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PREFACE

This volume of Reports and Transactions of the National Aerospace Laboratory NLR contains a selection of reports completed in recent years.

In reports TR F253 and TR T136 the results of investigations performed for the Netherlands Aircraft Development Board (NIV) are presented. Report TR 143 pertains to research work promoted by the Directorate of Materiel Air RNLAF. The permission for publication from these sponsors is herewith acknowledged.

In addition to the selected reports issued at more or less regular intervals in the series Reports and Transactions, numerous others are published on work carried out by NLR. A list of publications to date is available upon request.

Amsterdam, December 1969

A. J. Marx

(General Director)

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Analysis of pressure distributions measured on a wing with oscillating control surface in two-dimensional high subsonic and transonic flow

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At this moment little experimental material is available about the instationary aerodynamic forces on wing-control surface combinations in high subsonic and transonic flow. Up to now the investigations on this subject have been restricted mainly to the determination of overall hinge moments by means of the series and to the structure and to the series of the serie

moments by means of free oscillation techniques and to the study of the unsteady flow by optical methods. By aid of a special measuring technique, the NLR has succeeded to determine detailed instationary pressure distributions on a two-dimensional wing-control surface system in the Mach number range 0.5-1.02.

In addition to the pressure measurements the periodical motion of the shock waves has been studied in some cases by means of schlieren and shadowgraph observations.

The detailed results of the pressure measurements have been presented in a previous report. The present report, emphasizing the analysis and discussion, contains only a part of these results to illustrate the special features encountered. As far as possible comparisons are made with corresponding theoretical values.

This investigation has been performed under contract for the Netherlands Aircraft Development Board (NIV).

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	List of symbols	อธิย	Contents

n _c	instationary	hinge	moment
	derivative		
р	static pressure		
∆p	amplitude of ins	tationary	pressure
	disturbance		
p _r	stagnation press	ure	
$q = \frac{1}{2} \rho_{\infty} U_{\infty}^2$	dynamic pressur	e	
R	instationary nor	mal force	e on con-
	trol surface		
R	static normal t	force on	control
	surface		
r _c	instationary con	trol surfa	ace force
	derivative		
t	time		
U	flow velocity		
x	co-ordinate in	streamwi	se direc-
	tion		
ρ	air density		
ω	frequency of osc	illation (c/s)

Subscripts

loc.	local value
00	value in undisturbed flow
+	upper-surface of the model
-	lower surface of the model

Definitions



static and mean quantities :

$$\begin{split} \overline{K} &= C_{\overline{K}} q \, 2l \\ \overline{M} &= C_{\overline{M}} q \, (2l)^2 \\ C_{p_+} &= \frac{p_+ - p_{\infty}}{q} \\ C_{\overline{K}} &= -\int_0^1 (C_{p_+} - C_{p_-}) dx \quad C_{\overline{R}} = \int_{0.75}^1 (C_{p_+} - C_{p_-}) dx \\ C_{\overline{M}} &= -\int_0^1 (C_{p_+} - C_{p_-}) (x - 0.25) dx \\ C_{\overline{M}} &= -\int_{0.75}^1 (C_{p_+} - C_{p_-}) (x - 0.75) dx \\ M_{\text{loc.}} &= \left[\frac{2}{\gamma - 1} \left\{ \left(\frac{p_r}{p + qC_p} \right)^{(\gamma - 1)/\gamma} - 1 \right\} \right]^{\frac{1}{2}} \end{split}$$

$$K = 2\pi q l k_c e^{i\omega t}$$

$$M = 2\pi q l^2 m_c e^{i\omega t}$$

$$R = 2\pi q l r_c e^{i\omega t}$$

$$N = 2\pi q l^2 m_c e^{i\omega t}$$

$$C_{p_{c_+}} = -\frac{\Delta_{p_+}}{q \cdot C} \quad C_{p_{c_-}} = \frac{\Delta_{p_-}}{q \cdot C} \quad \Delta C_p = C_{p_{c_+}} - C_{p_{c_-}}$$

$$k_c = \frac{1}{\pi} \int_0^1 \Delta C_p dx$$

$$r_c = \frac{1}{\pi} \int_{0.75}^1 \Delta C_p dx$$

$$m_c = \frac{2}{\pi} \int_0^1 \Delta C_p \times \frac{1}{x(x-0.25)} dx$$

$$n_c = \frac{2}{\pi} \int_{0.75}^1 \Delta C_p \times \frac{1}{x(x-0.75)} dx$$

Introduction

For the flutter analysis of modern airplanes the insufficient knowledge about the instationary aerodynamic forces in the transonic speed regime forms a real problem. Up to now reliable theoretical results are absent and also experimental material is scarce, especially for wings with control surfaces. Concerning the latter subject the attention has been restricted mainly to the control surface buzz phenomenon. In relation hereto two ways of approach can be distinguished, viz. flow investigations by means of optical observations, and measurements of the instationary aerodynamic forces acting on the wing and the oscillating control surface.

Flow studies by optical methods of the behaviour of shock waves, boundary layer and the free motion of the control surface during buzz, have been performed e.g. by Lambourne (refs. 1, 2 and 3), Nakamura and Tanabe (ref. 4), and Loiseau (refs. 5 and 6). Especially Lambourne has revealed many essential details about the influences of fundamental parameters and the flow conditions around an aerofoil with oscillating control surface. His experiments, however, were limited to stationary buzz conditions, which means that only the limit cycle oscillations of the strongly non-linear buzz phenomenon were investigated. In an attempt to study also the transient phase, Nakamura and Tanabe took high speed schlieren pictures of the onset of control surface oscillations and the flow around it, covering in this way its growth from small amplitudes up to the limit cycle. An essential restriction imposed by the use of a free oscillating control surface, as was the case in the afore mentioned experiments, is that observations can be made only if buzz occurs, and that little or no control of the flap motion exists.

The other way of approach, namely the acquirement of information on the instationary aerodynamic forces acting on a wing with oscillating control surface, has been concentrated mainly on the determination of the instationary hinge moment by means of a free oscillation technique. For a wing-control surface system in two-dimensional transonic flow, results are also given

by Nakamura and Tanabe (ref. 4) and by Loiseau (refs. 5 and 6).

More thorough information about the unsteady aerodynamic forces can be obtained by measuring the detailed pressure distribution on a model in forced sinusoidal motion. Apart from the possibility to obtain various aerodynamic derivatives by integrating the pressure distribution, this technique offers a means to measure the upper and lower surface separately, as could be desirable in case of non-symmetric flow conditions.

A first attempt to determine local instationary pressures on an oscillating control surface in transonic flow has been made by Erickson and Robinson (ref. 7). Their method, using electrical pressure cells flush with the aerofoil surface, has been applied successfully by Wyss and Sorenson (ref. 8), who measured the control surface derivatives of a NACA 65_1 -213 aerofoil with a 25 percent control surface. Although they actually measured the pressure distribution over the control surface, only hinge moment derivatives were presented, except for some typical oscillograph records of the pressure fluctuations.

From the beginning of the NLR programme on instationary aerodynamics for transonic flow, the aim has been to measure detailed pressure distributions and to try to make additional flow observations (ref. 9). As a first attempt, instationary pressure distributions have been determined on a wing with harmonically oscillating control surface in two-dimensional subsonic and low transonic flows (refs. 10,11). However, the results of these tests, carried out on an 8 percent thick double circular arc profile with a 25 percent control surface, were strongly affected by severe flow separation already occurring at relatively low Mach numbers, and caused by the convex curvature on the control surface. For this reason, these experiments are considered to be more valuable because of the experience gained, especially with the rather unusual pressure measuring technique, than for the direct results.

In the present investigation, being an extension of the former work, a similar model was used with a NACA 65A006 profile which has better aerodynamic properties. Both the mean and the oscillatory pressure distributions over the wing and the control surface have been determined.

The control surface was forced to perform harmonic motions with frequencies equal to 30, 60, 90, 120 or 150 c/s at Mach numbers from 0.5 to 1.02, with small steps in the transonic region.

The detailed results of the pressure measurements have been presented without discussion in ref. 12. The present report, emphasizing the analysis and discussion, contains only a part of these results to illustrate the special features encountered. For some cases the measured pressure distributions have been compared with corresponding theoretical results.

In addition to the pressure distributions, the corresponding periodical motion of the shock waves, derived from schlieren and shadow observations, has been presented for a limited number of tests.

2 Apparatus and experimental methods

2.1 Wind tunnel

The tests have been performed in the Pilot tunnel of the NLR, being a closed circuit wind tunnel for Mach numbers up to about 1.0. Upper and lowerside of the test section (height 0.55 m, width 0.42 m) are fitted with longitudinal slotted walls. The open area ratio of these walls is 0.1. Further information about the Pilot tunnel can be found in ref. 13.

2.2 Model and excitation mechanism

The two-dimensional model, having a NACA 65A006 profile, has a chord length of 18 cm and is fitted with a control surface of 25 percent of the chord. The maximum thickness, 6 percent, is situated at 42 percent of the chord. The gap between the nose circle of the control surface and the wing is kept as small as possible and amounts to about 0.1 mm. Both upper and lower surface are provided with 19 pressure holes. A cross section of the wing with the location of these holes is given in fig. 1. At a distance of 6 percent of the chord from the leading edge both upper and lower surface are provided with a 10 mm wide transition strip of carborundum grains, to obtain a turbulent



PROFILE: NACA 65 A 006

Fig. 1. Chordwise location of pressure holes on both upper- and lowerside.



Fig. 2 Test set up in the wind tunnel.

boundary layer. The transition of the boundary layer has been checked by acenaphtene tests.

The model, that spans the test section horizontally, is clamped at both sides in perspex windows in the tunnel side walls. At each side of the model the control surface is extended through the perspex windows by means of a bar. The end of the bar is connected to an electrodynamic exciter by a lever (fig. 2). The exciters force the control surface into harmonic oscillations of the desired frequency and amplitude. The control surface motion is measured with displacement pickups connected to the bar just outside the wind tunnel.

Fig. 3a Model with connected scanningvalve.



Fig. 3b Model mounted in wind tunnel.

Fig. 3 gives photographs of the model with connected scanning valve, and of the model mounted in the wind tunnel.

2.3 Pressure measuring technique

The instationary pressure distributions have been obtained by aid of the method described in ref. 14. Its special feature is that pressure tubes are used to connect the pressure holes at the model surface to a scanning valve with one transducer, outside the wind tunnel.

The main problem in performing instationary

measurements in this way is to reduce the pressures recorded by the transducer to the actual pressures on the model surface with the help of the transfer functions of the tube-transducer system.

The transfer of oscillatory pressures through thin tubes depends a.o. on the mean pressure inside the tubes and as in transonic flow rather large differences in static pressure on the aerofoil occur, the oscillatory pressure perturbations have to be measured simultaneously with their corresponding mean pressure to determine the correct conditions for the reduction procedure. In behalf of this reduction procedure the



Fig. 4 Dynamic response characteristics of the pressure tubes.

transfer functions of the tubes have been determined experimentally as a function of the static pressure down to 0.35 ata.

Special attention has been paid to the choice of the pressure tube-transducer system. With the calculation method of ref. 15, tube dimensions are estimated such that for the frequencies to be measured a suitable response characteristic is obtained. The criterion applied is that at the test frequencies the corrections with respect to static pressure are small and that these frequencies do not coincide with the resonance frequencies of the tube-transducer system to suppress possible non-linearities in the oscillatory pressure transfer.

For the present investigation it was suitable to use two lengths of tubes; the shorter tubes for the measurements at 30 and 60 c/s and the longer ones for the measurements at higher frequencies (see fig. 4). In the latter case all tubes have been elongated simultaneously by placing one additional tube between the scanning valve and the pressure transducer. To distinguish nonsymmetrical flow patterns, the pressures on the upper and lower surface of the model have been measured separately, not their differences.

Finally it must be noted here that the equipment used measured only pressure perturbations having the same frequency as the control surface oscillation.

2.4 Schlieren and shadow graph observations

Parallel to the pressure measurements some schlieren and shadow graph pictures were taken using a stroboscopic light source, triggered by an electrical signal of one of the displacement pick-ups. By means of an adjustable phase shift in the circuit between displacement pick-up and light source, the picture of the oscillating model with its shock pattern could be fixed artificially in each desired position.

Due to the excitation mechanism only a part of the model could be observed, thus detailed shock motions on the control surface (M > 0.94) and boundary layer phenomena could not be studied.

3 Test programme

All pressure measurements have been performed with the model kept at zero incidence and the control surface oscillating about its midposition as good as possible. During the tests, frequencies of 30, 60, 90, 120 and 150 c/s were used at Mach numbers equal to 0.5(0.1)0.8; 0.825(0.025)0.90 and 0.92(0.02)1.02. The maximum values of the reduced frequency varied from k=0.478 at M=0.5 to k=0.187 at M=1.02.

For the major part of the tests the amplitude of oscillation was about 1.5 degrees. To study the influence of amplitude some additional measurements have been made for control surface amplitudes of about 0.8, 2.4 and 3 degrees.

The pressure distributions for the limiting case of zero frequency have been obtained from a series of static measurements. As in the instationary tests, these measurements have been restricted to zero angle of attack of the main surface.

To have some indication for the occurrence of wind

tunnel resonance, (see section 6) six additional pressures have been measured along the side wall of the test section in the neighbourhood of the slotted walls. The location of the additional pressure holes is given in fig. 5.



Fig. 5 Transonic test section of the Pilottunnel.

A complete survey of the pressure measurements is given in table 1.

For a limited number of tests the periodical motion of the shock waves has been studied by means of schlieren and shadow graph methods. The programme of these experiments is given in table 2.

4 Results

4.1 Accuracy

It is very difficult to estimate the errors in the final results, due to the complexity of the measuring equipment and the procedure used.

The pressures determined by the transducer in the scanning valve at the end of the tubes are estimated to have an error of about 2 percent for the average pressures and 2 degrees in phase angle. The same accuracy is supposed to exist for the transfer functions of the tubes used in the correction procedure. As in both cases the errors have a random character, the errors in the calculated magnitudes and phase angles of the pressures at the model surface are assumed to be well within 4 percent respectively 4 degrees.

Other inaccuracies may be introduced by an incorrect model motion. For the frequencies applied, the fixed wind part was sufficiently rigid to prevent any deformation. The control surface itself could deform elastically in torsion under the combined action of inertia and aerodynamic forces. This deformation leading to a small additional rotation of the measuring section with respect to the rotation measured just outside the test section, is taken into account by means of an iteration procedure, using the measured aerodynamic loads and the known elastic and inertia properties of the model. The maximum correction applied amounted to 5 percent in amplitude and 1 degree in phase angle. This correction is considered to be calculated with sufficient accuracy to have a negligible influence on the final error.

Based on the foregoing considerations, the final results are supposed to have errors in magnitude and phase angle well within 5 percent and 5 degrees respectively. However, it may not be excluded completely that one or more systematic errors have been overlooked, especially because the present investigation is one of the first applications of the NLR technique in transonic flow and no other experimental material for direct comparison is available. On the other hand should such errors still be present, they can be expected to affect only the quantitative level of the pressure coefficients and not the trend of the results with the various parameters, of which the Mach number is considered to be the most important.

Another question is how far results, obtained in a wind tunnel, have been influenced by interference of the tunnel walls. However, as no reliable method exists at this moment to calculate these interference effects for a slotted-wall test section, it is not possible to assess the error involved. The results have been presented without tunnel wall correction (see also section 6).

4.2 Static and mean aerodynamic properties

The distribution of local Mach number over the profile with undeflected control surface for several values of the free stream Mach numbers, is given in fig. 6. The distributions show that the critical Mach number is about 0.85. At M=0.875 the shock waves cause locally a considerable variation in local Mach number and at M=0.94 the shock waves stand near the hinge axis.

The variation with Mach number of the static aerodynamic coefficients for normal force, moment about the one quarter-chord axis, normal force on control surface, and hinge moment are shown in fig. 7 for control surface deflections of 1.4 and 2.75 degrees respectively. At Mach numbers above 0.875 the effectiveness of the control surface strongly decreases because on the wing part ahead of the shock waves, the change in pressure induced by the control surface deflection is diminished. Above M = 0.92 the hinge moment also decreases rapidly and even becomes negative at $M \sim 0.98$ for a deflection of 1.4 degrees. As this does not occur for a deflection of 2.75 degrees, it can be concluded that for this Mach number the control surface tends to take an equilibrium position either between +1.4 and +2.75 or between -1.4 and 2.75 degrees.

An interesting point forms the variation of the trailing edge pressure coefficient with Mach number, shown in fig. 8; it has been observed (refs. 1, 2, 16) that a sudden change of this pressure coincides with the onset of severe shock induced separation, a phenomenon which can be correlated with the critical Mach number for buffeting and buzz. The upper part of fig. 8 gives this pressure coefficient for the static tests with control surface deflections of 0, 1.4 and 2.75 degrees. Starting at M=0.5 the pressure increases in close agreement with the Prandtl-Glauert rule up to $M \sim 0.92$. In case of the undeflected control surface a sudden change in pressure occurs at M=0.92. At this Mach number the (stationary) shock waves stand near the hinge line. For a deflection of 1.4 and 2.75 degrees a similar behaviour is observed with somewhat lower values for the critical Mach number at which severe separation starts.

In the lower part of fig. 8 the trailing edge pressure coefficient, obtained from the static tests with undeflected control, is compared with the mean value of this coefficient in case of oscillating control. As is shown, the differences between both cases are very small. From the behaviour of the trailing edge pressures of the present wing it can be concluded that the instationary pressure measurements with a control surface amplitude of about 1.5 degrees have not been influenced by severe separation effects for Mach numbers up to 0.90.

It is interesting to know how the mean pressure distributions over the aerofoil with oscillating control deviate from the static pressure distributions, as this indicates to how far the unsteady problem can be considered as a small oscillatory perturbation superimposed on a stationary part. Therefore a comparison between the two pressure distributions is given in fig. 9 for some Mach numbers. From this figure it can be seen that differences occur only in cases where shock waves are present and that they are related directly to the trajectory of the shock motion (see also section 4.4). The discrepancies are largest in the low transonic speed range with relatively large shock displacements. At M = 0.94, where the shocks hardly move, the control surface oscillation again has a negligible influence on the static pressure distribution.

4.3 Instationary pressure distributions

The detailed results of the instationary pressure measurements have been collected in ref. 12. In the present report only a part of the measured pressure distributions will be given.

In the treatment of the results two flow regions will be distinguished, namely:

- region I : subsonic flow over the control surface. This region extends up to M=0.94 and shock waves, if present, remain ahead of the hinge axis.
- region II: mixed subsonic and supersonic, or fully supersonic flow over the control surface. In this region, corresponding to M > 0.94, the shock waves are behind the hinge axis.
- In the following the results for the two flow regimes

will be given and some peculiarities will be elucidated. A more thorough discussion of the measured instationary pressures, emphasizing the influence of Mach number, and a comparison with theory is given in section 5.

4.3.1 Results for region I

The results of the pressure measurements for frequencies of 30, 90 and 150 c/s are presented in the figs. 10, 11 and 12. As the distributions for upper and lower side agree well, only the instationary pressure jump across the chord ΔC_p is given, represented by its real part $\Delta C'_p$ and its imaginary part $\Delta C''_p$: the real part being in phase with the motion of the control surface and the imaginary part being in quadrature with it. In the figures also some theoretical distributions, calculated with the method of ref. 17, are given. In general the agreement between theoretical and measured distributions is reasonable at the lower Mach numbers. In section 5 this comparison will be discussed in more detail.

Considering the development of the measured pressure distributions with Mach number, the attention is drawn to the occurrence of a bubble near the place of maximum wing thickness, beginning already at M=0.80. This bubble grows larger at increasing Mach numbers and from the corresponding distributions of mean local Mach number it can be seen that its location coincides with the region in which the shock waves move. Above M=0.875 the instationary pressures in front of the shock waves start to decrease strongly and at M=0.94 and higher, no pressure disturbances are measured ahead of the hinge line.

The results for 30 c/s, given in fig. 10, show an unexpected, sharp peak at the hinge line in the imaginary part of the pressure distribution. This peak, leading to a negative value of the damping derivative, becomes less pronounced in the tests at higher frequencies. The test data and data reduction procedures have been carefully checked on this point, but no systematic errors or other errors, that might be responsible for this behaviour, could be detected. In connection with this it may be remarked that the same phenomenon has been experienced in the tests of ref. 10, where the same type of irregularity occurred in the measurements at 30 c/s, that disappeared for higher frequencies. At this moment no reasonable explanation of this phenomenon can be given.

