than the possible absolute error. For case B 1 the "strain concentration" near the rear spar seems to be somewhat underestimated in the calculations. In this connection it should, however, be remembered that the measured strains in the area around the intersection of the root rib and the rear spar are expected to be not quite reliable (see sec. 5.2, sub(e)).

For loading cases B 2 and B 3 the agreement is somewhat less than for the other loading cases. In both cases the tendency of the theoretical strain ε_x in the centre-section of the box beam is not in agreement with that of the test values. For this reason the strain concentrations near the rear spar are not indicated by the theoretical results.

The direct strains in y-direction in the skin

For all loading cases the agreement between test and theoretical results is satisfactory or good, with the exception of the strains in the region of the root rib.

In the cross-sections adjacent to the root rib, and for case C 1.2 also in the centre-section of the box beam, very large discrepancies occur.

For loading case B 1 the cause of these discrepancies can be indicated. The relation between the direct strains ε_x and ε_y , and the stress flow s_y as calculated in ref. 1 can be written in the form.

$$\varepsilon_y = \frac{1 - v^2}{E\hbar} \, s_y - v \varepsilon_x \tag{6.1}$$

The contribution of the term $v\varepsilon_x$ to the theoretical strains ε_y in the centre-section is smaller than 5% of ε_y . The large discrepancies between tests and theoretical results can thus be fully attributed to the calculated stresses s_y . The cause of the discrepancies is the fact that in the calculations of ref. 1 the direct stressflow in y-direction, s_y , are not allowed to vary in x-direction. This restriction does not allow the skin to be strained in compatibility with the boom of the root rib. The strains ε_y obtained from the calculated stresses in the root rib boom (in fig. 47, section 3, given as a dotted line) show much better agreement with the test results.

In cross-section 4 of the swept part adjacent to the root rib the discrepancy is somewhat less severe. Here the term containing \bar{s}_y in the expression.

$$\varepsilon_y = \frac{1 - \nu^2}{Eh} \, \bar{s}_y \, \sin(\pi/4) - \nu \varepsilon_x \tag{6.2}$$

amounts only to 23% of the value of $v\varepsilon_x$, and since the theoretical values ε_x show good agreement with the test results the effect of errors \bar{s}_y is not large.

For loading case C. 1.2 the fact that the calculated values ε_y in the centre-section of the box beam all lie about halfway between the minimum and the maximum test values also demonstrates the lack of freedom of variation for the stresses s_y in x-direction in the calculations.

From the character of the deviations between test

^{*} The values in parentheses given in the figs. 26 up to 45 inclusive are the calculated strains at the locations of the strain gauges. These values are obtained by graphical interpolation from the figs. 46 up to 55 inclusive. This explains the differences between the calculated strains shown in the figures of this report and in those of ref. 1.

and theoretical values for ε_y it can be seen that in the centre-section and in the first skin bay of the swept part internal systems of stress, allowing variation of s_y in x-direction, are required in order to obtain agreement with the test results.

The shear strains in the skin

For loading cases B 1 and B 2 the agreement between test and theoretical results is good or satisfactory, with the exception of the cross-sections adjacent to the root rib. For loading cases B 3 and C 1.2 the agreement is rather poor. For case B 3, however, the strain gradients are very large, so that small effects may have a strong influence on the differences. For case C 1.2 the test values in the centre-section are not very reliable because they were obtained after very large corrections had been applied.

Whether the discrepancies in the swept part of the box beam are due to the calculated stresses t or s_y can be estimated as follows. The shear strain γ_{xy} in the swept part depends on the stress flows \bar{s}_y and \bar{t} as calculated in ref. 1:

$$\gamma_{xy} = \frac{1}{Gh} \left(\bar{s}_y \cos(\pi/4) + \bar{i} \right) \tag{6.3}$$

For loading cases B 1 and C 1.2 the value of \bar{s}_y in cross section 4 amounts only to 10% of the value of \bar{t} , so that the large discrepancies in this section must be mainly due to the shear flow \bar{t} ; for loading cases B 2 and B 3 the value \bar{s}_y in cross section 4 amounts to 25% and 50% respectively of the value of \bar{t} , so that errors in \bar{s}_y can also have had influence.