4.3.2 Results for region II

The results of the pressure measurements for frequencies of 30, 90 and 120 c/s in flow regime II are given in the figs. 13, 14 and 15. Because the measured instationary pressure distributions on upper and lower surface of the model were found to be not symmetrical, especially at M = 0.96 and 0.98, the results for both sides have been given separately. To facilitate the comparison between upper and lower side pressures, the pressure coefficients C_{p_e} for the lower surface have been plotted with a minus sign. The differences between both sides are probably due to the fact that the control surface did not oscillate correctly about its midposition.

The pressure distributions for flow region II show that the pressure perturbations generated by the control surface oscillation are only felt on the control surface itself. For the various frequencies the pressure distributions qualitatively have the same shape and show the same development with Mach number. The main difference between the results for $\omega = 30$ c/s and those at the higher frequencies is that the sign of the imaginary part for $\omega = 30$ c/s is negative, leading to a negative damping, while the tests at the other frequencies show a positive value.

At M = 0.98 the real parts C'_{p_e} of the pressures, being the pressures at the moment of maximum control surface deflection, do not go to zero at the trailing edge any longer.

At M = 1.00 and 1.02 this is also the case for the imaginary parts.

4.4 Periodical shock motions

At about M=0.85 shock waves start to appear at the model surface, but only during a part of the period of oscillation, and alternately at the upper and lower surface of the aerofoil. At somewhat higher Mach numbers, dependent on the amplitude of oscillation, the shock waves exist during the entire oscillation period and show a periodical motion over the profile; the upper and lower side shock wave being in counter phase.

An example of the periodical shock motion is given in the schlieren photographs in fig. 16, giving the time history of the shock wave displacement during one oscillation period at M=0.90 and $\omega=30$ c/s. Unfortunately, due to the excitation mechanism, only a part of the model can be observed.

The influence of amplitude on the periodical motion of the shock waves is demonstrated in figs. 17 and 18. The figures present the instantaneous position of the shock at the upper surface, together with the corresponding motion of the control surface for M=0.875and 0.90 and frequencies of 30 and 90 c/s. These figures show that at M=0.875 the shock wave at each side disappears during a part of the cycle if the amplitude of the control surface deflection is raised from 1.5 to 3 degrees. Though the motion of the shock wave is periodical, it appears to be not sinusoidal. For the frequencies considered (30 and 90 c/s) only small time shifts occur between shock motion and control surface oscillation.

The influence of Mach number on the behaviour of

the shock waves is illustrated in fig. 19. It can be observed that an increase in Mach number leads to a rearward displacement of the region in which the shocks move, while simultaneously the extent of this region is shrinking. At M = 0.94 the shocks reach the hinge line and their periodical displacements become very small. This effect has also been mentioned by Lambourne (ref. 1), who suggests that, due to the small movement of the shocks in this case, a strong periodical variation in the severity of separation on the control surface is prohibited, which might well account for the absence of buzz in his tests under this condition. As mentioned before, the variation of trailing edge pressure with Mach number (fig. 8) leads to the conclusion that for the present investigation in flow region I no severe separation effects are present.

Another observation is (see fig. 16) that the backward and forward motion of the shocks is coupled with a cyclic change in strength. The more forward the shock is situated the weaker it is, just as in case of quasi-steady flow. Besides this quasi-steady effect, Lambourne (ref. 1) drew attention to the fact that in a uniform stream an upstream moving shock becomes stronger than a downstream moving one and he has shown experimentally that a shock moving forwards over an aerofoil is stronger than a similar situated shock moving rearwards. In the photographs presented in fig. 16, dealing with relatively low frequencies, this dynamic effect hardly can be observed, but it may be expected to play a more important role in case of higher shock velocities, i.e. at higher frequencies or larger amplitudes.

In contrast with the present results, Lambourne found that in his experiments the shock position varied approximately sinusoidally (see fig. 22 of ref. 1) and that the control surface did not oscillate about its midposition. One must be aware however that his tests were principally different from the present ones, because he investigated the free oscillations of a control surface during buzz, while in the present experiments the control surface was forced into harmonic oscillation about its midposition.

4.5 Aerodynamic derivatives

The aerodynamic derivatives, obtained by integrating the measured instationary pressure distributions, are presented in the tables 3–8. To show the dependence on Mach number, the derivatives have been plotted as a function of this parameter in the figs. 20–25. The theoretical curves, drawn also in these figures, have been obtained from the tables of ref. 18. As could be expected, the derivatives vary strongly with Mach number above the critical Mach number. The sharp decrease of the normal force derivative k_c and the moment derivative m_c above M=0.9 can be related to the drop in magnitude of the pressure disturbances in front of the shock waves, as can be seen from the pressure distributions of figs. 10, 11 and 12. The derivatives for the control surface $(r_c \text{ and } n_c)$ start to fall off at M = 0.94, being the Mach number at which the shock waves reach the hinge line. It must be noted here that above M = 0.94 the pressure perturbations only act on the control surface itself, implying that the derivatives k_c and r_c are identical and that between m_c and n_c only the momentum points differ.

At M = 1.0, the fact that the pressure difference does not go to zero any longer at the trailing edge (see figs. 13, 14, 15) causes the increase of the derivatives, especially of the hinge moment derivative n_c , at this Mach number.

The results of the repeated tests (fig. 22) show that a good repeatability of the test results has been obtained.

The theoretical curves in figs. 20–25, obtained from ref. 18, that gives two-dimensional derivatives up to M=0.8, are based on linearized potential theory and do not include thickness and viscosity effects. Compared with experiment in general the trend with Mach number is predicted accurately. The quantitative agreement becomes better as the frequency increases; the best agreement is obtained at the frequency of 150 c/s (fig. 25).

The influence of the amplitude of oscillation on the control surface derivatives is shown in figs. 26 and 27 for some values of the Mach number. Both figures indicate a tendency of decreasing values of the derivatives with increasing amplitude, except for M = 0.96. In the latter case a stronger non-linear behaviour occurs, that must be attributed probably to the large periodical motion of the shock waves on the control surface itself. From the results it can be concluded that, in case the shocks move ahead of the hinge line, the dependence on amplitude of the derivatives in the present tests is hardly influenced by the Mach number. This conclusion, however, is only valid in the range of the rather small amplitudes applied. According to ref. 19, dealing with measurements on a series of wing half models with oscillating control surface (a.o. NACA 65A006 profile), it has been experienced that at amplitudes larger than 3 or 4 degrees, the derivative $n_c^{\prime\prime}$ shows a strong non-linear variation with amplitude, particularly at Mach numbers above $M \sim 0.90$.

5 Further discussion of the instationary pressure distributions and comparison with theory

In analyzing the results for the instationary pressure distributions, the emphasis will be laid on the development with Mach number of the distributions, obtained for various frequencies.

As already mentioned in section 4.3, two flow conditions, related to the situation with undeflected control surface, will be distinguished.

- region I : subsonic flow over the control surface. In this region, extending to M = 0.94, the shock waves remain ahead of the hinge line and the pressure perturbations, generated by the oscillating control surface, are felt on both wing and control.
- region II: Mixed subsonic and supersonic or fully supersonic flow over the control surface (M > 0.94).

In this region the shock waves are behind the hinge axis and the pressure perturbations from the control surface act only on the control surface itself.

5.1 Region I

The influence of Mach number on the instationary pressure distributions can be illustrated best by taking fig. 28, that gives the distributions for both $\omega = 0$ and $\omega = 150$ c/s as a function of Mach number. The pressure coefficients for $\omega = 150$ c/s have been plotted as magnitude $|\Delta C_p|$ and phase angle ϕ , being the phase shift between the pressure difference ΔC_p and the motion of the control surface. In this figure also the theoretically predicted pressure distributions according to ref. 17 have been given.

Fig. 28 clearly shows that the measured distribution of the pressure magnitudes at $\omega = 0$ and at $\omega = 150$ c/s develop qualitatively equally with Mach number. At M = 0.80, when near to the point of maximum thickness the local airspeed on the model surface approaches the speed of sound, a bubble arises in the amplitude distribution. At M = 0.85, when during part of the oscillation period a shock wave occurs alternatively at model upper and lower surface, this bubble becomes more pronounced, due to the travelling shock wave. At still higher Mach numbers the pressure perturbations, evoked by the control surface, are increasingly hampered by the enlarged size of the supersonic region and the strengthened shock wave to reach the part of the wing ahead of the shock.

Regarding the development with Mach number of the measured phase curves, it can be seen that the part of the phase curves behind the hinge line hardly changes, but that the slope of the curve in front of the hinge axis increases with increased Mach number. Especially above M=0.85 a rather sharp increase in slope of the phase curve can be observed at the end of the supersonic region over the wing, becoming steeper with increasing Mach number.

Essential for the behaviour of the measured curves is the fact that in the stationary flow field around the aerofoil, velocity gradients both parallel and normal to the direction of the undisturbed flow exist, caused by the thickness distribution of the model. This will be elucidated qualitatively by considering in detail the pressure propagation in the flow field. For the sake of simplicity the oscillating control surface will be regarded as a unique source, located at the hinge axis. Then in a similar way as given by Spee (ref. 20), who studied the propagation of acoustic waves past twodimensional, quasi-elliptical aerofoils, the time history of the forward travelling wave front can be constructed rather easily, because the local propagation velocity of the wave front is equal to the vector sum of the local speed of sound, taken normal to the wave front and the flow velocity in the point considered. As for the present model only the velocity distribution at the model surface is known, an estimate has been made for the outer field. For simplicity, the reflected wave from the tunnelwalls also has been left out of consideration.

In case of an entirely subsonic flow, corresponding to a free stream Mach number of about 0.8, the position of an acoustic wave after equal time intervals Δt is indicated in fig. 29a. The solid lines at the upper surface show the behaviour of the wave in a flow field with velocity gradients parallel and normal to the undisturbed flow. For comparison, the dotted lines at the lower surface give the time history of such a wave in a uniform flow of M=0.8. The corresponding time lags, with respect to the time of departure from the source, are given in fig. 29b. In case of harmonically oscillating pressure disturbances, this time lag is a direct measure for the phase shift.

It can be concluded that for the non-uniform subsonic flow field the upstream propagation speed of the disturbance waves is mainly determined by the difference between the local speed of sound and the local flow velocity *at* the model surface. The velocity gradients normal to the chord cause a somewhat faster upstream travelling of the disturbances at some distance from the aerofoil, resulting in a forward inclination of the wave front. In other words, the forward travelling disturbances encounter more "headwind" in the vicinity of the aerofoil than further away.

At Mach numbers, at which a local supersonic region over the aerofoil is present, the inclination of the wave front is essential to make it possible for the wave to penetrate the supersonic region. This is demonstrated in fig. 30a, giving the time history of a disturbance wave at M = 0.875 for the estimated flow field. Now the lower part of the upstream moving wave merges with the relatively small shock wave, that terminates the supersonic region. The upper part of the wave front turns around the top of the finite shock wave and than moves into the supersonic region. As can be seen from

figs. 30a and 30b, a rather long travel time is needed in the vicinity of the shock wave, giving rise to a local steepening of the curve for the time lag. This is in qualitative agreement with the measured phase curve at this free stream Mach number, shown in fig. 28.

At increased Mach numbers both strength and length of the shock wave are increased and this is

considered to be the main reason for the observed decrease in amplitude of the pressure disturbances measured in front of the shock. Therefore it may be expected in free flight also, that the pressure disturbances in front of the shock wave rapidly decrease with increased Mach number. In the wind tunnel this may be more pronounced, because the inclined waves have to pass between the top of the shock wave and the tunnel wall. With increasing Mach number this passage becomes smaller and at M = 0.92 it apparently becomes impossible for the disturbances to reach the part of the wing ahead of the shock wave.

Comparing the experimental pressure distributions with the theoretical results, it must be kept in mind that the theoretical predictions are based on a numerical solution of the linearized potential equation for a uniform main flow and thus do not include the effects of thickness, shock waves and viscosity, except for the Kutta-condition.

So, a reasonable agreement between theory and experiment may be expected only at relatively low free stream Mach numbers, where the assumption of uniform basic flow is approximately valid and no shock waves are present.

For M = 0.5 this is certainly true and then theory and experiment (see fig. 28) are in good agreement, except for the more observed discrepancies, such as slightly smaller magnitudes and an increasing phase lead towards the trailing edge of the measured pressures. Both effects may be attributed mainly to viscosity.

Fig. 28 also shows that the influence of the nonuniform flow field in the experimental case starts to manifest itself already at M = 0.80 by a slight bubble in the amplitude distribution of the pressures near the maximum profile thickness and by an increased phase lag towards the leading edge. At higher Mach numbers these effects become more pronounced, resulting in increased differences between the theoretical and measured results. Yet, up to the critical Mach number $(M \sim 0.85)$ the agreement appears to be not too bad.

At supercritical Mach numbers the additional effect of shock waves with their periodic backward and forward motion increases the mentioned bubbles in the measured pressure amplitude distribution. Also this effect can not be described by the theory applied.

The qualitative agreement between the travel time of a wave, propagated by a single source in a nonuniform flow and the measured phase curves at corresponding Mach numbers (compare figs. 29b and 30b with fig. 28) suggests a simple modification of the theoretical phase curves in front of the hinge axis. This modification consists of increasing the theoretical time lag with the difference in travel time in nonuniform and in uniform flow of a wave, generated by a source at the hinge axis. This means the differences between the drawn and dotted curve of fig. 29b, respectively fig. 30b, expressed as phase angle, to be added to the corresponding theoretical phase curves. The results, presented in fig. 31, show the striking improvement obtained by this modification, especially for the supercritical Mach number.

5.2 Region II

In this flow regime flow separation plays an important role at the Mach numbers, whereby the shocks are moving on the control surface itself (M = 0.96 and 0.98). However, since no optical observations could be performed in this region, it is difficult to give a thorough explanation of the instationary pressures at this moment. The instationary pressure distributions for $\omega = 0$ and $\omega = 120$ c/s have been plotted versus Mach number in fig. 32. The results are given again as magnitude and phase angle.

As already mentioned in section 4.3, rather large differences between the pressures on the upper- and lower surface of the model have been measured, especially at M=0.96 and M=0.98. Probably the differences are due to the fact that the control surface did not oscillate correctly about its midposition, in spite of the precautions taken. The non-symmetrical distribution of the mean local Mach number on both model sides also indicates in this direction. It is thought that in this speed range the negative value of the mean hinge moment (fig. 7) is mainly responsible for this behaviour.

Comparing the amplitude distributions for $\omega = 0$ and for $\omega = 120$ c/s it appears that qualitatively they have the same shape and show the same development with Mach number. Between M = 0.94 and M = 1.00the phase curves behave rather irregularly; at M = 1.00and M = 1.02 the measured phase angles being rather small.

From the discussion in section 5.1, it is clear that the theory, used in refs. 17 and 18, is not adequate to describe the pressure distribution for region II in a reasonable way. On the other hand the piston theory is known to work only satisfactorily at rather high supersonic speeds. To gain experience on this point the experimental pressure distributions at M = 1.0have been compared with the results of piston theory, using the average Mach number over the chord. Furthermore the finite thickness of the control surface has been taken into account. The results given in fig. 33, show the rather good prediction of the instationary pressures obtained, especially for the amplitude distribution. The hinge moment derivative, predicted by piston theory, agrees reasonably with experiment. Especially above k = 0.15, this agreement is considerably better than the theoretical values, taken from ref. 18, even if in the latter case the average Mach number over the control surface is used. However, the material is too incidental to draw general conclusions at this moment.

6 Wind tunnelwall-interference

One of the problems encountered in unsteady wind tunnel tests may be the influence of the tunnelwalls on the results, especially for high subsonic and transonic flow. In this speed range the phenomenon of wind tunnel resonance may be encountered. This occurs if the disturbances from the model require an odd integer number of oscillation periods to travel from the model to the wall and back.

As shown by Runyan, Woolston and Rainey (ref. 21), the lowest resonance frequency for a uniform flow in a closed wall wind tunnel can be calculated from the formula

$$\omega_r = \frac{a_\infty}{2H} \sqrt{1 - M_\infty^2} \ c/s \, .$$

where *H* is the height of the test section.

Thus resonance happens for certain combinations of frequency, tunnel height and Mach number.

The possibility of resonance in a wind tunnel with slotted walls has been studied theoretically by Acum (ref. 22). It appears that resonance is anticipated at a frequency that depends also on the open area ratio of the slotted walls.

In applying Acum's results to the NLR Pilottunnel, having an open area ratio of 0.1, the lowest resonance frequency is estimated to be:

$$\omega_r = \frac{5.3a_\infty}{2\pi H} \sqrt{1 - M_\infty^2} \, c/s \, .$$

Fig. 34 gives the predicted values of the resonance frequency as a function of Mach number. The figure shows that resonance is only to be expected in the measurements near M = 1.0. It must be kept in mind, however, that the theoretical prediction assumes a constant subsonic Mach number across the test section and does not account for the influence of the plenum chamber, that also may be important.

To have some indication for the occurrence of resonance in the present tests six additional pressures have been measured along the sidewall of the test section in the neighbourhood of the slotted walls. The location of the additional holes is given in fig. 5.

As an example, the instationary pressure perturbations measured near the floor of the test section, during the tests at $\omega = 120$ c/s, are given in fig. 35. The pressures have been related to the average of the instationary pressure on the lower side of the control surface. In the figure also the phase lag ϕ has been given, calculated from the formula

$$\phi = \frac{\pi \omega H}{a_{\infty} \sqrt{1 - M_{\infty}^2}} \, \mathrm{rad}$$

derived by considering the travel time necessary for the disturbances, generated in a uniform flow by a source at the hinge axis, to reach the tunnel wall. Up to M = 0.875 the ratio of the pressures remains nearly constant and the phase angle increases gradually, in the same way as the theoretical curve. At higher Mach numbers the magnitude of the pressure perturbations and the phase angle become smaller with increasing Mach number. Probably this is caused by the occurrence of the shock waves. The pressure measured below the slots (n = 1) behaves qualitatively in the same way as the pressures just above the slots (n = 2, 3). At $M \sim 0.98$ the wall pressures become negligible because then the pressure holes come in front of the shock waves.

As it can be expected that at resonance conditions the fundamental of the measured signal will show a local peak in amplitude and a phase shift of 90 degrees with the pressure disturbances at the control surface, the variation of the measured wall pressures with Mach number gives no indication for the occurrence of resonance. Furthermore the comparison of the unsteady pressure distributions on the oscillating control surface with those for k=0, obtained from static measurements, also does not show pronounced irregularities that can be attributed to tunnel resonance, so it is concluded that resonance did not occur during the present investigation.

Recently it has been reported by Wight (ref. 23) that especially slotted wall test sections seem to induce large interference effects. This evidence is based on a comparison of aerodynamic derivatives ($k \le 0.05$), obtained from measurements under different wall conditions in various wind tunnels. Unfortunately the interference observed in these wind tunnels did not affect the results in the same way. Wight concludes, however, that the interference effects can not be attributed to tunnel resonance and that further experimental investigations are needed.

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In an attempt to explain the abovementioned interference effects, Garner and Moore (ref. 24) present an approximate method for the treatment of the problem for slotted walls. Their method is based on some interpolation between theoretical solutions for the limiting cases of closed and open roof and floor of a rectangular test section. It is shown that for the damping in pitch of some three-dimensional models, the right order of magnitude of the wall influence is predicted. A severe limitation is that the method is restricted to very low reduced frequencies.

Until now, however, a general and reliable method to reduce wind tunnel results for slotted and porous walls to free flight conditions is still lacking, as has been concluded also by Acum in his recent review (ref. 25) and therefore the results of the present report have been given without any correction for tunnel wall interference.

7 Comparison with other test results

As no other results for measured instationary pressure distributions are available, the comparison has to be restricted to the hinge moment derivative. Also in that case, the comparison is hampered by the fact that either the relative control surface chords or the maximum profile thicknesses differ.

The tests described by Loiseau (refs. 5 and 6) also deal with a 6 percent aerofoil, but the relative control surface chord is somewhat larger than for the present model, being 30 instead of 25 percent. The instationary hinge moment has been measured with the free oscillation technique and no results are given in the range $0.8 \le M < 1.0$. However, in the ranges of Mach number overlapping the present tests, qualitatively the same variation of the hinge moment derivative with Mach number can be observed (see fig. 36). The differences with the corresponding theoretical derivatives are in both cases of the same order of magnitude.

In fig. 37 a comparison is made with the results of Nakamura and Tanabe (ref. 4). The wing-control surface combination considered has the same chord ratio as the NLR model, but the maximum thickness is 10 percent. Their results have been obtained by means of free oscillation tests with a small model (chord 8 cm). The initial amplitude of 10 degrees, that has been applied, is thought to be somewhat large, especially because the tests of ref. 19 have indicated that, at amplitudes larger than about 3 degrees, the derivative $n_c^{"}$ shows a rather strong non-linear variation with amplitude above $M \sim 0.9$. Notwithstanding the different relative thicknesses and the different test conditions, the agreement between both test results is rather good.

A comparison with the results of Wyss and Sorenson (ref. 8), who determined the hinge moments by means of instationary pressure measurements on a 13 percent thick NACA 65₁-213 aerofoil with a flap of 25 percent, is given in fig. 38. The comparison has been restricted to M=0.7, though ref. 8 gives results up to M=0.8. However, at the latter Mach number the wing of ref. 8 is still supercritical, while the present wing is not. The values of n'_c show a reasonable agreement with the present results, but the imaginary parts n''_c differ widely. Probably this discrepancy must be attributed mainly to the large difference in profile thickness and camber. The values of n''_c of the present investigation appear to fit the theoretically predicted curve somewhat better. In fig. 38 also, some values have been plotted of ref. 4, that agree rather well with the results of the present tests. Furthermore a comparison is made with the results of former NLR tests (ref. 11), obtained from instationary pressure measurements on a 8 percent thick symmetrical circular arc profile. At the rear part of this aerofoil severe flow separation has been observed and probably this separation is responsible for the rather small values of n'_c , being considerably smaller than the values of refs. 4 and 8, and the present tests. Yet, the values of n'_c of ref. 11 appear to agree reasonably well with those of the present investigation and the corresponding theoretical prediction.

8 Concluding remarks

The results of a number of instationary pressure measurements and of some schlieren and shadow graph observations, both obtained for a two-dimensional wing with oscillating control surface in high subsonic and transonic flow (up to M = 1.02), have been presented and analysed.

The instationary pressure distributions are shown to depend strongly on Mach number, especially if local supersonic regions are present (M > 0.85). It appears that for Mach numbers higher than 0.94 the control surface oscillation is unable to generate pressure fluctuations at the aerofoil ahead of the hinge axis. The distribution of the amplitudes of the instationary pressures develops with Mach number qualitatively in the same way as the pressure distribution for the limiting case of zero frequency, obtained from static tests.

In case of supersonic regions located entirely in front of the hinge axis, the non-uniform basic flow around the aerofoil plays an essential role in the upstream propagation of the pressure disturbances evoked by the oscillating control surface. This can be elucidated by considering the propagation of the waves generated by an acoustic source at the hinge axis.

Up to the critical Mach number $(M \sim 0.85)$ the agreement between the measured instationary pressure distributions and results obtained from a linearized theory for subsonic, potential flow appears to be not too bad. The discrepancies between the theoretical and measured phase curves in front of the hinge axis are reduced considerably by taking into account, in a very approximate way, the extra time lag caused by the non-uniform flow field.

The oscillatory motion of the shock waves, that could be studied only up to M=0.94, appears to be periodic but not sinusoidal. For the frequencies considered, the time lag with respect to the control surface motion is very small.

Finally it has been concluded from the behaviour of the additional pressures, measured near the roof and floor of the test section, that tunnel resonance did not occur in the present investigation.

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TABLE 1
Summary of executed pressure measurements

	static tests					instatio	nary tests				
Mach		ω=	30 c/s	ω=	60 c/s	ω=	90 c/s	ω=1	20 c/s	$\omega = 1$	150 c/s
ber	С	 k	С	k	C	k	C	k	C	k	С
0.50	0; 1.4; 2.0; 2.75; -2.0	.096	1.46°	.191	1.91°	.287	1.50°	.382	 1.64°	.478	1.71°
0.60			_	.159	1.48		-				
0.70		.068 .068	1.45 1,46	.137 .137	0.85 1.91	.205	1.78	.273	1.60	.341	1.67
				.137 .137	2.34 3.14						
0.80		.060	-1.59	.119 .119	1.79 1.77	.179	1.72	.239	1.57	.299	1.77
0.825		.058	1.58	.116 .116	1.73 1.93	.174	1.67	.232	1.72	.290	1.82
0.85		.056	1.53	.112 .112	1.80 1.86	.169	1.71	.225 .225 .225	0.97 1.73 2.57	.282	1.63
0.875		.055	1.47	.109 .109	1.47 1.90	.164	1.75	.218	1.72	.273	1.69
0.90		.053	1.40	.106	1.83	.159	1.75	.212	1.72	.266	1.71
0.92		.052	1.47	.104 .104 .104 .104	0.84 1.80 2.44 3.18	.156	1.76	.208 .208 .208	0.84 1.70 2.39	.260	1.76
0.94		.050 .051 .051 .051	1.52 1.54 1.77 1.57	.101 .102 .102	1.43 1.74 1.74	.153 .152	1.55 1.59	.204 .203	1.61 1.68	.254	1.72
0.96		.050	1.76°	.100 .099 .100 .100 .100	0.85° 1.65 1.82 2.37 3.01	.149	1.71°	.200 .199 .199	0.88° 1.54 2.41		
0.98		.049	1.73	.097 .098	1.60 1.79	.147	1.73	.195 .194	1.77 0.95		
1.00		.047 .047	1.53 1.69	.096 .095 .096 .096	0.80 1.51 1.69 2.96	.144	1.60	.191 .191	1.57 2.46		
1.02	0; 1.4; 2.0 2.75; -2.0	.047 .046	1.70 1.49	.093 .094	1.41 1.59	.140	1.61	.187	1.49	-	

C = amplitude of control surface deflection

k = reduced frequency

 TABLE 2

 Executed test programme for shadow and schlieren observations.

Mach number		schlieren				
	$\omega = 30$	c/s	$\omega = 90$	c/s	$\omega = 2$	30 c/s
	, c	k	с	k	с	k
0.85	1.5; 3°	.056	1.5; 3°	.169	1.5°	.056
0.875	1.5; 3°	.055	1.5; 3°	.164	1.5°	.055
0.90	1.5; 3°	.053	$1.5; 3^{\circ}$.159	1.5°	.053
0.92	1.5; 3°	.052	1.5; 3°	.156	1.5°	.052
0.94	1.5; 3°	.051	1.5, 3°	.153	1.5°	.051

TABLE 3
Aerodynamic derivatives $\omega = 0$ c/s

M k	k	С	k,		m		r _c		n _c	
			re	im	re	im	ге	im	re	im
.50	0	2°	+ 1.036	0	+.455	0	+.2335	0	+.0315	0
.70	0	2	+1.126	0	+.502	0	+.2456	0	+.0335	0
.80	0	2	+1.218	0	+.583	0	+.2836	0	+.0361	0
.825	0	2	+1.250	0	+.611	0	+.2907	0	+.0369	0
.85	0	2	+1.302	0	+.654	0	+.2896	0	+.0362	0
.875	0	2	+1.394	0	+.787	0	+.2623	0	+.0331	0
.90	0	2	+1.253	0	+.940	0	+.2908	0	+.0364	0
.92	0	2	+ .768	0	+.715	0	+.3581	0	+.0493	0
.94	0	2	+ .386	0	+.402	0	+.3099	0	+.0474	0
.96	0	2		0	_	0	_	0		0
.98	0	2	+ .200	0	+.200	0	+.1680	0	+.0044	0
1.00	0	2	+ .370	0	+ .462	0	+.3700	0	+.0925	0
1.02	0	2	+ .362	0	+.456	0	+.3620	0	+.0940	0

TABLE 4 Aerodynamic derivatives $\omega = 30$ c/s

М	k	С	k _c		n	1 _c	r _c		n _c	
			re	im	re	im	re	im	re	im
.50	.096	1.46°	+ .933	356	+.448		+.2072	0452	+.0287	0055
.70	.068	1.45	+.891	474	+.462	115	+.2034	0698	+.0269	0074
.70	.068	1.46	+.936	503	+.491	132	+.2175	0787	+.0301	0095
.80	.060	1.59	+.893	618	+.521	181	+.2161	0936	+.0300	0110
.825	.058	1.58	+.891	641	-+ .544	189	+.2193	0915	+ .0301	0099
.85	.056	1.53	+.869	696	+.575	220	+.2236	0929	+.0311	0102
.875	.055	1.47	+.715	672	+.556	269	+ 1998	0607	+.0268	0057
.90	.053	1.40	+.609	709	+.588	470	+.2180	0648	+.0297	0037
.92	.052	1.47	+.443	418	+.471	367	+.2640	1615	+.0401	0190
.94	.050	1.52	+.291	170	+.328	168	+.2601	1316	+.0425	0172
.94	.051	1.54	+.288	167	+.328	162	+.2630	1266	+.0438	0169
.94	.051	1.77	+.276	161	+.312	155	+ .2482	- 1199	+.0421	0163
.94	.051	1.57	+.266	163	+.305	158	+.2473	1237	+.0413	0164
.96	.050	1.76	+.159	032	+.178	020	+.1477	0124	+.0261	+.0027
.98	.049	1.73	+.086	041	+.083	028	+.0790	0289	0025	+.0078
.997	.047	1.53	+.134	065	+.133	052	+.1171	0503	+.0082	+.0017
1.00	.047	1.69	+,264	109	+.333	136	+.2640	1090	+ .0686	0273
1.01	.047	1.70	+ .244	112	+.310	140	+,2440	1120	+.0659	0280
1.02	.046	1.49	+.248	091	+.315	114	+.2480	0910	+ .0670	0228

TABLE 5
Aerodynamic derivatives
$\omega = 60 \text{ c/s}$

M k	k	С	 /	k _e	'n	n _c		с	n _c	
			re	im	re	im	ге	im	re	im
.50	.191	1.91°	+.586	263	+.339	010	+.1374	0032	+.0193	+.0020
.60	.159	1.48	+.538	339	+.345	050	+.1405	0187	+.0196	+.0000
.70	.137	0.85	+.510	384	+.373	064	+.1550	0213	+.0210	0008
.70	.137	1.91	+ .465	365	+.323	054	+.1246	0163	+.0178	+.0002
.70	.137	2.34	+.448	360	+.312	062	+.1189	0182	+.0171	0001
.70	.137	3.14	+.421	357	+.293	065	+.1091 -	0189	+.0162	0003
.80	.119	1.79	+.394	411	+.336	081	+.1259	0179	+.0178	+.0003
.80	.119	1.77	+.394	420	+.337	091	+.1262	0215	+.0180	0001
.825	.116	1.73	+.365	430	+.341	099	+.1241	0197	+.0172	+.0002
.825	.116	1.93	+.357	435	+.344	098	+.1254	0192	+.0179	+.0002
.85	.112	1.80	+.325	430	+.353		+.1257	0158	+.0177	+.0009
.85	.112	1.86	+.323	451	+.361	118	+.1288.	0161	+.0183	+ .0007
.875	.109	1.47	+.299	466	+.391	184	+.1440	+.0019	+.0196	+.0038
.875	.109	1.90	+.224	457	+.345	192	+.1337	0061	+.0191	+.0027
.90	.106	1.83	+.130	390	+.250	276	+.1413	0170	+.0213	+.0026
.92	.104	0.84	+.230	247	+.304	231	+.2018	1082	+.0319	0108
.92	.104	1.80	+.168	229	+.235	196	+.1416	0674	+.0242	0054
.92	.104	2.44	+.143	208	+.202	185	+.1244	0664	+.0212	0067
.92	.104	3.18	+.127	196	+.183	170	+.1121	0564	+.0197	0057
.94	.101	1.43	+.151	078	+.198	071	+.1584	0547	+.0275	0063
.94	.102	1.74	+.148	060	+.187	046	+.1522	0344	+.0270	0028
.94	.102	1.74	+.136	070	+.176	057	+.1427	0427	+.0253	0042
.96	.100	0.85°	+.115	052	+.147	043	+.1261	0391	+.0254	0040
.96	.099	1.65	+.095	007	+.121	+.011	+.1015	+.0102	+.0182	+.0059
.96	.100	1.82	+.080	+.000	+.092	+.020	+.0769	+.0170	+.0139	+.0091
.96	.100	2.37	+.089	018	+.111	001	+.0933	+.0004	+.0174	+.0045
.96	.100	3.01	+.086	037	+.108	028	+ 0894	0223	+.0168	0015
.98	.097	1.60	+.042	004	+.040	+.010	+.0384	+.0040	0039	+ .0097
.98	.098	1.79	+.053	004	+.057	+.015	+.0520	+.0095	+.0016	+.0112
1.00	.096	0.80	+.159	056	+.201	070	+.1590	0560	+.0413	0139
1.00	.095	1.51	+.147	044	+.185	055	+.1470	+.0440	+.0382	0109
1.00	.096	1.69	+.139	046	+.175	057	+.1390	0460	+.0359	0114
1.00	.096	2.96	+.123	040	+.155	050	+.1230	0400	+.0325	0100
1.02	.093	1.41	+.139	048	+.176	059	+.1390	0480	+.0375	0119
1.02	.094	1.59	+.137	040	+.173	050	+.1370	0400	+.0356	0100

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 $\omega = 90 \text{ c/s}$

М	k	С	k,		n	1 _c	r	c	r	1 _c
			re	im	re	im	re	im	ге	im
0.50	.287	1.50°	+.763	+.022	+.411	+.175	+.1775	+.0995	+.0254	+.0171
0.70	.205	1.78	+.563	040	+.352	+.138	+.1458	+.0790	+.0218	+.0137
0.80	.179	1.72	+.503	~.098	+.372	+.135	+.1423	+.0823	+.0207	+.0143
0.825	.174	1.67	+.487	~.120	+.395	+.134	+.1461	+.0857	+.0212	+.0149
0.850	.169	1.71	+.439	138	+.418	+.122	+.1450	+.0917	+.0209	+.0161
0.875	.164	1.75	+.349	156	+.437	+.043	+.1576	+.1026	+.0233	+.0188
0.90	.159	1.75	+.264 •	084	+.354	020	+.1768	+.0866	+.0269	+.0184
0.92	.156	1.76	+.269	034	+.347	004	+.2148	+.0266	+.0353	+.0092
0.94	.153	1.55	+.214	+.032	+ .268	+ .061	+.2162	+.0510	+.0386	+.0110
0.94	.152	1.59	+.195	+.032	+.245	+.061	+.1983	+.0465	+.0357	+.0099
0.96	:149	1.71	+.106	+ .072	+.128	+.112	+.1039	+.0976	+.0174	+.0216
0.98	.147	1.73	+.057	+.049	+.050	+.080	+.0481	+.0649	0070	+.0192
1.00	.144	1.60	+.207	+ .056	+.261	+.070	+.2070	+.0560	+.0537	+.0139
1.02	.140	1.61	+.191	+.048	+.244	+.060	+.1910	+.0480	+.0526	+.0120

TABLE 7 Aerodynamic derivatives $\omega = 120 \text{ c/s}$

М	k	С	k _c		m _e		r _c		n _c	
			re	im	re	im	rê	im	re	îm
0.50	.382	1.64°	+.648	028	+.361	+.174	+.1504	+.0942	+.0205	+.0158
0.70	.273	1.60	+.552	080	+.373	+.156	+.1490	+.0922	+.0206	+.0154
0.80	.239	1.57	+.480	116	+.413	+.158	+.1504	+.1028	+.0204	+.0170
0.825	.232	1.72	+.449	136	+.442	+.155	+.1525	+.1105	+.0209	+.0185
0.85	.225	0.97	+.485		+.576	+.130	+.1998	+.1370	+.0265	+.0228
0.85	.225	1.73	+.397	148	+.487	+.124	+.1594	+.1212	+.0220	+.0206
0.85	.225	2.57	+.349	075	+.413	+.163	+.1245	+.1285	+.0177	+.0215
0.875	.218	1.72	+.279	110	+.438	010	+.1827	+.1225	+.0261	+.0218
0.90	.212	1.72	+.243	066	+.337	053	+.1960	+.0837	+.0297	+.0182
0.92	.208	0.84	+.336	059	+.430	060	+.2633	+.0122	+.0400	+.0098
0.92	.208	1.70	+.251	043	+.330	047	+.2077	+.0052	+.0336	+.0065
0.92	.208	2.39	+.226	018	+.304	022	+.1863	+.0230	+.0309	+.0086
0.94	.204	1.61	+.197	+.029	+.255	+.040	+.2069	+.0331	+.0364	+ .0085
0.94	.203	1.68	+.197	+.034	+.259	+.049	+.2115	+.0410	+.0374	+.0098
0.96	.200	0.88	+.180	+ .049	+.235	+.064	+.1936	+.0515	+.0399	+ .0129
0.96 ·	.199	1.54	+ .144	+.071	+.181	+.104	+.1486	+.0893	+.0274	+.0229
0.96	.199	2.41	+.126	+.064	+.157	+.094	+.1280	+.0797	+.0231	+.0197
0.982	.195	1.77	+.079	+.037	+.079	+ .054	+.0731	+.0436	0031	+.0129
0.986	.194	0.95	+.228	+.031	+.290	+.043	+.2280	+.0347	+.0615	+ .0080
1.00	.191	1.57	+.229	+.044	+ .289	+ .044	+.2290	+.0440	+.0595	+.0109
1.00	.191	2.46	4.203	+.040	+.256	+.050	+.2030	+.0400	+.0527	+.0100
1.02	.187	1.49	+.215	+.032	+.273	+.040	+.2150	+.0320	+.0580	+.0060

TABLE 8 Aerodynamic derivatives $\omega = 150 \text{ c/s}$

<i>М</i> .	k	С	k _c		m _c		r _c		n _c	
			re	im	re	im	re	im	te	im
0.50	.478	1.71°	+.699		+.439	+.200	+.1705	+.1160	+.0216	+.0229
0.70	.341	1.67	+.674	317	+.519	+.097	+.1926	+.0887	+.0256	+.0190
0.80	.299	1.77	+.466	-,411	+.544	022	+.1959	+.0720	+.0268	+.0165
0.825	.290	1.82	+.410	429	+.578	074	+.2078	+.0739	+.0290	+.0171
0.85	.282	1.63	+.265	370	+.540	191	+.2206	+.0689	+.0310	+.0169
0.875	.273	1.69	+ .194	210	+.327	205	+.2304	+ .0552	+ .0336	+.0154
0.90	.266	1.71	+.223	163	+.262	145	+.2254	+.0350	+.0350	+.0120
0.92	.260	1.76	+.274	125	+.300	128	+.2070	0281	+.0351	+.0034
0.94	.254	1.72	+.214	+.048	+ 261	+.060	+.2088	+.0506	+.0374	+.0135





Fig. 6 Chordwise distribution of local Mach number for the wing with undeflected control surface.

Fig. 7 Static aerodynamic coefficients versus Mach number.



Fig. 8 Variation of trailing edge pressure coefficient with Mach . number.

Fig. 9 Comparison between the chordwise distributions of the static pressure for the undeflected control and the mean pressure in case of oscillating control.

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Fig. 10 Chordwise distributions of mean local Mach number and instationary pressures for $\omega = 30$ c/s.













Fig. 14 Chordwise distributions of mean local Mach number and instationary pressures for $\omega = 90$ c/s.



Fig. 16 Time history of shock wave displacement during one period of oscillation $(M = 0.90, \omega = 30 \text{ c/s})$.



Fig. 15 Chordwise distributions of mean local Mach number and instationary pressures for $\omega = 120$ c/s.





Fig. 17 Influence of control surface amplitude on the periodical motion of the shock wave for $\omega = 30$ c/s.

Fig. 18 Influence of control surface amplitude on the periodical motion of the shock wave for $\omega = 90$ c/s.





Fig. 21 Aerodynamic derivatives versus Mach number for $\omega = 30$ c/s.



Fig. 20 Aerodynamic derivatives versus Mach number for $\omega = 0$ c/s.





Fig. 19 Shock wave motion on the upper surface at various Mach numbers.



Fig. 23 Aerodynamic derivatives versus Mach number for $\omega = 90$ c/s.