The discrepancies between test and theoretical values in the first skin bay of swept part for loading cases B 1 and C 1.2 lead to the conclusion that the calculations of ref. 1 still possess insufficient degrees of freedom for the stressflow \bar{t} to enable a reliable calculation of the actual shear flow distribution.

Especially the freedom of variation of t in x-direction in the skin bays adjacent to the root rib is insufficient.

The shear strains in spar and rib webs

The agreement between test and theory is extremely good for loading case B 1; for the other cases it is satisfactory or good. The discrepancies for the loading cases B 2 and B 3 show again much the same tendency.

In summary it can be said that with reference to the most important strains (ε_x in skin and booms, and γ_{xz} in the webs) the agreement between test and theoretical results for the loading cases B 1 and C 1.2 is quite satisfactory, provided that due consideration is given to the incompatibilities inherent in the theoretical solution (e.g. the boom strains being approximated with less accuracy than the skin strains). However, the ex-

pectation seems legitimate that by adding one or more internal systems of stress in the centre-section and the first bay of the swept part to the unknowns of the problem, the theoretical results could still be improved.

For the loading cases B 2 and B 3 the agreement between test and theoretical results is less satisfactory than for loading cases B 1 and C 1.2. In both bays adjacent to the root rib rather important differences occur. These differences point to a lack of degrees of freedom in the calculations for the stresses in the centre-section and the first bay of the swept part. In this context it is interesting to note that for loading case B 2 the participation factors for the two stress systems which allow for stress distributions s_x , that are non linear in x-direction in the first bay of the swept part (see ref. 1, table 14, stress systems nrs. 16 and 17), are very much larger than those of most other stress systems in the swept part (nrs. 12 ... 29), and are of about the same magnitude as the participation factors of the stress systems covering one bay of the centresection and the first bay of the swept part (nrs. $6 \dots 9$). For loading case B 3 about the same indications though less pronounced, can be found.

This points to the importance of the stress systems with non-linear stress distributions in x-direction (which are also the only stress systems that include stresses s_y and allow variation of t in x-direction) for these loading cases. It should therefore be possible to improve the theoretical results by adding some internal systems of stress for the centre-section and the first bay of the swept part, which extend the number of degrees of freedom for the stresses in this region. The results of an improved calculation, using the stress-distribution obtained in ref. 1 as a basic stress system, will be reported later.

7 Conclusions

- 7.1 In the loading cases where "torsional moments" are applied at the tips (fig. 1, cases B 1 and C 1.2) the agreement between test and theoretical results is very satisfactory for the most important strains (the direct strains parallel to the spars in the skin and booms, and the shear strains in the webs), provided that due consideration is given to the incompatibilities inherent in the theoretical solution. In the asymmetric loading case (case C 1.2) the agreement is somewhat less than in the symmetric loading case (case B 1).
- 7.2 In the loading cases where "bending moments" are applied (fig. 1, cases B 2 and B 3) the agreement between test and theoretical results is satisfactory in the swept part at some distance from the root rib (the rib at the root of the swept part). In the centre section and the first bay of the swept

part however rather important differences occur. The discrepancies show much the same tendencies in both loading cases.

- 7.3 In all loading cases indications are present that a large part of the differences still present between test and theoretical results are caused by the fact that the calculations allowed only a limited number of degrees of freedom for the direct stresses in chordwise direction (s_y) and the shear stresses (t) in the skin bays adjacent to the root rib (the rib at the root of the swept part). It should be possible to improve the theoretical results by adding some internal systems of stress in the centre section and the first bay of the swept part.
- 7.4 The method of applying constant deflections as used in the tests reduces the effects of visco-elasticity on the results of strain measurements appreciably.