Fig. 25 Aerodynamic derivatives versus Mach number for $\omega = 150$ c/s







Fig. 26 Influence of control surface amplitude on the aerodynamic derivatives for $\omega = 60$ c/s.



w= 120 cps ----- Re ----- m



Upstream propagation of disturbance waves generated by a source at the hinge axis a Wave propagation for non-uniform and uniform basic flow. b Time lag, derived from wave patterns.

(Des) DAD BMIT

(Seec) SAL BMIT



Fig. 32 Instationary pressure distributions for $\omega = 0$ and $\omega = 120$ c/s as a function of free stream Mach number.





Fig. 33a Instationary pressure distributions at sonic speed and comparison with piston theory.

PHASE ANGLE

Fig. 31 Improvement in phase lag, ahead of the hinge axis, obtained by correcting the theoretical results for the additional time losses in non-uniform flow.



Fig. 33b Hinge moment derivative at sonic speed and comparison with theory.



Fig. 34 Predicted tunnel resonance frequencies, using ref. 22.



Fig. 35 Instationary pressures near the floor of the test section compared with the average of the instationary pressure on the lower side of the control surface.



EXPERIMENT :	Ŕ.	τ	MAX. THICKNESS
	~ 0.1	0.3	6 */+
	0.191-0.093	0.25	6•/•

Fig. 36 Comparison with the measured hinge moment derivatives of refs. 5 and 6.





EXPERIMENT:	PROFILE	MAX. THICKNESS	M
X REF.0	NACA 651 - 213	13 %.	0.70
AREF.4	NACA 64 A 010	10 %	0.74
D N.L.R.	CIRCULAR BICONVEX	8 °/•	0.70
o N.L.R.	NACA 65 A 006	6 %	0.70

Fig. 37 Comparison with the measured hinge moment derivatives of ref. 4.

Fig. 38 Comparison with the measured hinge moment derivatives of refs. 4, 8 and 11.
gnibsol lsixsid rebnu Buckling of orthotropic core sandwich panels

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G. Bartelds

Summary

results for biaxial loading in conjunction with exact solutions for uniaxial loading cases appears to be promising. A computer routine for automatic generation of exact interaction curves is outlined. For many practical applications the use of approximate conformance is obtained for the general results when compared with an approximate solution for orthotropic panels also presented in this report. Comparison of the results for the special case of core isotropy with previous approximate results reveals good agreement in most cases. Similar Exact buckling loads and interaction curves are derived for orthotropic sandwich plates with two clamped and two simply supported edges.

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⁶ a ⁶ h to sing y pairs and ⁶ house of the statistical of th	LGL LGL	saund (C
	a ga gu	-bqmoo onanona loi magor (, a vianoque
Plate (see ng. 1).	*8 8 6	-ungroo offemotice rol mersory (O vibrand A
	6	St Monte billio and home o bobbot
Co-ordinates in middle surface of	. A X	loaded orthotronic sandwhich
	**	buckling noblem of a biaking
Deformation vector with components	ü	Appendix C : Approximate solution for the
	a i m	(1 suger on the management of
$= \tilde{u}_2$ stremense displacements $2\tilde{u}_2$	ų n	Appendix B: Buckling conditions for two types
i = 2; lower face laver.		loaded sandwich plate.
i = 1; upper face laver.		Appendix A: Buckling equations for a biaxially
nents.	• • •	5 References. 10
Face parallel displacement compo-	'a ''n	4 Concluding remarks. 10
plate.		3.4 Applicability of present results.
at the middle surface of the composite		3.3 Efficiency of approximate and exact methods 9
Displacement components of points	м 'a 'n	3.2 Results for orthotropic plates. 8
Thickness of face layers.	1	for isotropic plates. 8
$\cdot 0 = x^{d} (x^{d})$		3.1 Comparison of previous and present results
compression, $\tilde{p}_{x} = (p_{x})_{p_{x}=0}$, $\tilde{p}_{y} = 0$	£ 1	3 Discussion of results. 8
Values of load parameters for uniaxial	*₫ ¤₫	6 cores. 6
$b^{\lambda} = \frac{\mu_{z} D^{z}}{2}$		2.5 Buckling of sandwich plates with isotropic
$z^{p^{a}N}$		of two edges.
$D_{\mu\nu} = \frac{\mu_{\tau}}{2} D_{\tau}^{\mu}$	бл схл	2.4 Buckling conditions for relaxed clamping
pue $\frac{zq^x}{N} = 0$ sustaineren peo I	u u	of two edges
Integer buckle wave numbers.	น 'พ	2.3 Buckling conditions for complete clamping
(54).		along two edges.
Coefficients appearing in equation	24	2.2 Solution for one type of simple support
(11), (12) and (13).		£ suoit
Coefficients appearing in equations	14	2.1 Coverning equations and boundary condi-
Complex unit, integer subscript.	1	2 Analysis 3
Coefficients defined in appendix C.	10	
I nickness of core layer.	Э	
Panel dimensions (see fig. 1).	qʻp	əbvə
	, ,	-
	slodmys to teld	stastao.D

С	Extensional rigidity of face layer, $=$	U_i, V_i, W_i	Deformation amplitudes (see eq. (10)).
	$\frac{Et}{1+t^2}$	U V	Potential functional for applied loads
Ds	Bending rigidity of sandwich plate, = $(a + b)^2$	α	Parameter involving plate aspect ratio, $-n^2 a^2/b^2$
	$2C\left(\frac{c+i}{2}\right)^{2}$.	α.α	$-\pi u/b$ Normalized buckling load coefficients
F	Voung's modulus of face layer mate-	ω_x, ω_y	$\alpha = p / p - \alpha = p / p$
L	rial	V V	$\sum_{x} p_{x} p_{x}, \sum_{y} p_{y} p_{y}.$
GG	Transverse shear moduli of core laver	7x77y VV.	Average shear angles.
O_x, O_y	material	12719	
I _i	Surface integrals defined in appendix		$\bar{\gamma}_x = \frac{c}{c+t} \gamma_x, \bar{\gamma}_y = \frac{c}{c+t} \gamma_y.$
- *	C.		
L*	Matrix of linear operators.	δ_i	Kronecker symbol.
$L, L_{\rm b}, L_{\rm H}$	Linear operators.	3	Load configuration parameter,
M_i	Matrices of boundary condition coeffi-		$=p_x/p_y$
	cients, $i = 1$: symmetric buckling mo-	Eris Evis Yrvi	Plane-strain components of a face
	des, $i = 2$: antisymmetric buckling mo-		layer.
	des.	η	Core orthotropy parameter,
M_i, M_i''	Real and imaginary parts, respectively,		
-	of M _i .		$= \sqrt{G_x/G_y}$.
$ M'_{i} , M''_{i} $	Determinant values of M'_i and M''_i .	.,	Wavelength parameter
$ M'_{i}('') , M''_{i}(') $	Quasi-determinant values defined in	μ_i	Real and imaginary parts of a respect
	appendix B.	μ_i, μ_i	Rear and imaginary parts of μ_i , respect-
N_x, N_y	Edge loads per unit width of sand-		Deisgen's rotic for fore lavor material
•	wich plate.	v	Poisson's ratio for face layer material.
R_s	Core shear stiffness parameter,	ρ	Square of plate aspect ratio a/b .
	$= \sqrt{G_{\mathbf{x}}G_{\mathbf{y}}} \frac{(c+t)^2}{c} .$		$\partial \partial \partial^2$
S. S*	Stiffness parameters.	,x ² ,y ² ,xx ³ ···	$\partial x' \partial y' \partial \overline{x^2}' \cdots$
.,	$2S^* \pi^2 D$	17 2 1 4	7 1
	$S = \frac{2B}{1-v} = \frac{\pi B_s}{a^2 R}.$	v², v²,	Laplace operator applied once, twice,
	$1 - \mathbf{v} \mathbf{u} \mathbf{R}_{\mathbf{S}}$		

1 Introduction

The analysis presented in this report was initiated by the late Dr. Plantema. During the preparation of his wellknown book on sandwich construction (ref. 1), he noticed the lack of reliable information on the elastic buckling behaviour of biaxially compressed rectangular sandwich plates, especially for boundary conditions other than those of simple support. In particular, the most advanced results for plates with two or all four edges clamped, as reported by Guest and Solvey (ref. 2) gave rise to serious doubts. As a result an investigation treating the same problem was initiated by De Jonge and Plantema, using a different approach. The inadequacies of reference 2 were brought to the attention of its authors which led to a reconsideration of the problem by Green and Solvey. The results of these two independent investigations are reported in references 3 and 4.

In the analysis of reference 3 Bylaards method of split rigidities is employed in conjunction with an energy approach to obtain an approximate solution to the buckling problem of a biaxially compressed rectangular sandwich plate with two edges clamped and two edges simply supported or with all four edges clamped. The facings are treated as isotropic membranes, the core is considered to be isotropic in transverse shear properties.

In references 2 and 4 an approximate solution is effected by means of the minimum total potential principle. Although the material of both face and core layers is initially treated as an orthotropic medium, results are given for isotropic facings and core only. The effect of finite face bending stiffness is indicated. The principal results of references 3 and 4 are the combinations of dimensionless load parameters p_x and p_y , characterising the biaxial loading condition for which buckling occurs. In both publications it is proposed to construct interaction curves using the critical load parameter combinations of the approximate analysis by multiplying them by the ratio of exact and approximate values for the cases of *uniaxial compression* in either of the two coordinate directions. These values are available from existing literature. Interaction curves constructed in this manner then intersect the co-ordinate axes in the correct points; only the shape is not exact. Although it is plausible to use these "curves" of polygonal shape for design purposes they nevertheless represent the results of a somewhat crude analysis. For example, the true shape of such

2

curves for other than simply supported plates should consist of smooth line segments.

However, an exact solution to the buckling problem of a biaxially compressed rectangular plate of sandwich construction, with two edges clamped and two edges simply supported is readily available. The present report gives the results of such a solution for two types of clamped coundary conditions.

The facelayers are treated as isotropic membranes. This procedure has not only been shown to be fully acceptable in an analysis of classical longwave instability of sandwich plates of practical dimensions but is in fact mandatory if the transverse compressibility of the core is neglected. Inclusion of the face bending stiffness only will lead to slightly unconservative results in many cases. The core is considered to be an orthotropic layer with respect to transverse shear as is appropriate for honeycomb type core materials. The solution for an isotropic core is obtained as a special case.

The careful programming of present solutions for automatic numerical computation by Mr. G. Doekes of the Applied Mathematics and Data Processing Department is gratefully acknowledged.

2 Analysis

2.1 Governing equations and boundary conditions

As shown in appendix A the linear buckling problem of a flat sandwich plate under biaxial inplane loading (see fig. 1) is governed by the sixth-order partial differential equation (A5)

$$\frac{D_s}{R_s} \nabla^4 \left[\bar{u} - \frac{D_s}{R_s} \frac{1 - \nu}{2} \left\{ \eta \bar{u}_{,xx} + \eta^{-1} \bar{u}_{,yy} \right\} \right] + \left[\frac{N_x}{R_s} ()_{,xx} + \frac{N_y}{R_s} ()_{,yy} \right] \\ \left[\left(\frac{D_s}{R_s} \right)^2 \frac{1 - \nu}{2} \nabla^4 \bar{u} - \frac{D_s}{R_s} \left\{ \eta \bar{u}_{,yy} + \eta^{-1} \bar{u}_{,xx} + \frac{1 - \nu}{2} (\eta \bar{u}_{,xx} + \eta^{-1} \bar{u}_{,yy}) \right\} + \bar{u} \right] = 0 \quad (1)$$

where u is a deformation vector with components $\tilde{u}\left(=\frac{u_1-u_2}{2}\right)$, $\tilde{v}\left(=\frac{v_1-v_2}{2}\right)$ and w (see fig. 1). The parameters D_s , R_s and η are defined as

$$D_{s} = \frac{Et(c+t)^{2}}{2(1-v^{2})} = \frac{C(c+t)^{2}}{2} \quad \text{(bending stiffness of sandwich plate with membrane facings)}$$

$$R_{s} = \sqrt{G_{x}G_{y}} \frac{(c+t)^{2}}{c} \quad \text{(core shear stiffness parameter)}$$

$$\eta = \sqrt{G_{x}/G_{y}} \quad \text{(core orthotropy parameter)}$$

The components of \bar{u} are further related by the two auxiliary conditions

$$\frac{2}{c+t} \left[\left(\frac{D_s}{R_s} \right)^2 \frac{1-v}{2} \nabla^4 \tilde{u} - \frac{D_s}{R_s} \left\{ \eta \tilde{u}_{,yy} + \eta^{-1} \tilde{u}_{,xx} + \frac{1-v}{2} \left(\eta \tilde{u}_{,xx} + \eta^{-1} \tilde{u}_{,yy} \right) \right\} + \tilde{u} \right] = \left[\frac{D_s}{R_s} \left\{ (\eta - \eta^{-1}) w_{,yy} + \frac{1-v}{2} \left(\eta w_{,xx} + \eta^{-1} w_{,yy} \right) \right\} - w \right]_{,x}$$
(2)

and

$$\frac{2}{c+t} \left[\left(\frac{D_s}{R_s} \right)^2 \frac{1-v}{2} \nabla^4 \tilde{v} - \frac{D_s}{R_s} \left\{ \eta \tilde{v}_{,yy} + \eta^{-1} \tilde{v}_{,xx} + \frac{1-v}{2} \left(\eta \tilde{v}_{,xx} + n^{-1} \tilde{v}_{,yy} \right) \right\} + \tilde{v} \right] = \\ = \left[\frac{D_s}{R_s} \left\{ (\eta^{-1} - \eta) w_{,xx} + \frac{1-v}{2} \left(\eta w_{,xx} + \eta^{-1} w_{,yy} \right) \right\} - w \right]_{,y}.$$
(3)

Three sets of boundary conditions, each prescribing either an edge load or the corresponding edge displacement, are also listed in appendix A (eq. (A4)).

The present analysis will be restricted to a rectangular sandwich plate. The edges parallel to the X-axis are simply supported and the edges parallel to the Y-axis are clamped. Simple support is defined here as the condition of vanishing lateral edge displacement and edge bending moment similar to the case of a homogeneous plate. Then, for $y = \pm b/2$

$$w = 0$$
 (4)

$$C(c+t)(\tilde{v}_{,y}+v\tilde{u}_{,x})=0.$$
⁽⁵⁾

and

 $\frac{2\tilde{u}}{c+t} = 0$

The third boundary condition for the plate with finite shear stiffness can be prescribed as either

or

$$C \frac{1-v}{2} (c+t) (\tilde{v}_{x} + \tilde{u}_{y}) = 0.$$
 (6b)

The last condition (6b) indicates a vanishing edge twisting moment while condition (6a) reflects the prevention of the corresponding edge rotation.

A "clamped" boundary will be defined here as the condition of non-vanishing edge bending moment. Then, according to equation (A4), for $x = \pm a/2$

$$\tilde{a} = 0. \tag{7}$$

(6a)

Of the remaining possible conditions the realistic situation of vanishing lateral edge displacement is chosen, that is

v

$$v = 0. (8)$$

Both the case of a vanishing edge rotation and of a vanishing edge twisting moment are considered, corresponding to the conditions $\frac{2\tilde{v}}{c+t} = 0$ (9a)

and

$$C \frac{1-\nu}{2} (c+t) (\tilde{u}_{,\nu} + \tilde{v}_{,x}) = 0.$$
(9b)

The edge situation, corresponding to equation (7), (8) and (9a) will be termed "completely clamped" and obviously is the more realistic case. The case of relaxed clamping, implied by equation (9b) is included only for comparison with previous results (refs. 3 and 4).

2.2 Solution for one type of simple support along two edges

1

A solution, satisfying the governing equations (1), (2) and (3) as well as the essential simple support conditions (4) and (5) is

$$\tilde{u} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ w \end{bmatrix} = \sum_{i=1}^{6} \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} e^{\pi \mu i x/a} \cos \left[\frac{n\pi}{b} (y+b/2) \right]$$
(10)

where n is an integer. The boundary condition (6a) is implied by this solution and, in view of the neglect of face bending stiffness, the problem under consideration may be assumed to correspond to the structural end fixtures sketched in figure 2.

The part of the solution, depending on x only, consists of 6 linearly independent functions, each satisfying equations (1), (2) and (3). A characteristic equation for μ_i is obtained from equation (1) as

$$k_1 \mu_i^6 + k_2 \mu_i^4 + k_3 \mu_i^2 + k_4 = 0 \tag{11}$$

while the substitution of (10) into the auxiliary relations (2) and (3) yields

$$A_{i} = \frac{2a}{c+t} \frac{U_{i}}{W_{i}} \approx \mu_{i} \cdot \frac{k_{5} \mu_{i}^{2} + k_{6}}{k_{7} \mu_{i}^{4} + k_{8} \mu_{i}^{2} + k_{9}}$$
(12)

and

$$B_{i} = \frac{2b}{c+t} \frac{V_{i}}{W_{i}} \approx \frac{k_{10}\mu_{i}^{2} + k_{11}}{k_{7}\mu_{i}^{4} + k_{8}\mu_{i}^{2} + k_{9}}$$
(13)

where

$$k_{1} = S^{*}(p_{x}\rho S - \eta)$$

$$k_{2} = -p_{x}\rho[\eta S^{*} + \eta^{-1}S + 2\alpha SS^{*}] - p_{y}\alpha SS^{*} + \alpha S^{*}(2\eta + \eta^{-1}) + 1$$

$$k_{3} = p_{x}\rho[S^{*}\eta^{-1}\alpha + S\eta\alpha + SS^{*}\alpha^{2} + 1] + p_{y}\alpha[S^{*}\eta + S\eta^{-1} + 2\alpha SS^{*}] - S^{*}\alpha^{2}(\eta + 2\eta^{-1}) - 2\alpha$$

$$k_{4} = -p_{y}\alpha[S^{*}\eta^{-1}\alpha + S\eta\alpha + SS^{*}\alpha^{2} + 1] + S^{*}\alpha^{3}\eta^{-1} + \alpha^{2}$$

$$k_{5} = S^{*}n\pi$$

 $k_{6} = \pi [(S - S^{*})\eta^{-1}\alpha - (1 + S\eta\alpha)]$ $k_{7} = SS^{*}$ $k_{8} = - [S^{*}\eta + S\eta^{-1} + 2\alpha SS^{*}]$ $k_{9} = S^{*}\eta^{-1}\alpha + S\eta\alpha + SS^{*}\alpha^{2} + 1$ $k_{10} = n\pi [S\eta^{-1} - (S - S^{*})\eta]$ $k_{11} = -n\pi [1 + S^{*}\eta^{-1}\alpha]$

and

The dimensionless parameters appearing in these expressions are defined as

$$p_x = \frac{N_x b^2}{\pi^2 D_s}, \quad p_y = \frac{N_y a^2}{\pi^2 D_s} \quad \text{(load parameters)}$$

$$S = \frac{\pi^2 D_s}{R_s a^2} = \frac{2S^*}{1-v} \quad \text{(stiffness parameters)}$$

$$\rho = \frac{a^2}{b^2} = \frac{\alpha}{n^2} \quad \text{(aspect ratio parameters)}$$

The characteristic equation (11) in general yields three nonzero "roots" μ_i^2 (except when $k_4=0$) resulting in three pairs of roots μ_i of opposite sign. As all coefficients k are real valued the square of at least one pair of roots is real and the corresponding roots are either real or imaginary. The other two pairs can be complex. Hence the following classes of solutions must be considered.

Class	$\mu_1(=-\mu_2)$	$\mu_3(=-\mu_4)$	$\mu_5 (\approx -\mu_6)$
1	μ_1^\prime	$\mu_{\beta}^{\prime}+i\mu_{3}^{\prime\prime}$	$\mu'_3 - i\mu''_3$
2	μ_1'	μ'3	μ_5'
- 3	μ'_1	μ'_3	$i\mu_5''$
4	μ'_1	iμ ₃ ″	$i\mu_5''$
5	$i\mu_1''$	$i\mu_3''$	iμ"5
6	$i\mu_1''$	$\mu'_3 + i\mu''_3$	$\mu'_3 - i\mu''_3$

Next, it is evident from equations (12) and (13) that A_i occurs in pairs of opposite sign and B_i in pairs of equal sign.

Having established the general solution (10) together with the auxiliary relations (12) and (13) the only admissible solution, satisfying the homogeneous boundary conditions (7), (8) and (9a) or (9b), is determined by the condition that the determinant value of the matrix of coefficients of the related homogeneous equations vanishes. For the case of the completely clamped edge these equations are

$$\sum_{i=1}^{6} A_{i} W_{i} e^{\pi \mu_{i} x/a} = 0$$

$$\sum_{i=1}^{6} B_{i} W_{i} e^{\pi \mu_{i} x/a} = 0 \quad \text{at } x = \pm a/2,$$

$$\sum_{i=1}^{6} W_{i} e^{\pi \mu_{i} x/a} = 0$$
(15a)

In case of a vanishing edge twisting moment at the clamped edge the homogeneous boundary conditions lead to

$$\sum_{i=1}^{6} A_i W_i e^{\pi \mu_i x/a} = 0$$

$$\sum_{i=1}^{6} B_i^* W_i e^{\pi \mu_i x/a} = 0 \quad \text{at } x = \pm a/2 ,$$

$$\sum_{i=1}^{6} W_i e^{\pi \mu_i x/a} = 0 \quad (15b)$$

where $B_i^* = \mu_i B_i + nA_i$. Clearly B_i^* , similar to A_i and μ_i occurs in pairs of opposite sign.

As demonstrated in appendix B the buckling conditions deriving from the boundary conditions (15a) or (15b) can be written

$$\{|M'_{1}| - |M'_{1}('')|\} \{|M'_{2}| - |M'_{2}('')|\} - \{|M''_{1}| - |M''_{1}(')|\} \{|M''_{2}| - |M''_{2}(')|\} - i [\{|M''_{1}| - |M''_{1}(')|\} \{|M''_{2}| - |M''_{2}('')|\} + \{|M'_{1}| - |M'_{1}('')|\} \{|M''_{2}| - |M''_{2}(')|\}] = 0$$
(16)

In this expression $|M'_1|$, $|M''_1|$, $|M''_2|$ and $|M''_2|$ are the determinant values of the real and imaginary parts, respectively, of the two matrices M_1 and M_2 into which the matrix of coefficients of equations (15a) and (15b) can be partitioned after some elementary row and column operations. The quantity |M'(")| is a quasi-determinant defined as the sum of products of elements of M'_1 and the corresponding cofactors of M''_1 .

2.3 Buckling conditions for complete clamping of two edges

The matrices M_1 and M_2 deriving from equation (15a) are

$$M_{1} = \begin{bmatrix} A_{i} \sin \pi \mu_{i}/2 \\ B_{i} \cosh \pi \mu_{i}/2 \\ \cosh \pi \mu_{i}/2 \end{bmatrix} \quad (i = 1, 3, 5)$$
(17)

and

$$M_{2} = \begin{bmatrix} A_{i} \operatorname{ch} \pi \mu_{i}/2 \\ B_{i} \operatorname{sh} \pi \mu_{i}/2 \\ \operatorname{sh} \pi \mu_{i}/2 \end{bmatrix} \quad (i = 1, 3, 5).$$
(18)

The third row of both matrices stems from the boundary condition regarding the lateral deflection w. Clearly M_1 corresponds to a deflection pattern that is symmetric with respect to the line x = 0, passing through the center of the plate. M_z represents antisymmetric patterns. Evaluation of the buckling condition (16) for the different classes of solutions (see page 5) shows that in each case the left hand side of buckling condition reduces to the product of one symmetric and one antisymmetric determinant value only, leading to the appropriate result that the buckling pattern is either purely symmetric or purely antisymmetric (see (B8)).

The buckling conditions for the case of completely clamped edges are listed in table 1.

2.4 Buckling conditions for relaxed clamping of two edges

For this case the matrices M_1 and M_2 are

$$M_{1} = \begin{bmatrix} A_{i} \text{ sh } \pi \mu_{i}/2 \\ B_{i}^{*} \text{ sh } \pi \mu_{i}/2 \\ \text{ ch } \pi \mu_{i}/2 \end{bmatrix} \quad (i = 1, 3, 5)$$
(19)

anđ

$$M_{2} = \begin{bmatrix} A_{i} \operatorname{ch} \pi \mu_{i}/2 \\ B_{i}^{*} \operatorname{ch} \pi \mu_{i}/2 \\ \operatorname{sh} \pi \mu_{i}/2 \end{bmatrix} \quad (i = 1, 3, 5).$$
(20)

The corresponding buckling conditions for symmetric and antisymmetric buckling modes are listed in table 2.

2.5 Buckling of sandwich plates with isotropic cores

As indicated in appendix A the governing equations (1), (2) and (3) of the buckling problem simplify considerably if the core is isotropic in shear properties ($\eta = \sqrt{G_x/G_y} = 1$). Substitution of $\eta = 1$ into these equations results in the governing equations

$$\left[1 - \frac{D_s}{R_s} \frac{1 - v}{2} \nabla^2(\cdot)\right] \left[\frac{D_s}{R_s} \nabla^4 \bar{u} + \left\{\frac{N_x}{R_s}(\cdot)_{,xx} + \frac{N_y}{R_s}(\cdot)_{,yy}\right\} \left\{\bar{u} - \frac{D_s}{R_s} \nabla^2 \bar{u}\right\}\right] = 0$$
(21)

and the auxiliary relations

$$\frac{2}{2+t} \left[1 - \frac{D_s}{R_s} \frac{1-\nu}{2} \nabla^2 (\cdot) \right] \left[\tilde{u} - \frac{D_s}{R_s} \nabla^2 \tilde{u} \right] = - \left[w - \frac{D_s}{R_s} \frac{1-\nu}{2} \nabla^2 w \right]_{,x}$$
(22)

and

$$\frac{2}{c+t} \left[1 - \frac{D_s}{R_s} \frac{1-\nu}{2} \nabla^2 (\cdot) \right] \left[\tilde{\nu} - \frac{D_s}{R_s} \nabla^2 \tilde{\nu} \right] = - \left[w - \frac{D_s}{R_s} \frac{1-\nu}{2} \nabla^2 w \right]_{,\nu}$$
(23)

In general again a solution for \bar{u} , consisting of six linearly independent functions will be obtained, each of which satisfies the equations (21), (22) and (23).

For a tentative solution as given by equation (10) a characteristic equation is now obtained as

$$\left[\mu^{2} - \left(\alpha + \frac{1}{S^{*}}\right)\right] \left(\underline{k}_{1}' \mu^{4} + \underline{k}_{2}' \mu^{2} + \underline{k}_{3}'\right) = 0$$
(24)

where $k'_1 = 1 - p_x \rho S$

 $k'_2 = p_x \rho (1 + S\alpha) + p_y S\alpha - 2\alpha$

and $k'_3 = -p_y \alpha (1 + S\alpha) + \alpha^2$.

For $\eta = 1$ the relations (12) and (13) reduce to

$$A_{i} = \frac{2a}{c+t} \frac{U_{i}}{W_{i}} = \pi \mu_{i} \frac{S^{*}(\mu_{i}^{2}-\alpha)-1}{[S^{*}(\mu_{i}^{2}-\alpha)-1][S(\mu_{i}^{2}-\alpha)-1]} = \frac{\pi \mu_{i}}{S(\mu_{i}^{2}-\alpha)-1}$$
(25)

and

$$B_{i} = \frac{2b}{c+t} \frac{V_{i}}{W_{i}} = n\pi \frac{S^{*}(\mu_{i}^{2}-\alpha)-1}{[S^{*}(\mu_{i}^{2}-\alpha)-1][S(\mu_{i}^{2}-\alpha)-1]} = \frac{n\pi}{S(\mu_{i}^{2}-\alpha)-1}$$
(26)

provided that $S^*(\mu_i^2 - \alpha) - 1 \neq 0$. However, if $\eta = 1$ two of the roots μ_i are obtained from the condition $S^*(\mu_i^2 - \alpha) - 1 = 0$. In this case limit values are obtained from the expression for the orthotropic plate (eqs. (12) and (13)), using l'Hospitals rule, as

$$\lim_{\substack{\eta \to 1 \\ S^*(\mu_i^2 - \alpha) \to 1}} A_i = \pm \frac{\pi \sqrt{\alpha + \frac{1}{S}}}{S - S^*} \cdot \frac{S^*(\mu_i^2)_{,\eta} + 1 - 2\alpha(S - S^*)}{(\mu_i^2)_{,\eta} + 2\alpha + \frac{1}{S^*}}$$
(27)

and

$$\lim_{\substack{\eta \to 1 \\ S^*(\mu_i^2 - \alpha) \to 1}} B_i = \frac{n\pi}{S - S^*} \frac{S^*(\mu_i^2)_{\eta} + 1 - 2\alpha(S - S^*) - 2S/S^*}{(\mu_i^2)_{,\eta} + 2\alpha + \frac{1}{S^*}}$$
(28)

From the characteristic equation for the orthotropic case (eq. (11)) the limit value of $(\mu_i^2)_n$ results as

$$\lim_{\substack{\eta \to 1 \\ S^{*}(\mu_{i}^{2}-\alpha) \to 1}} (\mu_{i}^{2})_{,\eta} = \lim_{\substack{\eta \to 1 \\ S^{*}(\mu_{i}^{2}-\alpha) \to 1}} - \frac{\mu_{i}^{6}k_{1,\eta} + \mu_{i}^{4}k_{2,\eta} + \mu_{i}^{2}k_{3,\eta} + k_{4,\eta}}{3\mu_{i}^{4}k_{1} + 2\mu_{i}^{2}k_{2} + k_{3}} = -\left(2\alpha + \frac{1}{S^{*}}\right).$$
(29)

Obviously, A_i and B_i tend to infinity as η and $S^*(\mu_i^2 - \alpha)$ approach unity. As all deformations are bounded these results imply that W_i tends to zero in this case. Using these previous results the ratio A_i/B_i is determined as:

$$\lim_{\substack{\eta \to 1 \\ S^*(\mu_1^-,\alpha) \to 1}} \frac{A_i}{B_i} = \pm \frac{\sqrt{\alpha + \frac{1}{S^*}}}{n} \frac{S^*(\mu_i^2)_{,\eta} + 1 - 2\alpha(S - S^*)}{S^*(\mu_i^2)_{,\eta} + 1 - 2\alpha(S - S^*) - 2S/S^*} = \pm \frac{\alpha}{n\sqrt{\alpha + \frac{1}{S^*}}}.$$
(30)

In case of clamped edges, allowing edge parallel shear the ratio A_i/B_i^* , associated with the root $\mu_i = \sqrt{\alpha + 1/S^*}$, reaches a limit value

$$\lim_{\substack{\eta \to 1 \\ S^*(\mu_i^2 - \alpha) \to 1}} \frac{A_i}{B_i^*} = \lim_{\substack{\eta \to 1 \\ S^*(\mu_i^2 - \alpha) \to 1}} \frac{A_i/B_i}{\mu + nA_i/B_i} = \pm \frac{\alpha}{n(2\alpha + 1/S^*)}.$$
(31)

For the sake of simplicity the parameter S and S*, both depending on the core shear stiffness moduli G_x and G_y have been considered to be constant in the limiting process. This implies that only the ratio G_x/G_y changes, the product G_xG_y is constant. If either G_x or G_y is kept constant during the limiting process then

$$\lim_{\substack{\eta \to 1 \\ S^*(\mu_i^2 - \alpha) \to 1}} (\mu_i^2)_{,\eta} = -\left(2\alpha + \frac{2}{S^*}\right)$$
(32)

and .

$$\lim_{\substack{n \to 1 \\ 5^*(\mu_i^2 - \alpha) \to 1}} (\mu_i^2)_{,\eta} = -2\alpha$$
(33)

$$G_v = \text{constan}$$

respectively. The limit values of A_1 , B_1 and A_1/B_1 are, of course, independent of the way in which the isotropic state is approached. The simplifications in the buckling conditions due to isotropy are indicated in tables 1 and 2.

3 Discussion of results

3.1 Comparison of previous and present results for isotropic plates

As was pointed out earlier a comparison of approximate and exact results for isotropic sandwich panels was the initial aim of the current investigation. For a typical sandwich panel the interaction diagram in terms of the dimensionless load parameters p_x and p_y is shown in figure 3. In this case not only the solution of reference 4, based on a stationary total potential approach, but also the split rigidity treatment of reference 3 yields an upper bound solution in the domain of p_x and p_y values considered. In both of these approximate solutions the assumed deflected shape implies a vanishing edge shear parallel to the clamped edges but also a vanishing of the twisting moment. The overconstrained nature of deformation pattern used in reference 4 is even more pronounced as in addition to the lateral displacement also the transverse shear force vanishes at the clamped edges. To evaluate the effect of a vanishing twisting moment an exact solution corresponding to classical clamped edge conditions but allowing edge parallel shear is indicated in figure 3 by a dashed line. In the present example the nonclassical boundary conditions relative to the edge twisting moment appear to have little effect with regard to buckling behaviour.

The significance of the two approximate solutions of reference 3 and 4 for practical applications as proposed in both publications can be checked in figure 4 for the example discussed here. The results of exact and approximate solutions have been presented in a normalized form. Apparently, for the present case the energy method yields an interaction diagram that shows very good agreement with the exact interaction curve. The method of split rigidities tends to overestimate the buckling loads in a wide range. Especially for α_x values in the order of unity (a case of practical interest) the allowable α_y values preserving panel stability are severely overestimated.

A similar comparison of normalized interaction curves for a wide range of panel configurations confirms these initial conclusions. Some of these results are shown in figures 5 through 8. Again, the results for clamping allowing edge parallel shear (relaxed clamping) are indicated by a dashed line. As anticipated the reduction of buckling load values for p_x and p_y due to this relaxation are insignificant for relatively stiff panels (see figures 5 and 7) and for rectangular panels in which the clamped edges are separated by a greater length (figure 8). In some instances both solutions corresponding to complete and relaxed clamping are represented by a single curve in the normalized interaction diagrams.

In all of these examples the energy method yields a better approximation to the true interaction curve than the method of split rigidities. Although both methods might be accepted for preliminary design purposes only the method based on the stationary character of the total potential yields results that are of sufficient accuracy to serve all engineering design purposes.

3.2 Results for orthotropic plates

So far the comparison between approximate and exact results has been confined to isotropic sandwich panels, the only class treated in the analyses of references 3 and 4. However, most sandwich panels of actual construction utilize a core layer with orthotropic shear properties. The practice of replacing a set of different shear moduli G_x and G_y by their geometric mean $\sqrt{G_x G_y}$ to obtain buckling loads from analysis based on core isotropy in general does not lead to useful results. This conclusion is evident from figure 9. The exact results for two values of the core orthotropy

parameter $\eta = \sqrt{G_x/G_y}$ but the same value of the stiffness parameter $S = \frac{\pi^2 D_s c}{\sqrt{G_x G_y a^2 (c+t)^2}}$ are compared with a

result obtained from an isotropic panel analysis $(\eta = 1)$ using the geometric mean value of the core shear moduli, that is, for the same value of S. The values for η^2 of 2 and 2^{-1} are typical for honeycomb core material oriented with the ribbon direction normal or parallel to the clamped edges, respectively. Clearly, the curves for the actual orthotropic panels deviate considerably from the result for the isotropic dummy. Very importantly, the ratio of \tilde{p}_x/\tilde{p}_y is drastically changed. For the same reason the exact normalized interaction curves for orthotropic panels are poorly represented by the result obtained for the so called representative isotropic panel model, as is evidenced in figures 10 and 11.

However, a better approximation should result from an approximate analysis taking account of core orthotropy. Although the initial considerations of reference 4 include orthotropic core shear properties the actual analysis is confined to isotropic core shear behaviour. For the class of sandwich panels treated here (orthotropic core layer, identical isotropic facings) an approximate solution to the buckling problem for biaxial loading, based on the principle of stationary total potential, is formulated in Appendix C. Results according to this method, using the same one term approximation to the buckled shape as suggested in reference 4, are also shown in figure 9. The interaction diagram shows a pattern similar to that of the exact curves. The upper bound characteristics are apparent and are more pronounced in the region of combined tensile and compressive loading ($\alpha_x \alpha_y < 0$). The normalized shapes of these approximate interaction curves are introduced in figures 10 and 11 and also, for a panel with a relatively stiffer

core, in figures 12 and 13. Especially for panels with a relatively stiff core the energy method accounting for core orthotropy is certainly preferable to the original method restricted to core isotropy. This result has been evidenced very strongly for a wide range of parameter configurations analysed and, in fact, the poorest improvement met is shown in figure 10. It is noted that exact solutions for panels with relaxed clamping are not considered to be of practical interest. Thus, all of the exact results apply to the case of complete clamping along two panel edges.

3.3 Efficiency of approximate and exact methods

The obvious advantage relative to accuracy of exact solutions as compared to approximate results, even for a mathematical model that only approximately represents a structural problem, is in general only obtained at the cost of greater analytical complexity and inherent computational efforts. Especially in the present case considerable difficulties were encountered during the computation of exact results. An ALGOL procedure was developed to generate interaction curves automatically in the manner indicated in Appendix D. Although the problem in fact only consists of finding the zero's of presumably smooth continuous functions (see tables I and II) the essential difficulty stems from the occurrence of many sign changes in a very small domain, all corresponding to different buckling modes. This effect is encountered especially in case of relatively weak core configurations. A similar effect is noted in reference 5. It indicates that the panel is susceptible to the classical phenomenon of shear failure accompanied by typically short wavelength buckle patterns. Routinely used iteration procedures (method of bisection, regula falsi) all proved insatisfactory due to the rapidly changing gradients. For the same reason methods employing the first derivative are not considered as in the present problem no closed form expression can be obtained for this derived function. Instead, a direct trial and error method was used and considerable attention was paid to the interpretation of the results. In the procedure used step sizes in the search process are a rather direct measure of the computational times involved and are chosen to produce reliable results within an acceptable time. If final results, labelled according to the symmetric or antisymmetric character of the buckling mode, still revealed unreliable interaction curves the step size in the automatic process was reduced for the particular configuration under consideration. In case of an illbehaved exact solution a valuable check of the reliability is readily available in the form of an approximate solution according to appendix C.

On the other hand, for a one term energy type solution interaction curves are obtained in a very fast straightforward manner. Also, only two points are needed for each straight line segment of the approximate interaction diagram while many separate points are needed for a satisfactory definition of the exact curve. On a CDC-3300 computer (with floating point unit) solutions according to exact and approximate methods respectively for one set of parameter values required 240 seconds and 24 seconds. Considerable improvement in accuracy must be expected if more terms are included in an energy method while the time required for automatic computation will still be a fraction of the time necessary to generate an exact interaction curve. It must further be noted that exact closed form solutions can only be obtained for a limited class of problems with simple boundary conditions. If for example transverse compressibility of the core layer together with face bending stiffness are introduced in a treatment of e.g. edge zone problems for sandwich shells the analytical complexity will in general exclude an exact solution.

In view of the accuracy achieved by the approximate solution to the interaction problem under consideration, in many instances within the presently required tolerances, and on the basis of considerations mentioned earlier it is doubtful, for many practical cases, whether the exact solution warrants the efforts involved.

3.4 Applicability of present results

For the design of biaxially loaded sandwich panels with orthotropic cores but identical isotropic facings, having two completely clamped edges and two simply supported edges, a method similar to the one proposed in references 2 and 4 can be used with confidence. The approximate shape of the interaction diagram can be obtained using the total potential energy principle. For the present problem the expressions presented in appendix C can be utilized. For relatively stiff core panels a one term solution provides results of sufficient accuracy; for panels with core layers that are relatively weak in shear more terms should be included.

The approximate result of an upper bond character can be either normalized in the manner shown in figures 4 through 8 or reduced in size to intersect the co-ordinate axes at the correct points, using the values of \tilde{p}_x and \tilde{p}_y that can be readily extracted from the present exact solution. Thus to obtain useful data for design purposes from the approximate results the exact values of \tilde{p}_x and \tilde{p}_y must be available for a practical range of parameter values relative to panel stiffness properties, core orthotropy and panel aspect ratio (S, η and ρ , respectively). Results for the case of panels loaded along the two clamped edges ($p_y = 0$) are shown in figures 15 through 18 and tables 3 through 6. Similar results for the same panel configurations but loaded uniaxially along two simply supported edges ($p_x=0$) are given in figures 19 through 22 and in tables 7 through 10.