8 References

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APPENDIX A

Corrections for strain gauge stiffness

Because of the low value of Young's modulus for polystyrene the attachment of strain gauges to a structure consisting of this material can have a considerable influence on the local stiffness. Especially in the case of thin panels (in the order of 0.1 cm) with strain gauges bonded at both sides this stiffening effect can result in an error in the measured strains of up to about 20%.

A correction for this effect is made by taking into account the stiffness $(EA)_g$ of the strain gauges as determined experimentally in the following way*.

Two strain gauges of the type used on the test-model were bonded back to back with the same cement as was used to bond the gauges on the model. These strain gauges were loaded in tension by means of dead weight, while the resulting strain was measured. As dummies were used two identical strain gauges, also cemented back to back. From 7 measurements on each of the two pairs of strain gauges (dimensions cut down to 6×20 mm; see also sec. 3.3) the stiffness was found to be

$$(EA)_g = (371 \text{ à } 397) \frac{\text{kg}}{\text{cm/cm}}$$
 per bonded gauge.
(A.1)

The relation between the measured strain, ε_m , and the strain corrected for strain gauge stiffness, ε_c , in the case of a simple structural member loaded in tension is (strain gauges at both sides and local effects neglected):

$$\varepsilon_c = \varepsilon_m \left[1 + \frac{2(EA)_g}{(EA)_0} \right]$$
 (A.2)

in which $(EA)_g$ represents the stiffness of the bonded strain gauge and $(EA)_0$ represents the stiffness of the structural member considered. Application of this correction formula to the results of strain measurements on skin and web panels presents the difficulty of determining the value of $(EA)_0$ that has to be used for each strain gauge location.

Considering a strain gauge bonded to a thin flat plate the following can be stated:

1. the disturbance of the strain field due to the presence of the strain gauge will extend itself at most 5 times the width of the strain gauge $(5b_g)$. This gives a minimum for the correction factor:

$$\left[\frac{(EA)_g}{(EA)_0}\right]_{\text{minimum}} = \frac{(EA)_g}{5Ehb_g}$$
(A.3)

2. the minimum of the above meant width can be considered to be equal to the width of the strain gauge (b_g) . This gives a maximum for the correction factor:

$$\left[\frac{(EA)_g}{(EA)_0}\right]_{\text{maximum}} = \frac{(EA)_g}{Ehb_g}$$
(A.4)

In the present case the value of the correction factor for the strain gauge locations on skins and webs has

^{*} This method was derived from ref. 7.

been taken as the mean of its maximum and its minimum value.

For skin and webs with strain gauges bonded at both sides the applied correction thus is:

,
$$\varepsilon_c = \varepsilon_m \left[1 + 2 \cdot \frac{3(EA)_g}{5 Ehb_g} \right]$$
 (A.5)

For the strain gauge locations on the spar and rib booms the value of A_0 is taken equal to the crosssectional area of the boom in question:

$$\varepsilon_c = \varepsilon_m \left[1 + \frac{(EA)_g}{EA_{\text{boom}}} \right] .$$
 (A.6)

The numerical values of the applied correction are:

skins and webs with
$$h = 0.2 \text{ cm}$$
 $\varepsilon_c = 1.119 \varepsilon_m$
webs with $h = 0.1 \text{ cm}$ $\varepsilon_c = 1.238 \varepsilon_m$
booms $(A = 0.81 \text{ cm}^2)$ $\varepsilon_c = 1.014 \varepsilon_m$ (A. 7)

Concerning the accuracy of these corrections it may be assumed that this is in any case better than 50%. The maximum error in the strain measurements due to the stiffness of the strain gauges amounts than to less than

6% for skin and webs with h = 0.2 cm

12% for rib webs with h = 0.1 cm

0.7% for the booms.