As mentioned earlier, in some instances the approximate shape of the interaction curve tends to overestimate the

load carrying capacity in case of combined tensile and compressive loading, that is $\alpha_x \alpha_y < 0$ (see figure 12). However, in these regions of the $\alpha_x - \alpha_y$ plane the validity of the present linearly elastic solution to the buckling problem may also be affected by plasticity. If for example the Tresca yield criterion is assumed to be applicable a simple hexagonal domain is defined in the $p_x - p_y$ plane, within which the elastic solution remains valid. For typical values of 5×10^{-3} for the ratio of uniaxial yield stress to Young's modulus and of 40 for the width to thickness ratio of a square panel the yield values of p_x and p_y are obtained as

$$(p_x)_{yield} = (p_y)_{yield} = \frac{4(1-v^2)}{\pi^2} \frac{\sigma_{yield}}{E} \left(\frac{a}{c+t}\right)^2 = 2.95.$$
 (34)

For a square orthotropic panel the interaction curve together with the hexagonal intersection of Tresca's yield surface and the $p_x - p_y$ plane are shown in figure 14. If a value of 25 is assumed for the ratio of core to face thickness then the chosen value of .4 for the stiffness parameter S implies that a relatively weak core sandwich panel is considered, namely

$$\frac{E}{\sqrt{G_x G_y}} = \frac{2S(1-v^2)}{\pi^2} \frac{a^2}{tc} = \frac{2S(1-v^2)}{\pi^2} \left(\frac{a}{c+t}\right)^2 \frac{(1+c/t)^2}{c/t} = 3.2 \times 10^3 \,. \tag{35}$$

For panels with a relatively stiffer core the increased buckling loads may be limited by plasticity effects to an even larger extend.

4 Concluding remarks

This report presents an exact solution to the elastic buckling problem of a biaxially loaded sandwich plate with two clamped and to simply supported edges. The two identical membrane face layers are isotropic but for the core layer shear orthotropy is assumed. Two types of edge clamping are considered.

For the special case of core isotropy the approximate solution of reference 4 based on the minimum total potential principle shows good agreement with the present exact solution. An approximate solution, presented in reference 3, utilizing the method of split rigidities tends to be unconservative in certain cases.

An approximate energy solution is presented for plates with orthotropic cores. Again very good agreement is evidenced for plates of practical construction. Only for very weak core sandwiches the simple one term energy solution is of insufficient accuracy.

In view of the much larger computational efforts required to obtain an exact interaction curve and on the basis of the accuracy achieved by the approximate method a combination of exact results, for uniaxial compression in directions parallel and normal to the clamped edges and of the approximate shape of the interaction curve provided by the energy solution appears to be a promising proposition for many engineering design applications. Load configurations for which this method may prove to be unconservative (combined tensile and compressive loading) are indicated but, also, plasticity effects may render the elastic solution invalid in these cases.

An outline is given of an automatic computational procedure for the generation of exact interaction curves in a chosen domain of parameter values.

5 References

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APPENDIX A

Buckling equations for a biaxially loaded sandwich plate

Plates with orthotropic cores

Equations for the bending and buckling of sandwich plates having isotropic face layers and an orthotropic core can be derived from the basic equilibrium equations for sandwich plates given in reference 1. The theory presented in this standard reference uses as deformation parameters the lateral deflection w and two average shear angles $\bar{\gamma}_x$ and $\bar{\gamma}_y$. Equilibrium equations and boundary conditions for a number of typical edge situations are listed.

Reference 5 presents a sandwich shell theory of somewhat more generality than included in the case considered here. It reduces to the present case if radii of curvature and the core extensional stiffness in normal direction are set equal to infinity and the face bending stiffness is neglected. The deformation parameters employed in this theory in reduced form are the lateral deflection w, the actual core shear angles γ_x and γ_y and two relative in plane face displacements $\tilde{u} = (u_1 - u_2)/2$ and $\tilde{v} = (v_1 - v_2)/2$ where u_1, v_1, u_2 and v_2 are the displacement components in X and Y direction of the two face layers, respectively. In the classical treatment of sandwich plates these parameters are related by the two simple continuity conditions:

 $\tilde{u} = \frac{u_1 - u_2}{c\gamma_x} = \frac{c\gamma_x}{c+t}$

and

$$\tilde{v} = \frac{v_1 - v_2}{2} = \frac{c\gamma_y}{2} - \frac{c + t}{2} w_{,y}.$$
(A.1)

The quantities $\frac{2\tilde{u}}{c+t}$ and $\frac{2\tilde{v}}{c+t}$ are recognized as the rotations of a plate normal section. In the variationally derived

equations of reference 5 the relevant boundary conditions are also expressed in terms of these quantities.

The governing equations and boundary conditions used in this analysis are obtained directly from reference 5. Where applicable, the similarity of the present notation and that of reference 1 can be demonstrated by use of the relations $\bar{\gamma}_x = \frac{c}{c+t} \gamma_x = \frac{2\tilde{u}}{c+t} + w_{,x}$

and

 $\bar{\gamma}_{y} = \frac{c}{c+t} \gamma_{y} = \frac{2\tilde{v}}{c+t} + w_{,y}. \tag{A.2}$

The equations of moment equilibrium about the X and Y directions and of lateral equilibrium of a plate element are, in terms of *buckling* deformations \tilde{u} , \tilde{v} and w:

$$C\left[\tilde{u}_{,xx} + \frac{1-\nu}{2}\tilde{u}_{,yy}\right] - \frac{2G_x}{c}\tilde{u} + C\frac{1+\nu}{2}\tilde{v}_{,xy} - \frac{c+t}{c}G_xw_{,x} = 0$$

$$C\frac{1+\nu}{2}\tilde{u}_{,xy} + C\left[\tilde{v}_{,yy} + \frac{1-\nu}{2}\tilde{v}_{,xx}\right] - \frac{2G_y}{c}\tilde{v} - \frac{c+t}{c}G_yw_{,y} = 0$$

$$-\frac{c+t}{c}G_x\tilde{u}_{,x} - \frac{c+t}{c}G_y\tilde{v}_{,y} + \frac{N_x}{2}w_{,xx} + \frac{N_y}{2}w_{,yy} - \frac{(c+t)^2}{2c}[G_xw_{,xx} + G_yw_{,yy}] = 0$$
(A.3)

where C is the plate extensional stiffness $\left(=\frac{Et}{1-v^2}\right)$ and G_x and G_y are the core shear moduli for the XZ and YZ plane respectively.

Appropriate boundary conditions are (for x = constant)

$$\begin{aligned} \tilde{u} &= 0 \quad \text{or} \quad C(c+t) \left[\tilde{u}_{,x} + v \tilde{v}_{,y} \right] &= 0 \\ \tilde{v} &= 0 \quad \text{or} \quad C \frac{1-v}{2} (c+t) \left[\tilde{u}_{,y} + \tilde{v}_{,x} \right] &= 0 \end{aligned}$$

$$(A.4)$$

$$w &= 0 \quad \text{or} \quad N_x w_{,x} + (c+t) G_x \gamma_x = \left[N_x - G_x \frac{(c+t)^2}{c} \right] w_{,x} + \frac{2\tilde{u}}{c+t} \frac{G(c+t)^2}{c} = 0.$$

Similar conditions for an edge y = constant are obtained by interchange of u and v and of x and y.

The set of governing linear partial differential equations (A.3) can be written

$$L^*\bar{u} = 0 \tag{A.5}$$

where \bar{u} is a deformation vector with components \tilde{u} , \tilde{v} and w and L^* is a symmetric matrix of linear operators. By means of a well known linear operation (A.5) can be transformed into

$$L\bar{u} = 0 \tag{A.6}$$

where L is the linear operator corresponding to the determinant of L^* . Each of the components of \bar{u} satisfies (A.6) while \tilde{u} , \tilde{v} and w are further related by

$$L_{I}\left(\frac{2\tilde{u}}{c+t}\right) = L_{II}(w_{,x})$$
$$L_{I}\left(\frac{2\tilde{v}}{c+t}\right) = L_{III}(w_{,y}). \tag{A.7}$$

and

In the present notation the linear operators L, L_{I} , L_{II} and L_{III} are

$$L() = \frac{D_s}{R_s} \nabla^4 \left[1 - \frac{D_s}{R_s} \frac{1 - v}{2} \left\{ \eta(\cdot)_{,xx} + \eta^{-1}(\cdot)_{,yy} \right\} \right] + \left[\frac{N_x}{R_s}(\cdot)_{,xx} + \frac{N_y}{R_s}(\cdot)_{,yy} \right] \left[\left(\frac{D_s}{R_s} \right)^2 \frac{1 - v}{2} \nabla^4(\cdot) - \frac{D_s}{R_s} \left\{ \eta(\cdot)_{,yy} + \eta^{-1}(\cdot)_{,xx} + \frac{1 - v}{2} (\eta(\cdot)_{,xx} + \eta^{-1}(\cdot)_{,yy}) \right\} + 1 \right] (A.8)$$

$$L_t(\cdot) = \left(\frac{D_s}{R_s} \right)^2 \frac{1 - v}{2} \nabla^4(\cdot) - \frac{D_s}{R_s} \left[\eta(\cdot)_{,yy} + \eta^{-1}(\cdot)_{,xx} + \frac{1 - v}{2} \left\{ \eta(\cdot)_{,xx} + \eta^{-1}(\cdot)_{,yy} \right\} \right] + 1$$

$$L_{\eta}(\cdot) = \frac{D_s}{R_s} \left\{ \eta - \eta^{-1} \right\} (\cdot)_{,yy} + \frac{1 - v}{2} \frac{D_s}{R_s} \left\{ \eta(\cdot)_{,xx} + \eta^{-1}(\cdot)_{,yy} \right\} - 1$$

$$L_{m}(\cdot) = \frac{D_s}{R_s} \left\{ \eta^{-1} - \eta \right\} (\cdot)_{,xx} + \frac{1 - v}{2} \frac{D_s}{R_s} \left\{ \eta(\cdot)_{,xx} + \eta^{-1}(\cdot)_{,yy} \right\} - 1$$

$$(a + v)^2$$

where $D_s = \frac{C(c+t)^2}{2}$ is the bending stiffness of the sandwich plate and $R_s = \sqrt{G_x G_y} \frac{(c+t)^2}{c}$ is a core shear stiffness parameter. $\eta = \sqrt{G_x/G_y} \neq 1$ reflects core orthotropy.

Plates with an isotropic core.

The equation for the case of core shear isotropy are obtained from the general orthotropic theory by setting η equal to unity. Then the operators of the govering equations rewrite:

$$L() = \left[1 - \frac{D_s}{R_s} \frac{1 - \nu}{2} \nabla^2\right] \left[\frac{D_s}{R_s} \nabla^4() + \left\{1 - \frac{D_s}{R_s} \nabla^2\right\} \left\{\frac{N_x}{R_s}()_{,xx} + \frac{N_y}{R_s}()_{,yy}\right\}\right]$$

$$L_l() = \left[1 - \frac{D_s}{R_s} \frac{1 - \nu}{2} \nabla^2\right] \left[() - \frac{D_s}{R_s} \nabla^2()\right]$$

$$L_{II}() = L_{III}() = -\left[() - \frac{D_s}{R_s} \frac{1 - \nu}{2} \nabla^2()\right].$$
(A.9)

This apparently simpler set of equations also gives rise to some singularities in the general solution that are discussed in the section "Analysis" of this report.

APPENDIX B

Buckling conditions for two types of edge clamping

General considerations

The general form of the matrix of coefficients, taken from the homogeneous boundary conditions for a clamped edge at $x = \pm a/2$ is, for a biaxially compressed plate, (see eqs. (15)).

$$\begin{bmatrix} A_{i} e^{\pi \mu_{i}/2} \\ A_{i} e^{-\pi \mu_{i}/2} \\ B_{i} e^{\pi \mu_{i}/2} \\ e^{\pi \mu_{i}/2} \\ e^{-\pi \mu_{i}/2} \end{bmatrix}$$
 (*i* = 1, 2, ..., 6). (B.1)

The values of μ_i are the roots of the characteristic equation (see eqn. (11))

$$k_1 \mu_i^6 + k_2 \mu_i^4 + k_3 \mu_i^2 + k_4 = 0.$$
(B.2)

(B.3)

(B.5)

All coefficients k in this equation are real valued functions of the geometrical, material and load parameters pertaining to the problem. As only even powers of μ_i appear the roots occur in pairs of equal modulus but opposite sign. Further, the square of at least one pair of roots $(\mu_1, \mu_2 = -\mu_1)$ is real, the squares of the other two pairs can be complex. On the basis of these observations the possible solutions are classified in the following manner.

Class	$\mu_1(=-\mu_2)$	$\mu_3(=-\mu_4)$	$\mu_5(=-\mu_6)$	
1	μ_1'	$\mu'_3 + i\mu''_3$	$\mu_3'-i\mu_3''$	
2	μ'_1	μ'_3	μ'_5	
3	μ'_1	μ'_3	$i\mu_5''$	
4	μ'_1	$i\mu_3''$	iµ''5	
5	$i\mu_1''$	$i\mu_3''$	$i\mu_5''$	-
6	iμ''	$\mu'_{3} + i\mu''_{3}$	$\mu'_{3} - i\mu''_{3}$	

The case of completely clamped edges (vanishing edge rotations and lateral displacement)

The quantities A_i and B_i of (B.1) are given for this case by (see eqs. (12) and (13))

$$A_{i} = \mu_{i} \frac{k_{5} \mu_{i}^{2} + k_{6}}{k_{7} \mu_{i}^{4} + k_{8} \mu_{i}^{2} + k_{9}} = A_{i}' + i A_{i}''$$

$$B_{i} = \frac{k_{10} \mu_{i}^{2} + k_{11}}{k_{10} \mu_{i}^{2} + k_{11}} = B_{i}' + i B_{i}''$$
(B.4)

and

and

$$B_i = \frac{k_{10}\mu_i^2 + k_{11}}{k_7\mu_i^4 + k_8\mu_i^2 + k_9} = B'_i + iB''_i$$

where again all coefficients k are real valued functions. Clearly A_i occurs in pairs of opposite sign and in a form identical to that of the corresponding root μ_i (see (B.3)). B_i occurs in pairs of equal sign. Moreover B_i is real valued in all cases except 1 and 6: then B_3 and B_5 are conjugate complex.

By the nature of the solution to the present problem, reflected in the special form of the roots (see (B.3)) and of the quantities A_i and B_i the matrix of coefficients (B.1) can be shown to be equivalent to

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

where the submatrix

$$M_{1} = \begin{bmatrix} A_{i} \, \mathrm{sh} \, \pi \mu_{i}/2 \\ B_{i} \, \mathrm{ch} \, \pi \mu_{i}/2 \\ \mathrm{ch} \, \pi \mu_{i}/2 \end{bmatrix} \quad (i = 1, \, 3, \, 5)$$

 $M_{2} = \begin{bmatrix} A_{i} \operatorname{ch} \pi \mu_{i}/2 \\ B_{i} \operatorname{sh} \pi \mu_{i}/2 \\ \operatorname{sh} \pi \mu_{i}/2 \end{bmatrix} \quad (i = 1, 3, 5).$

The last row of both M_1 and M_2 stems from the boundary condition for the lateral displacement w and reflects the form of the displacement pattern. Clearly M_1 corresponds to a wave form that is symmetric with respect to the line x = 0, passing through the center of the plate. M_2 represents buckling modes that are antisymmetric with respect to the Y-axis.

Then, the buckling condition (the condition that the determinant value of the equivalent matrix (B.5) vanishes) is

$$|M_1| \cdot |M_2| = 0. (B.6)$$

All entries of both M_1 and M_2 can be complex, that is, the submatrices can be written $M_i = M'_i + iM''_i$. Then, the buckling condition in final form reads:

$$\{ |M'_{1}| - |M'_{1}(")| \} \{ |M'_{2}| - |M'_{2}(")| \} - \{ |M''_{1}| - |M''_{1}(')| \} \{ |M''_{2}| - |M''_{2}(')| \} - i [\{ |M''_{1}| - |M''_{1}(')| \} \{ |M'_{2}| - |M'_{2}(")| \} + \{ |M'_{1}| - |M'_{1}(")| \} \{ |M''_{2}| - |M''_{2}(")| \}] = 0.$$
 (B.7)

In this equation $|M'_i|$ and $|M''_i|$ are the determinant values of the real and imaginary parts of a submatrix M_i while $|M'_i('')|$ is a quasi-determinant formed by summing all the products of entries of M'_i and the corresponding cofactors of M''_i (each product having a + sign if the sum of row number and column number of the entry involved is even and a - sign otherwise). |M''(')| is similarly defined.

For the six classes of solutions indicated by (B.3) the buckling condition reduces to the following simple conditions:

Case	Buckling condition	
1	$- M_1''(') \cdot M_2''(') = 0$	
2	$ M_1' \cdot M_2' = 0$	
3	$i M'_1 \cdot M''_2(') = 0$	(B.8)
4	$- M'_1 \cdot M'_2('') =0$	
5	$- M'_1 \cdot M''_2 =0$	
6	$-i M_1''() \cdot M_2''() = 0$	

The expanded forms of these conditions are presented in table 1. As the buckling condition requires the vanishing of either a "symmetric" (subscript 1) or an "antisymmetric" (subscript 2) determinant, the buckling conditions are presented in two columns labelled "symmetric modes" and "antisymmetric modes". Only a numerical evaluation can indicate which condition governs the buckling problem under consideration.

The clamped edge allowing edge parallel shear

For this case similar considerations hold as discussed in connection with the completely clamped edge. The only difference is that B_i is replaced by

$$B_i^* = \mu_i B_i + nA_i = B_i^{*'} + iB_i^{*''}. \tag{B.9}$$

Clearly B_i^* occurs in pairs of opposite sign and in a form equivalent to that of μ_i as given in (B.3). The effect of the substitution indicated in (B.9) is implemented by interchanging the second rows of M_1 and M_2 , that is

$$M_{1} = \begin{bmatrix} A_{i} \sin \pi \mu_{i}/2 \\ B_{i}^{*} \sin \pi \mu_{i}/2 \\ ch \pi \mu_{i}/2 \end{bmatrix} \quad (i = 1, 3, 5)$$

$$M_{2} = \begin{bmatrix} A_{i} ch \pi \mu_{i}/2 \\ B_{i}^{*} ch \pi \mu_{i}/2 \\ sh \pi \mu_{i}/2 \end{bmatrix} \quad (i = 1, 3, 5).$$
(B.10)

and

Expanded forms of the buckling conditions for symmetric or antisymmetric patterns are given in table 2.

Isotropic core sandwich plates

In case of an isotropic core material $(\eta^2 = G_x/G_y = 1)$ the squares of the roots μ_1 and $\mu_2 = -\mu_1$ are positive and independent of the loadparameters p_x and p_y (see eq. (24)). In addition $W_1 = W_2 = 0$ and

$$B_1 = \frac{n\mu_1}{\alpha} A_1 = B_2$$

in case of a completely clamped edge while

$$B_1^* = \frac{n\mu_1}{\alpha} \left[1 + \frac{\alpha}{\mu_1^2} \right] A_1 = -B_2^*$$

APPENDIX C

Approximate solution for the buckling problem of a biaxially loaded orthotropic sandwich plate

Total potential functional

For the problem under consideration an upper bound solution for the buckling loads is obtained by use of the direct variational method commonly referred to as the minimum total potential principle. For an analysis of the equilibrium system of internal stresses associated with the buckling deformations and the external buckling loads the variation due to panel flexure and shear, of the strain energy functional U, assuming linearly elastic material behaviour, is appropriately defined as

$$\delta U = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left[\frac{C}{2} \left(\varepsilon_{xi}^2 + \varepsilon_{yi}^2 + 2\nu \varepsilon_{xi} \varepsilon_{yi} + \frac{1-\nu}{2} \gamma_{xyi}^2 \right) + \frac{C}{2} \left(G_x \gamma_x^2 + G_y \gamma_y^2 \right) \right] dx \, dy \qquad (i = 1, 2)$$
(C.1)

where the index i takes the values 1 and 2 for upper and lower face layers respectively. In this expression $C = \frac{Et}{1-v^2}$

is the extensional stiffness of either of two similar face layers and G_x and G_y are the transverse shear moduli for the orthotropic core; γ_x and γ_y are the corresponding actual shear angles generated during buckling. Face bending stiffness and transverse core compressibility are neglected. The plain strain components of the face layers are defined:

$$\varepsilon_{xi} = u_{i,x}$$

$$\varepsilon_{yi} = v_{i,y} \qquad (i = 1, 2)$$

$$\gamma_{xyi} = u_{i,y} + v_{i,x}$$

(C.2)

For the panel with similar face layers considered here the buckling displacements of the faces and the core shear angles are related by continuity conditions:

$$u_{1} = \frac{c}{2}(\gamma_{x} - w_{,x}) - \frac{t}{2}w_{,x}$$

$$v_{1} = \frac{c}{2}(\gamma_{y} - w_{,y}) - \frac{t}{2}w_{,y}$$

$$u_{2} = -\frac{c}{2}(\gamma_{x} - w_{,x}) + \frac{t}{2}w_{,x}$$

$$v_{2} = -\frac{c}{2}(\gamma_{y} - w_{,y}) + \frac{t}{2}w_{,y}.$$
(C.3)

Obviously, a neutral surface with respect to buckling displacements u and v is, for the compound panel with similar facings, taken halfway between the face layers.

In reference 4 also dealing with a panel with identical facings, the distances of neutral surfaces for displacement components u and v to one of the faces are introduced as separate variables apart from the core shear angles γ_x and γ_y . In the variational process utilized in this reference these additional unknowns can only be solved in a very late stage of the analysis after γ_x and γ_y have been determined. A direct variation with respect to these variables immediately yields the trivial value of one half of the core depth for both and considerably simplifies the subsequent analytical steps. From (C.3) the buckling strains of the face layers result as

$$\epsilon_{x1} = \frac{c}{2} \gamma_{x,x} - \frac{c+t}{2} w_{,xx} = -\epsilon_{x2}$$

$$\epsilon_{y1} = \frac{c}{2} \gamma_{y,y} - \frac{c+t}{2} w_{,yy} = -\epsilon_{y2}$$

$$\gamma_{xy1} = \frac{c}{2} (\gamma_{x,y} + \gamma_{y,x}) - \frac{c+t}{2} \cdot 2w_{,xy} = -\gamma_{xy2}.$$
(C.4)

A solution satisfying at least the geometric boundary conditions $\tilde{u} = \tilde{v} = 0$ (at $x = \pm \frac{a}{2}$, see appendix A) is defined by

and

$$y_{x} = \widetilde{A} \frac{c+t}{c} w_{,x}$$

$$y_{y} = \widetilde{B} \frac{c+t}{c} w_{,y}.$$
(C.5)

Then the face strains are simply

$$\varepsilon_{x1} = \frac{c+t}{2} (\tilde{A}-1) w_{,xx} = -\varepsilon_{x2}$$

$$\varepsilon_{y1} = \frac{c+t}{2} (\tilde{B}-1) w_{,yy} = -\varepsilon_{y2}$$

$$(C.6)$$

$$y_{xy1} = \frac{c+t}{2} (\tilde{A}+\tilde{B}-2) w_{,xy} = -\gamma_{xy2}$$

and the strain energy variation δU can be written as

$$\delta U = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left[\frac{C(c+t)^2}{4} \left\{ (\tilde{A}-1)^2 w_{,xx}^2 + (\tilde{B}-1)^2 w_{,yy}^2 + 2(\tilde{A}-1)(\tilde{B}-1) w_{,xx} w_{,yy} + \frac{1-v}{2} (\tilde{A}+\tilde{B}-2)^2 w_{,xy}^2 \right\} + \frac{cG_x}{2} \frac{(c+t)^2}{c^2} \tilde{A}^2 w_{,x}^2 + \frac{cG_y}{2} \frac{(c+t)^2}{c^2} \tilde{B}^2 w_{,y}^2 \right] dx \, dy \,.$$
(C.7)

The variation of the potential functional V for the applied buckling loads N_x and N_y is defined as

$$\delta V = \int_{-b/2}^{b/2} \left[N_x \int_{-a/2}^{a/2} - \frac{w_{,x}^2}{2} dx \right] dy + \int_{-a/2}^{a/2} \left[N_y \int_{-b/2}^{b/2} - \frac{w_{,y}^2}{2} dy \right] dx .$$
(C.8)

Finally the variation of the total potential functional which is required to vanish is determined by the sum

$$\delta(U+V) = \frac{C(c+t)^2}{4} \left[(\tilde{A}-1)^2 I_1 + (\tilde{B}-1)^2 I_2 + 2(\tilde{A}-1)(\tilde{B}-1) I_3 + \frac{1-\nu}{2} (\tilde{A}+\tilde{B}-2)^2 I_4 + \frac{(c+t)^2}{c^2} \left[\frac{cG_x}{2} \tilde{A}^2 I_5 + \frac{cG_y}{2} \tilde{B}^2 I_6 \right] - \frac{N_x}{2} I_5 - \frac{N_y}{2} I_6 = 0 \right]$$
(C.9)

where I_1 through I_6 are the surface integrals of $w_{,xx}^2$, $w_{,yy}^2$, $w_{,xx}w_{,yy}$, $w_{,xy}^2$, $w_{,xy}^2$, $w_{,yy}^2$, respectively, evaluated over the domain occupied by the plate.

Deflection pattern

As suggested in reference 4 a simple one-parameter solution satisfying the geometric boundary conditions (and some of the natural boundary conditions) can be taken as

$$w = \tilde{C} \sin \frac{\pi}{a} \left(x + \frac{a}{2} \right) \sin \frac{m\pi}{a} \left(x + \frac{a}{2} \right) \sin \frac{n\pi}{b} \left(y + \frac{b}{2} \right).$$
(C.10)

Then the surface integrals appearing in the total potential expression (C.9) are:

$$I_{1} = \frac{\pi^{4}}{a^{4}} \frac{ab}{16} \cdot 2(m^{4} + 6m^{2} + 1)\tilde{C}^{2}$$

$$I_{2} = \frac{\pi^{4}}{b^{4}} \frac{ab}{16} \cdot n^{4}(2 + \delta_{1m})\tilde{C}^{2}$$

$$I_{3} = I_{4} = \frac{\pi^{4}}{a^{2}b^{2}} \frac{ab}{16} \cdot 2n^{2}(1 + m^{2})\tilde{C}^{2}$$

$$I_{5} = \frac{\pi^{2}}{a^{2}} \frac{ab}{16} \cdot 2(1 + m^{2})\tilde{C}^{2}$$

$$I_{6} = \frac{\pi^{2}}{b^{2}} \frac{ab}{16} \cdot n^{2}(2 + \delta_{1m})\tilde{C}^{2}$$
(C.11)

where δ_{1m} is the Kronecker symbol.

Introducing dimensionless parameters as used in the exact solution, namely

$$S = \frac{\pi^2 D_s}{a^2 R_s} = \frac{\pi^2}{a^2} \cdot \frac{Cc}{2\sqrt{G_x G_y}}$$
$$\eta = \sqrt{G_x/G_y}$$
$$\rho = \frac{a^2}{b^2}$$

and

1

together with a load configuration parameter

$$\varepsilon = \frac{p_x}{p_y} = \frac{N_x b^2}{N_y a^2}$$

expression (C.9) can be written after multiplication by a common factor

$$\frac{32}{\pi^4} \frac{ab}{C(c+t)^2} \,\delta[U+V] = \tilde{C}^2 \left[(\tilde{A}-1)^2 \,\frac{m^4+6m^2+1}{\rho} + (\tilde{B}-1)^2 \,\frac{\rho n^4}{2} (2+\delta_{1m}) + (\tilde{A}-1)(\tilde{B}-1)2\nu n^2(1+m^2) + (\tilde{A}+\tilde{B}-2)^2 \,\frac{1-\nu}{2} n^2(1+m^2) + \frac{\tilde{A}^2\eta}{S\rho} (1+m^2) + \frac{\tilde{B}^2n^2}{2S\eta} (2+\delta_{1m}) - p_y \left\{ (2+\delta_{1m}) \frac{n^2}{2} + \varepsilon(1+m^2) \right\}^2 \right] 0$$
(C.12)

For nonvanishing buckle amplitude \tilde{C} this identity defines the buckling load configurations as a function of \tilde{A} and \tilde{B} . Variation with respect to the free parameters \tilde{A} , \tilde{B} results in two equations from which \tilde{A} and \tilde{B} can be solved, thus defining the critical load configurations:

$$\begin{bmatrix} c_3 + 2c_4 & 2c_2 + 2c_4 + 2c_6 \\ 2c_1 + 2c_4 + 2c_5 & c_3 + 2c_4 \end{bmatrix} \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} = \begin{bmatrix} 2c_2 + c_3 + 4c_4 \\ \vdots \\ 2c_1 + c_3 + 4c_4 \end{bmatrix}$$
(C.13)

Then the final expression for the determination of buckling loads results as:

$$c_{0}p_{y} = (c_{1} + c_{4} + c_{5})\tilde{A}^{2} + (c_{3} + 2c_{4})\tilde{A}\tilde{B} + (c_{2} + c_{4} + c_{6})\tilde{B}^{2} - (2c_{1} + c_{3} + 4c_{4})\tilde{A} - (2c_{2} + c_{3} + 4c_{4})\tilde{B} + c_{1} + c_{2} + c_{3} + 4c_{4}$$

$$(C.14)$$

where

$$c_{0} = n^{2}(2+\delta_{1m})+2\varepsilon(1+m^{2})$$

$$c_{1} = 2(m^{4}+6m^{2}+1)/\rho$$

$$c_{2} = n^{4}\rho(2+\delta_{1m})$$

$$c_{3} = 4\nu n^{2}(1+m^{2})$$

$$c_{4} = (1-\nu)n^{2}(1+m^{2})$$

$$c_{5} = \frac{2\eta}{\rho S}(1+m^{2})$$
and $c_{6} = \frac{n^{2}}{\eta S}(2+\delta_{1m})$.

APPENDIX D

Program for automatic computation of interaction curves

AUXILIARY PROCEDURES

An ALGOL program is available for automatic computation of interaction curves for any given set of parametervalues. The principal blocks of the program are the following procedures:

PROCEDURE FIXCOEF

Computes the coefficients k_s through k_{11} of equation (14) for given values of η , S, ρ , v and n. These coefficients are independent of p_x and p_y .

PROCEDURE SOLVE

Solves, for given p_x and p_y values, the characteristic equation (11) by Cardanus' method. The ratios A and B (see equations (12) and (13)) are computed using the results of FIXCOEF and the class of solution is identified according to the classification used in this report. Values and sign of the associated conditions for symmetric and antisymmetric buckling modes are determined. The procedure takes account of simplifications due to isotropy.

PROCEDURE RANGE

Determines the maximum value of the wave number n to be considered in a chosen domain of the p_x-p_y plane in the manner indicated in the diagram.



For each segment of the interaction curve two exact points are determined in the number order indicated. For the determination of each point only one of the parameters, p_x or p_y , is increased stepwise until a sign change of either of the two buckling conditions is found. In the example shown the segment corresponding to the wave number 4 can be ignored.

Next, the estimated points of intersection are determined for successive segments, each considered to be a straight line. Further the p_x and p_y values to be used for normalization of the exact interaction curve (point numbers 2 and 6, respectively, in the example) are saved. Both FIXCOEF and SOLVE are frequently called during the execution of RANGE.

PROCEDURE INTERACTION (PXB, PXE, PYB, PYE)

Provides preliminary information for the determination of an exact interaction curve segment between two estimated points of intersection, (PXB, PYB) and (PXE, PYE)



For a steep segment ($|PYE-PYB| \ge |PXE-PXB|$) points will be determined for fixed p_y values in the search interval by stepwise varying p_x . For a flat segment (|PYE-PYB| < |PXE-PXB|) a similar procedure is followed in which p_x and p_y are interchanged. It is noted that the formal parameters of this procedure may be only estimates of exact curve coordinates. For the final determination of segments the procedure CURVE is called.

PROCEDURE CURVE (X, Y, YB, YE, start)

This procedure determines exact points of interaction curve segments in the manner shown in the diagram.

Using a chosen START-value the formal variable X is stepped for a fixed value of Y = YB until a zero crossing is found. Next, a second point is determined at a slightly higher Y value; X is increased through a chosen safety margin starting at a value determined by the slope of the estimated segment. Then, Y is increased again and the starting value of X is determined by the direction defined by the first two exact points again observing a safety margin. For each point the character of the critical buckling mode (symmetric or antisymmetric) is identified.



It is recognized that in general the convex side of the curve is turned away from the origin. This fact is used to limit the search range of X in order to allow a reduction of the step size without undue sacrifice relative to machine time: for each step SOLVE must be executed. If, however, too narrow a safety margin is selected the procedure may loose track of the curve in case of strong curvature. The phenomenon is reported by means of an error message.

AUTOMATIC DETERMINATION OF INTERACTION CURVES

An outline of the programmed instructions is shown in the block diagram below. A complete text of the ALGOL program is available upon request.



Class of solution	Buckling conditions: (see eqs. (15	a))
(see page 5)	Symmetric modes	Antisymmetric modes
	$A_1'B_3' \sinh \frac{\pi \mu_1'}{2} (\cosh \pi \mu_3 + \cos \pi \mu_3') + (A_3'B_3' - A_3'B_3') \cosh \frac{\pi \mu_1'}{2} \sinh \pi \mu_3 +$	$A_1 B_3 \cosh \frac{\pi \mu_1'}{2} (\sin \pi \mu_3 - \cos \pi \mu_3') + (A_3 B_3 - A_3 B_3' - A_3 B_1') \sin \frac{\pi \mu_1'}{2} \sin \pi \mu_3 + A_3 B_3' \sin \pi \mu_3' \sin $
	$+ (\underline{A''_3 B''_3 + A'_3 B'_1}) \operatorname{ch} \frac{\pi \mu'_1}{2} \sin \pi \mu''_3 = 0$	$-\left(\underline{A_3'B_3'} + \underline{A_3'B_3'} - A_3'B_1\right) \sin \frac{\pi\mu_1}{2} \sin \pi\mu_3' = 0$
2	$A_1(B_3 - B_5)$ sh $\frac{\pi\mu_1}{2}$ ch $\frac{\pi\mu_3}{2}$ ch $\frac{\pi\mu_5}{2} + A_3(\underline{B_5} - B_1)$ ch $\frac{\pi\mu_1}{2}$ sh $\frac{\pi\mu_3}{2}$ ch $\frac{\pi\mu_5}{2} + B_1$	$A_1(B_3 - B_5) \ \text{ch} \ \frac{\pi \mu'_1}{2} \ \text{sh} \ \frac{\pi \mu'_3}{2} \ \text{sh} \ \frac{\pi \mu'_3}{2} \ + A_3(\underline{B}_3' - B_1') \ \text{sh} \ \frac{\pi \mu'_1}{2} \ \text{ch} \ \frac{\pi \mu'_3}{2} \ \text{sh} \ \frac{\pi \mu'_5}{2}$
	$+ A_5'(B_1' - \underline{B_3}) \operatorname{ch} \frac{\pi \mu_1}{2} \operatorname{ch} \frac{\pi \mu_2}{3} \operatorname{sh} \frac{\pi \mu_3}{2} = 0$	+ $A'_5(B'_1 - \frac{B'_3}{3})$ sh $\frac{\pi\mu'_1}{3}$ sh $\frac{\pi\mu'_3}{2}$ ch $\frac{\pi\mu'_5}{2} = 0$
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~	$A_1(B_3 - B_5)  \sinh \frac{\pi \mu_1}{2}  \cosh \frac{\pi \mu_2}{2}  \cos \frac{\pi \mu_3'}{2} + A_3(B_3 - B_1)  \cosh \frac{\pi \mu_1}{2}  \sinh \frac{\pi \mu_2}{2}  \cos \frac{\pi \mu_3'}{2} + \frac{\pi \mu_3}{2}  \cosh \frac{\pi \mu_3}{2} + \pi \mu_$	$A_1'(B_3 - B_5')$ ch $\frac{\pi\mu'_1}{2}$ sh $\frac{\pi\mu'_3}{2}$ sin $\frac{\pi\mu'_3}{2} + A_3'(B_5' - B_1')$ sh $\frac{\pi\mu'_1}{2}$ ch $\frac{\pi\mu'_3}{2}$ sin $\frac{\pi\mu'_3}{2}$
	$-A''_{5}(B'_{1} - \underline{B'_{3}}) \operatorname{ch} \frac{\pi \mu'_{1}}{2} \operatorname{ch} \frac{\pi \mu'_{3}}{2} \sin \frac{\pi \mu'_{3}}{2} = 0$	+ $A_5''(B_1' - \underline{B}_3')$ sh $\frac{\pi \mu_1}{2}$ sh $\frac{\pi \mu_3}{2} \cos \frac{\pi \mu_3}{2} = 0$
4	$A_1'(B_3'-B_5) \sin \frac{\pi \mu_1}{2} \cos \frac{\pi \mu_3'}{2} \cos \frac{\pi \mu_3'}{2} - A_3'(B_5'-B_1') \sin \frac{\pi \mu_1}{2} \sin \frac{\pi \mu_3'}{2} \cos \frac{\pi \mu_3'}{2} + \frac$	$A'_1(B_3 - B_5) \operatorname{ch} \frac{\pi \mu'_3}{2} \sin \frac{\pi \mu'_3}{2} \sin \frac{\pi \mu'_3}{2} + A'_3(\underline{B_5} - B'_1) \operatorname{sh} \frac{\pi \mu'_1}{2} \cos \frac{\pi \mu'_3}{2} \sin \frac{\pi \mu'_3}{2}$
	$-A''_{5}(B'_{1} - \underline{B'_{2}}) \operatorname{ch} \frac{\pi \mu'_{1}}{2} \cos \frac{\pi \mu''_{3}}{2} \sin \frac{\pi \mu''_{5}}{2} = 0$	$+ A_5'(B_1 - B_2')$ sh $\frac{\pi \mu_1'}{2}$ sin $\frac{\pi \mu_3'}{2} \cos \frac{\pi \mu_3'}{2} = 0$
م	$A_1''(B_3 - B_5) \sin \frac{\pi \mu_1'}{2} \cos \frac{\pi \mu_2'}{2} \cos \frac{\pi \mu_3'}{2} + A_3''(B_5 - B_1') \cos \frac{\pi \mu_1'}{2} \sin \frac{\pi \mu_3'}{2} \cos \frac{\pi \mu_3'}{2} + A_3''(B_5 - B_1') \cos \frac{\pi \mu_1'}{2} \sin \frac{\pi \mu_3'}{2} + A_3''(B_5 - B_1') \cos \frac{\pi \mu_3''}{2} + A_3''$	$A_1''(B_3 - B_5) \cos \frac{\pi \mu_1''}{2} \sin \frac{\pi \mu_3''}{2} \sin \frac{\pi \mu_3''}{2} + A_3''(B_5 - B_1') \sin \frac{\pi \mu_1''}{2} \cos \frac{\pi \mu_3''}{2} \sin \frac{\pi \mu_3''}{2}$
	$+ A''_{5}(B'_{1} - B'_{3}) \cos \frac{\pi \mu''_{2}}{2} \cos \frac{\pi \mu''_{3}}{2} \sin \frac{\pi \mu''_{3}}{2} = 0$	+ $A''_5(B'_1 - B'_3) \sin \frac{\pi \mu''_1}{2} \sin \frac{\pi \mu''_3}{2} \cos \frac{\pi \mu''_3}{2} = 0$
ę	$-A_1^n B_3^r \sin \frac{\pi \mu_1^n}{2} (\cosh \pi \mu_3 + \cos \pi \mu_3) + (A_3^n B_3^n + A_3^n B_3^n - A_3^n B_1) \cos \frac{\pi \mu_1^n}{2} \sin \pi \mu_3^n +$	$A_1''B_3'\cos\frac{\pi\mu_1''}{2}\left( \operatorname{ch} \pi\mu_3 - \cos\pi\mu_3' \right) - \left(A_3'B_3' + A_3'B_3'' - A_3'B_1' \right) \sin\frac{\pi\mu_1'}{2}\sin\pi\mu_3' + $
	$+(A''_3B'_3-A'_3B''_3-A''_3B'_1)\cos\frac{\pi\mu'_1}{2}\sin \pi\mu'_3=0$	$+ (A'_3B'_3 - A'_3B''_3 - A''_3B'_1) \sin \frac{\pi\mu'_1}{2} \sin \pi\mu'_3 = 0$

TABLE 1

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Buckling conditions for biaxially loaded sandwich plates with two simp	ly supported edges and two edges having relaxed clamping (The complete expressions apply to plates with an orthotropic core	layer; in
	case of core isotropy the underlined terms must be omitted).	