APPENDIX B

The scale factors to be used for the comparison of the test results and the theoretical results

As mentioned in sec. 3.1 the geometrical scale factors used for the model are

 $-k_m/k_p = q = \frac{1}{2}$ for all length dimensions.

 $-h_m/h_p = r = 1$ for the cross-sectional thickness of the stress carrying members.

The cross-sectional areas thus have a scale factor of $q \cdot r = \frac{1}{2}$.

The other scale factors are determined by means of a dimensional analysis (see for instance ref. 8). The pertinent variables which are of importance in the problem of the analysis of a beam with the two above mentioned geometrical scale factors, loaded by shear loads or moments (fig. 1), are:

- k span of the box beam
- h -- cross-sectional thickness of a stress carrying member (web, skin, stiffener etc.)

M -- loading moment

- P shear load
- E Young's modulus
- ν Poisson's ratio
- σ stress
- y deflection or displacement

the unit of force [F]
the unit of length [L] of all dimensions except the cross-sectional thickness
the unit of length [l] of the cross-sectional thickness

The dimensions of the pertinent variables are now tabulated as follows

		k	h	М	P	E	v	σ	у
exponent	(force [F]			1	1	1		1	
of the unit	length [L]	1		1		—1		—1	1
of	[length [1]		1			1		—1	•

With the help of this table the following five (8-3) independent dimensionless groups (π -factors) can be formed (of course also other sets of groups are possible):

$$\pi_{1} = \nu \qquad \pi_{4} = \frac{M}{Ek^{2}h}$$

$$\pi_{2} = \frac{y}{k} \qquad \pi_{5} = \frac{P}{Ekh}$$

$$(B.1)$$

$$\pi_{3} = \frac{\sigma}{E} (=\epsilon)$$

In the following the index m will be added if the model box beam used in the tests is referred to; the index p will indicate the prototype box beam of ref. 1.

Equations (B.1) give the scale factors to be applied in the model test:

(a) $v_m = v_p$; as $v_m \approx 0.31$ and $v_p = 0.3$ this condition is satisfied to a sufficient degree of accuracy

(b)
$$y_p = \frac{k_p}{k_m} y_m = \frac{1}{q} y_m = 2 y_m$$

(c) $\sigma_p = \frac{E_p}{E_m} \sigma_m$, or $\varepsilon_p = \varepsilon_m$
(d) $M_p = \frac{(Ek^2h)_p}{(Ek^2h)_m} M_m = \frac{E_p}{E_m} \frac{1}{q^2r} M_m$
(e) $P_p = \frac{(Ekh)_p}{(Ekh)_m} P_m = \frac{E_p}{E_m} \frac{1}{qr} P_m$

The conditions (c), (d) and (e) of (B.2) yield the conversion factors to be applied to the test results:

$$\begin{pmatrix} \frac{\varepsilon}{M} \end{pmatrix}_{p} = \frac{E_{m}}{E_{p}} q^{2} r \left(\frac{\varepsilon}{M} \right)_{m}$$

$$\begin{pmatrix} \frac{\varepsilon}{P} \end{pmatrix}_{p} = \frac{E_{m}}{E_{p}} q r \left(\frac{\varepsilon}{P} \right)_{m} ,$$

$$(B.3)$$

or, with $q = \frac{1}{2}$, r = 1, $E_m = 0.336 \cdot 10^5 \text{ kg/cm}^2$, $E_p = 7 \cdot 10^5 \text{ kg/cm}^2$

$$\begin{pmatrix} \frac{\varepsilon}{M} \\ p \end{pmatrix}_{p} = 0.012 \begin{pmatrix} \frac{\varepsilon}{M} \\ m \end{pmatrix}_{m} ,$$

$$\begin{pmatrix} \frac{\varepsilon}{P} \\ p \end{pmatrix}_{p} = 0.024 \begin{pmatrix} \frac{\varepsilon}{P} \\ m \end{pmatrix}_{m} ,$$

to be applied in the loading cases B 1, B 2 and C 1.2 (B.4) to be applied in loading case B 3 Application of these conversion factors transforms the test results into strains that would occur in the prototype box beam of ref. 1 under loads as given in fig. 1.

		box beam	ideal	actual test- after milling	model measured and assembling
		01101.1	leat-model	min	max.
d	cm	18.0	9.0	8.98	9.02
b	cm .	56.56	28.28	28.21	28.27
h .	cm	0.2	0.2	0.198	0.207
Ab	cm ²	1.25	0.625	0.63	0.63
Astr	cm ²	1.24	0.62	0.649	0.666
$I_{\rm skin} = \frac{1}{4} bhd^2$	çm ⁴	916.27	114.2	112.60	119.03
$I_{\rm web} = \frac{1}{12} h d^3$	cm ⁴	97.20	12.15	11.95	12.66
$I_{\text{boom}} = A_b \cdot \frac{1}{4} d^2$	cm ⁴	101.25	12.66	12.70	12.81
$I_{\text{stringer}} = A_{\text{str}} \cdot \frac{1}{4} d^2$	cm4	100.44	12.55	13.08	13.55
Icrosss-ection	cm ⁴	4038.90	504.87	509.18	531.42
				520.	30
$J = \frac{2b^2 d^2 h}{b + b + d}$	cm ⁴	5562.23	695.27	683.32	727.12
o + d				705	.22

	ŗĥ	Astr	АР
<u>h</u> ,	•		d
	Ь		

 $\frac{I_{\text{actual}}}{I_{\text{ideal}}} = 1.031$ $\frac{J_{\text{actual}}}{J_{\text{ideal}}} = 1.014$

TABLE 2

The magnitudes of the applied maximum loads and deflect	ctions on the model

Loading case		Maximum load 1)	Maximum deflection		
	Type of loading	Moment (kgcm) Shearing Force (kg)	Vertical translation (cm)	Rotation ²) (rad)	
B 1	torsional moment (symmetrical)	2070	0.757	+ 0.0196	
В 2	bending moment	1885	0.719	- 0.0047	
В 3	shear load	26.3	1.41	+ 0.0071	
C 1.2	torsional moment (asymmetrical)	2330	0.731	0.0217	

¹) measured directly after application of the deflections.

2) signconvention: + starboard tip front spar up

--- starboard tip front spar down

TABLE 3

Transformation formulas used to transform the results of ref. 1, into strains related to cartesian co-ordinates

			idealized structure of ref. 1	
spar booms root-rib booms	$10^5 \varepsilon_x = 10^5 \varepsilon_x =$	0.116144 <i>N</i> 0.077220 <i>N</i>	$A = 1.23 \text{ cm}^2$ $A = 1.85 \text{ cm}^2$	$\left.\right\rangle \qquad \varepsilon_x = \frac{N}{EA}$
spar- and rib webs $(h = 0.2 \text{ cm})$ rib webs $(h = 0.1 \text{ cm})$	$10^{5} \gamma_{xz} =$ $10^{5} \gamma_{xz} =$	1.85714 <i>t</i> 3.71428 <i>t</i>	h = 0.2 cm h = 0.1 cm	$\Big\}\gamma_{xz}=rac{2(1+v)}{Eh}t$
skin of box beam centre-section (cartesian- co-ordinates)	$ \left(\begin{array}{ccc} 10^5 \varepsilon_x &= \\ 10^5 \varepsilon_y &= \\ 10^5 \gamma_{xy} &= \end{array}\right) $	$\begin{array}{l} 0.42079 \ s_x = - \ 0.12624 \ s_y \\ - 0.12624 \ s_x + \ 0.68787 \ s_y \\ 1.8571 \ t \end{array}$	$h = 0.2 ext{ cm}$ $h_s = 0.139500 ext{ cm}$	$\begin{cases} \varepsilon_i = A_{ij} S_j^{-1} \\ i = 1, 2, 3 \\ j = 1, 2, 3 \end{cases}$
skin of the swept part of the box beam (in ref. 1 oblique co-ordinates are used)	$\begin{cases} 10^5 \varepsilon_x = \\ 10^5 \varepsilon_y = - \\ 10^5 \gamma_{xy} = - \end{cases}$	$\begin{array}{c} 0.50853\bar{s}_x+0.17799\bar{s}_y+0.719\\-0.15255\bar{s}_x+0.40623\bar{s}_y-0.215\\1.3132\bar{s}_y+1.857\end{array}$	$\begin{array}{l} 17 \ l \ h = 0.2 \\ 575 \ l \ h_s = 0.197283 \ \mathrm{cm} \\ 11 \ l \end{array}$	$\begin{cases} \varepsilon_i = A_{ij}^* \bar{s}_j^2 \\ i = 1, 2, 3 \\ j = 1, 2, 3 \end{cases}$

¹) where A_{ij} is the matrix of coefficients of the stress-strain relations for the composite skin-stiffener plate (see ref. 2 and 3):

$$A_{ij} = \frac{1}{Eh(1+h_s/h)} \qquad \begin{vmatrix} 1 & -\nu & 0 \\ -\nu & [1+h_s/h(1-\nu^2)] & 0 \\ 0 & 0 & 2(1+\nu)(1+h_s/h) \end{vmatrix} \text{ and } \begin{cases} s_1 = s_x \\ s_2 = s_y \\ s_3 = t \end{cases} \begin{cases} \varepsilon_1 = \varepsilon_x \\ \varepsilon_2 = \varepsilon_y \\ \varepsilon_3 = \gamma_{xy} \end{cases}$$

*) $\begin{cases} \varepsilon_1 = \varepsilon_x & \text{for cartesian} \\ \varepsilon_2 = \varepsilon_y & \text{co-ordinates,} \\ \varepsilon_3 = \gamma_{xy} & \end{cases}$ $\begin{cases} s_1 = \overline{s}_x & \text{for the oblique co-ordinates} \\ \overline{s}_3 = \overline{t} & \text{used in ref. 1} \end{cases}$

The matrix A_{ij}^{\dagger} is obtained by substitution of the equations $\tilde{\varepsilon}_i = \bar{A}_{ij} \bar{s}_j$ for oblique co-ordinates, obtained from ref. 1, into the equations for transformation from oblique to cartesian co-ordinates:

TABLE 4

Comparison between the strains $(10^{5}\varepsilon)$ in the booms and skin, measured on the right and left half of the model

TABLE 5





TABLE 6 Comparison between the strains $(10^5\varepsilon)$ in the booms and skin, measured on the right and left half of the model



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RIGHT

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3

б

R

-8.0

[+8.4]

-19.1

[+184]

-20.8

[+21.6]

-29.7

[+31.0]



TABLE 8 Comparison between test and theoretical results in summary **Boom-strains**

		Maximal deviati test and theorem	ion between . tical results		
Boom	Loading	both cross-sect. adjacent to root rib	all other cross sections	Overall quality of agreement	See fig.
case .	case	% of max. measured strain in all booms	% of max. measured strain in all booms		
	(B1	36	15	very good, except near root rib	46
Front	B 2	5	· 14	good	50
Spar	(В3	19	38	bad; strain gradients are overestimated by ref. 1	51
_	C 1.2	22	14	quite good	· 55
	(81	31	25	satisfactory	46
Pear	B 2	50	26	bad; test values near root rib are not in equilib-	50
Spar)	51	30	bad, strain gradients are overestimated by ref. 1	51
араг	C 1.2	56	17	good, except near root rib	55
	(B1		. 11	satisfactory	46
	B 2	•	65	bad	50
Root	B 2		30	rather bad; strain gradients are overestimated by	51
boom	(C 1.2		14	quite good except near front spar	55

Comparison between test and theoretical results in summary					
Skin strains					

-		Maximal deviat test and theore	tion between tical results		
Direction Loading of strain case	both cross-sect. adjacent to root rib	all other cross-sections	Overall quality of agreement	See	
	% of max. measured strain in skin	% of max. measured strain in skin		ng.	
	(B1	20	9	very good	47
e_x	B 2	20	11	satisfactory	49
	B 3	21	9	satisfactory	52
	C 1.2	14	11	quite good, near root rib somewhat less	54
	(B1)	140	13	very good, except near root rib	47
	B 2	64	33	good, except near root rib	49
Fy	B 3	100	35	satisfactory, except near root rib	52
	C 1.2	39	42	good, except near root rib and in centre-section	54
	(B1	33	10	satisfactory	48
Yzy ·	B 2	18	14	satisfactory, except in first skin bay of swept part	48
) B 3	27	26	not good, strain gradients are overestimated by ref. 1	53
	(C 1.2	21	24	not good, probably due to large applied correc- tions on test results	53

TABLE 10

Comparison between test and theoretical results in summary Shear strains in spar and rib webs

	Maximal deviation between	test and theoretical results		
Loading case	in webs of, and adjacent to root rib	in all other webs	Overall quality of agreement	See fig.
	% of max. measured strain in all webs	% of max. measured strain in all webs	-	
B 1	2	. 3	very good	30
B 2	12	10	satisfactory	35
B 3	13	7	good	40
C 1.2	9	6	good	45





Fig. 26 Normal strains $10^{5} \epsilon_{x}$ in spar- and rib booms.



LOADING CASE B 1



Fig. 27 Normal strains $10^5 \varepsilon_x$ in skin and stringer.



CALCULATED VALUES ARE GIVEN IN PARENTHESES

Fig. 28 Normal strains $10^5 \varepsilon_y$ in the skin.





Fig. 29 Shear strains $10^5 \gamma_{xy}$ in the skin.





Fig. 30 Shear strains $10^5 \gamma_{xx}$ in spar- and rib webs.



Fig. 31 Normal strains $10^5 \varepsilon_x$ in spar- and rib booms.





Fig. 32 Normal strains $10^5 \epsilon_x$ in skin and stringers.









LOADING CASE B 2









Fig. 35 Shear strains $10^5 \gamma_{xz}$ inspar- and rib webs.





Fig. 36 Normal strains $10^5 \epsilon_x$ in spar- and rib booms.





Fig. 37 Normal strains $10^5 \epsilon_x$ in skin and stringers.





Fig. 38 Normal strains $10^5 \varepsilon_y$ in the skin.







Fig. 39 Shear strains $10^5 \gamma_{xy}$ in the skin.



(-7.1)

(+43,1 (+43,1)

-64.2

THE STRAINS WHICH ARE NOT YET CORRECTED FOR THE FINITE STIFFNESS OF THE SUPPORTS ARE GIVEN BETWEEN BRACKETS.

CALCULATED VALUES ARE GIVEN IN PARENTHESES.

-**68,**4 [-66,8] [-72,8)

Fig. 45 Shear strains $10^5 \gamma_{xx}$ in spar- and rib webs.



Fig. 42 Normal strains $10^5 \varepsilon_x$ in skin and stringers.

-125



Fig. 46 Direct strains $10^5 \varepsilon_x$ in the spar and rib booms. Loading case B 1.



FS 1 2 3 4 5 6 7 8 RS FS 1 2 3 4 5 6 7 8 RS STIFFENERS STIFFENERS

Fig. 47

126



Fig. 49

LZI





RECTION





821'





Fig. 55 Direct strains $10^{5}\varepsilon_{x}$ in the spar and rib booms. Loading case C 1.2.

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