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Class of solution	Buckling conditions (see equation (15b))					
(see page 10)	Symmetric modes	Antisymmetric modes				
1	$\left(A_{3}''B_{3}^{*'} - A_{3}'B_{3}^{*'}\right) \operatorname{ch} \frac{\pi \mu_{1}'}{2} \left(\operatorname{ch} \pi \mu_{3}' - \cos \pi \mu_{3}''\right) + \left(A_{1}'B_{3}^{*'} - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sin} \pi \mu_{3}'' + \left(A_{1}'B_{3}^{*'} - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}^{*'} - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}^{*'} - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}^{*'} - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}^{*'} - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}'' - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}'' - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}'' - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}'' - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}'' - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}'' - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}'' - A_{3}'B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}'' - A_{3}''B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}'' - A_{3}''B_{1}^{*'}\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}'B_{3}'' - A_{3}''B_{1}''\right) \operatorname{sh} \frac{\pi \mu_{1}'}{2} \operatorname{sh} \pi \mu_{3}'' + \left(A_{1}''B_{3}'' - A_{3}''''''''''''''''''''''''''''''''''''$	$(A_3''B_3^{*'} - A_3'B_3^{*''})$ sh $\frac{\pi\mu_1'}{2}$ (ch $\pi\mu_3' + \cos \pi\mu_3'') + (A_3'B_1^{*'} - A_1'B_3^{*'})$ ch $\frac{\pi\mu_1'}{2}$ sin $\pi\mu_3'' +$				
	$+(A'_1B^{*''}_3-A''_3B^{*'}_1)$ sh $\frac{\pi\mu'_1}{2}$ sh $\pi\mu'_3=0$	$- + (A'_1 B_1^{*''} - A''_3 B_1^{*'}) \text{ ch } \frac{\pi \mu'_1}{2} \text{ sh } \pi \mu'_3 = 0$				
2	$(A'_{2}B^{\bullet'}_{1} - A'_{1}B^{\bullet}_{3})$ sh $\frac{\pi\mu'_{1}}{2}$ ch $\frac{\pi\mu'_{3}}{2}$ sh $\frac{\pi\mu'_{3}}{2} + (A'_{1}B^{\bullet}_{3} - A'_{3}B^{\bullet}_{1})$ sh $\frac{\pi\mu'_{1}}{2}$ sh $\frac{\pi\mu'_{3}}{2}$ ch $\frac{\pi\mu'_{3}}{2}$	$(A'_{5}B^{*'}_{1} - A'_{1}B^{*'}_{5})$ ch $\frac{\pi\mu'_{1}}{2}$ sh $\frac{\pi\mu'_{3}}{2}$ ch $\frac{\pi\mu'_{5}}{2} + (A'_{1}B^{*'}_{3} - A'_{3}B^{*'}_{1})$ ch $\frac{\pi\mu'_{1}}{2}$ ch $\frac{\pi\mu'_{3}}{2}$ sh $\frac{\pi\mu'_{5}}{2}$				
	+ $(A'_3B''_3 - A'_5B''_3)$ ch $\frac{\pi\mu'_1}{2}$ sh $\frac{\pi\mu'_3}{2}$ sh $\frac{\pi\mu'_5}{2} = 0$	+ $(A'_3B_5^{*'} - A'_5B_5^{*'})$ sh $\frac{\pi\mu'_1}{2}$ ch $\frac{\pi\mu'_3}{2}$ ch $\frac{\pi\mu_5}{2}$ = 0				
3	$(A'_1B_5^{**} - A''_5B_1^{*}) \text{ sh } \frac{\pi\mu'_1}{2} \text{ ch } \frac{\pi\mu'_3}{2} \sin \frac{\pi\mu''_5}{2} + (A'_1B_5^{**} - A'_3B_1^{**}) \text{ sh } \frac{\pi\mu'_1}{2} \text{ sh } \frac{\pi\mu'_3}{2} \cos \frac{\pi\mu''_5}{2}$	$(A_5''B_1^{*'} - A_1'B_1^{*''})$ ch $\frac{\pi\mu_1'}{2}$ sh $\frac{\pi\mu_3'}{2^4}\cos\frac{\pi\mu_3''}{2} + (A_1'B_3^{*'} - A_3'B_1^{*'})$ ch $\frac{\pi\mu_1'}{2}$ ch $\frac{\pi\mu_3'}{2}\sin\frac{\pi\mu_3'}{2}$				
	+ $(A_5'B_3'' - A_3'B_3'')$ ch $\frac{\pi\mu_1'}{2}$ sh $\frac{\pi\mu_3'}{2}$ sin $\frac{\pi\mu_3'}{2} = 0$	+ $(A'_{3}B^{*''}_{5} - A''_{5}B^{*'}_{5})$ sh $\frac{\pi\mu'_{1}}{2}$ ch $\frac{\pi\mu'_{3}}{2}\cos\frac{\pi\mu'_{3}}{2} = 0$				
4	$(A_3'B_1^{*\prime} - A_1'B_3^{*\prime\prime}) \text{ sh } \frac{\pi\mu_1'}{2} \sin \frac{\pi\mu_3''}{2} \cos \frac{\pi\mu_5''}{2} + (A_1'B_3^{*\prime\prime} - A_5'B_1^{*\prime}) \text{ sh } \frac{\pi\mu_1'}{2} \cos \frac{\pi\mu_3''}{2} \sin \frac{\pi\mu_5''}{2}$	$(A_1'B_3^{*''} - A_3''B_1^{*'}) \text{ ch } \frac{\pi\mu_1'}{2}\cos\frac{\pi\mu_3''}{2}\sin\frac{\pi\mu_5''}{2} + (A_5''B_1^{*'} - A_1'B_5^{*''}) \text{ ch } \frac{\pi\mu_1'}{2}\sin\frac{\pi\mu_3''}{2}\cos\frac{\pi\mu_5''}{2}$				
	+ $(A_3''B_3^{*''} - A_3''B_3^{*''})$ ch $\frac{\pi\mu_1'}{2}\sin\frac{\pi\mu_3''}{2}\sin\frac{\pi\mu_3''}{2} = 0$	+ $(A_3'' B_3^{*''} - A_3'' B_3^{*''})$ sh $\frac{\pi \mu_1'}{2} \cos \frac{\pi \mu_3''}{2} \cos \frac{\pi \mu_3''}{2} = 0$				
5	$(A_3''B_5''' - A_5''B_3''')\cos\frac{\pi\mu_1''}{2}\sin\frac{\pi\mu_3''}{2}\sin\frac{\pi\mu_3''}{2} + (A_5''B_1''' - A_1''B_5''')\sin\frac{\pi\mu_1''}{2}\cos\frac{\pi\mu_3''}{2}\sin\frac{\pi\mu_3''}{2}$	$(A_3''B_5''' - A_3''B_5''')\sin\frac{\pi\mu_1''}{2}\cos\frac{\pi\mu_3''}{2}\cos\frac{\pi\mu_3''}{2} + (A_5''B_1''' - A_1''B_5''')\cos\frac{\pi\mu_1''}{2}\sin\frac{\pi\mu_3''}{2}\cos\frac{\pi\mu_3''}{2}$				
	+ $(A_1'' B_3^{*''} - A_3'' B_1^{*''}) \sin \frac{\pi \mu_1''}{2} \sin \frac{\pi \mu_3''}{2} \cos \frac{\pi \mu_5''}{2} = 0$	+ $(A_1''B_3^{*''} - A_3''B_1^{*''})\cos\frac{\pi\mu_1'}{2}\cos\frac{\pi\mu_3'}{2}\sin\frac{\pi\mu_3'}{2} = 0$				
6	$(A'_{3}B_{1}^{*''} - A''_{1}B_{3}^{*'}) \sin \frac{\pi \mu_{1}''}{2} \sin \pi \mu_{3}'' + (A''_{3}B_{1}^{*''} - A''_{1}B_{3}^{*''}) \sin \frac{\pi \mu_{1}''}{2} \sin \pi \mu_{3}' +$	$(A'_{3}B^{*''}_{1} - A''_{1}B^{*}_{3})\cos\frac{\pi\mu''_{1}}{2}\sin\pi\mu''_{3} + (A''_{1}B^{*''}_{3} - A''_{3}B^{*''}_{1})\cos\frac{\pi\mu''_{1}}{2}ch\pi\mu'_{3}$				
	+ $(A_3''B_3^{*'} - A_3'B_3^{*''})\cos\frac{\pi\mu_1''}{2}(\operatorname{ch}\pi\mu_3 - \cos\pi\mu_3') = 0$	$+(A_3''B_3^{*'}-A_3'B_3^{*'})\sin\frac{\pi\mu_1''}{2}(\operatorname{ch}\pi\mu_3+\cos\pi\mu_3'')=0$				

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# TABLE 2

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TABLE 3 Values of  $\tilde{p}_x$  for orthotropic plate in uniaxial compression  $(\eta = 1.5, p_y = 0)$ 

<u></u>		`		<i>,</i>	_	
$\overline{\ }$	S=.1	.2	.3	.4	.6	1
$\rho =$	<u> </u>					
.5	14.069					
.6	10.261	8.360				
.7	7.991	6.475	5.449	4.704		
.8	6.543	5.264	4.406	3.787	2.949	
.9	5.571	4.442	3.692	3.153	2.429	1.644
1	4,895	3.856	3.174	· 2.688	2.043	1.362
1.1	4,407	3.419	2.779	2.329	1.744	1.147
1.2	4.042	3.076	2.461	2.038	1.506	.981
1.3	3.756	2.790	2.194	1.798	1.314	.850
1.4	3.518	2.541	1.966	1.597	1.157	.744
1.5	3,308	2.303	1.765	1.428	1.028	.656
1.6	3.047	2.096	1.596	1.284	.920	.583
1.7	2.822	1.920	1.451	1.162	.826	.525
1.8	2.630	1.769	1.327	1.055	.746	.470
1.9	2.466	1.636	1.215	.963	.680	.422
2	2.321	1.521	1.120	.883	.619	.379
2.1	2.191	1.412	1.032	.811	.575	.349
2.2	2.073	1.314	.956	.747	.522	.317
2.3	1.963	1.226	.886	.690	.479	.291
2.4	1.854	1.147	.824	.645	.441	.265
2.5	1.755	1.074	.768	.615	.406	.250
2.6	1.665	1.008	.718	.563	.377	.223
2.7	1.582	.948	.679	.543	.354	.214
2.8	1,504	.892	.662	.484	.331	.199
2.9	1.432	.841	.605	.453	.300	.179
3.—	1.364	.793	.570	.423	.282	.176

TABLE 4 Values of  $\tilde{p}_x$  for orthotropic plate in uniaxial compression.  $(n = 1.3, p_n = 0)$ 

			η - 1.2, py	0,		
$\overline{\ }$	S = .1	.2	.3	.4	.6	1.—
$\cdot \rho = $	<u> </u>					
.5	13.670					
.6	9.985	7.995				
.7	7,792	6.210	5.167	4.422		
.8	6.394	5.066	4.193	3.572	2.740	1.842
.9	5.439	4.289	3.523	2.979	2.254	1.490
1.—	4.808	3.734	3.033	2.537	1.889	1.232
1.1	4.339	3.315	2.651	3.190	1.606	1.037
1.2	3,987	2.978	2.338	1.908	1.384	.886
1.3	3.707	2.691	2.075	1.677	1.205	.766
1.4	3.466	2.421	1.851	1.485	1.060	.670
1.5	3.221	2.197	1.661	1.325	.937	.588
1.6	2.966	2.001	1.499	1.189	.832	.515
1.7	2.752	1.832	1.359	1.070	.756	.457
1.8	2.570	1.684	1.237	.969	.682	.408
1.9	2.410	1.552	1.129	.879	.610	.367
2.—	2.268	1.436	1.034	.806	.554	.328
2.1	2.137 -	1.329	.950	.742	.499	.298
2.2	2.014	1.233	.880	.683	.456	.274
2.3	1.897	1.146	.808	.623	.414	.251
2.4	1.791	1.068	.759	.573	.388	.230
2.5	1.693	.997	.699	.528	.355	.215
2.6	1.603	.937	.648	.483	.330	.197
2.7	1.518	.876	.604	.455	.300	.181
2.8	1.439	.818	.564	.419	.288	.168
2.9	1.366	.788	.521	.395	.261	.157
3.—-	1.297	.735	.489	.368	.252	.147

TABLE 5 Values of  $\tilde{p}_x$  for orthotropic plate in uniaxial compression.  $(\eta = .75, p_y = 0)$ 

$\swarrow$	S = 1	.2	.3	.4	.6	1
$\rho =$	\					
.4	17.612					
.5	11.793	8.758				
.6	8.665	6.439	5.132	4.264	3.168	
.7	6.811	5.059	4.022	3.323	2.430	1.534
.8	5.635	4.173	3.288	2.683	1.927	1.180
.9	4.852	3.557	2,754	2.215	1.547	.933
1.—	4.305	3.091	2.341	1.855	1.254	.759
1.1	3.900	2.710	2.009	1.563	1.037	.624
1.2	3.572	2.389	1.742	1.306	.873	.522
1.3	3.275	2.114	1.481	1.116	.746	.456
1.4	2.979	1.881	1.279	.965	.639	.390
1.5	2.735	1.671	1.119	.845	.564	.337
1.6	2.524	1.467	.981	.736	.498	.266
1.7	2.334	1.302	.882	.655	.445	.240
1.8	2.159	1.167	.785	.586	.390	.211
1.9	1.996	1.040	.698	.528	.353	.190
2.—	1.845	.941	.630	.473	.324	.174
2.1	1.710	.857	.571	.440	.284	.158
2.2	1.552	.784	.525	.394	.266	.148
2.3	1.428	.725	.478	.363	.250	.133
2.4	1.312	.656	.441	.328	.220	.126
2.5	1,211	.605	.403	.305	.206	.114
2.6	1.126	.559	.375	.288	.191	.113
2.7	1.039	.518	.352	.271	.185	.098
2.8	.975	.483	.335	.244	.167	.092
2.9	.893	.453	.301	.231 .	.154	_
3.—	.836	.423	.283	.214	.147	

TABLE 6 Values of  $\tilde{p}_x$  for orthotropic plate in uniaxial compression.  $(\eta = .65, p_v = 0)$ 

		· ·	,, <u>r</u> y	-)		
$\overline{}$	S=.1	.2	.3	.4	.6	1
$\rho = 0$	<u> </u>					
.4	16.748					
.5	11.229	8.161 .	6.426			
.6	8.263	6.017	4.740	3.907	2.863	1.811
.7	6.507	4.741	3.722	3.042	2.191	1.334
.8	5.395	3.916	3,037	2.447	1.705	1.021
.9	4.654	3.334	2.532	2.022	1.345	.811
1.—	4.132	2.883	2.139	1.635	1.088	.658
1.1	3.737	2.513	1.800	1.353	.901	.540
1.2	3.407	2,200	1.509	1.131	.766	.456
1.3	3.089	1.927	1.290	.970	.644	.341
1.4	2.813	1.670	1.112	.836	.557	.295
1.5	2.578	1.450	.968	.724	.485	.256
1.6	2.372	1.278	.853	.638	.432	.229
1.7	2.178	1.138	.753	.568	.378	.206
1.8	1.996	1.011	.674	.505	.342	.184
1,9	1.813	.903	.604	.459	.310	.167
2.—	1.630	.814	.545	.422	.288	.151
2.1	1.478	.741	.494	.371	.255	.140
2.2	1.360	.677	.456	.353	.227	.132
2.3	1.234	.620	.415	.324	.214	
2.4	1.130	.570	.382	.291	.192	
2.5	1.050	.531	.351	.264	.178	
2.6	.971	.484	.330	.244	.170	
2.7	.895	.451	.300	.225	.154	
2.8	.835	.420	.294	.213		
2.9	.780	.392	.261	.199		
3	.728	.368	.246	.186		_

TABLE 7Values of  $\tilde{p}_y$  for orthotropic panel in uniaxial compression. $(\eta = 1.5, p_x = 0)$ 

TABLE 8
Values of $\tilde{p}_y$ for orthotropic panel in uniaxial compression.
$(\eta=1.3, p_x=0)$

		. (				
$\overline{}$	S=.1	.2	.3	.4	.6	1.—
$\rho = $	<u> </u>				<u> </u>	
.3	4.317	3.014	2.214.	1.668	1.113	.670
.4	4.333	3.012	}	1	1	1
.5	4.333	3.013	)		Ì	
.6	4.316	3.012	}			ļ
.7	4.361	3.028				
.8	4.383	3.012	}		)	}
.9	4.316	3.023	ļ	ļ	]	
1.—	4.333	3.045	l			ļ
1.1	4.399	3.018	2.214			
1.2	4.492	3.012	2.215	{	ĺ	
1.3	4.600	3.017	2.215		Í	1
1.4	4.594	3.029	2.214		Ì	
1.5	4.465	3.045	2.214		Ì	Í
1.6	4.383	3.062	2.215	1		
1.7	4.336	3.079	2.215			1
1.8	4.316	3.096	2.216	1		{
1.9	4.317	3.070	2.216			1
2	4.333	3.045	2.214			
2.1	4.361	3.028		1	1	1
2.2	4.399	3.018	Į	1	1	}
2.3	4.443	3.013	ļ			
2.4	4.492	3.012	2.214		}	
2.5	4.545	3.013	2.215			
2.6	4.600	3.017	2.215		ļ	j
2.7	4.657	3.023	2.216	ŀ	j	]
2.8	4.714	3.029	2.216			
2.9	4.772	3.037	2.217	ſ	ſ	1
3.—	4.829	3.045	2.218	1.668	1.113	.670

<u> </u>						
$\searrow$	S = .1	.2	.3	.4	.6	1
$\rho = $	\					
.3	4.390	3.147	2.401	1.904	1.283	.770
.4	4.394	3.139	2,402		{	1
.5	4.427	3,139	2.401			
.6	4.369	3.164	2.402			
.7	4.476	3.140	2.405			
.8	4.394	3.170	2.402	ļ	j	ļ
.9	4.369	3.164	2.402			
1	4,427	3.139	2.412			
1.1	4.534	3.146	2.411	Í	Í	
1.2	4.669	3.170	2.402		Ì	Í
1.3	4.728	3.230	2.401		l l	Í
1.4	4.561	3.241	2.405		l l	Ì
1.5	4.455	3.279	2.412		1	1
1.6	4.394	3.245	2.420		{	}
1.7	4.368	3.196	2.428		- { ·	}
1.8	4.369	3.164	2.437		ł	}
1.9	4.390	3.147	2.445	}	Į	
2.—	4.427	3.139	2.436			
2.1	4.476	3.140	2.421		ļ	
2.2	4.534	3.146	2.411	1		
2.3	4.599	3.156	2.405			
2.4	4.669	3.170	2.402		j	
2.5	4.743	3.186	2.401			
2.6	4.818	3.203	2.401			
2.7	4.895	3.222	2.402	{	1	(
2.8	4.972	3.241	2.405	{	1	( ·
2.9	5.050	3.260	2.408	1	ĺ	1
3.—	5.127	3.279	2.412	1.904	1.283	.770

TABLE 9

Values of  $\tilde{p}_y$  for orthotropic panel in uniaxial compression. ( $\eta = .75, p_x = 0$ )

TABLE 10
Values of $\tilde{p}_y$ for orthotropic panel in uniaxial compression.
$(\eta = .65, p_x = 0)$

$\angle$	S=.1	.2	3	.4	.6	1
$\rho =$	<u> </u>					
.3	4.270	3.224	2.629	2.234	1.730	1.207
.4	4.277	3.229	2.632	2.236	1.731	
.5	4.256	3.211	2.620	2.229	1.731	
.6	4.414	3.320	2.694	2.277	1.749	}
.7	4.276	3.220	2.627	2.335	1.740	
.8	4.277	3.229	2.632	2.236	1.731	[
.9	4.414	3.320	2.694	2.277	1.749	(
1.—-	4.636	3.457	2.783	2,338	1.778	(
1.1	4.751	3.522	2.847	2.406	1.811	(
1.2	4.502	3.360	2.728	2.317	1.802	1
1.3	4.353	3.265	2.659	2.262	1.761	1
1.4	4.276	3.220	2.627	2.235	1.740	ł
1.5	4.256	3.211	2.620	2.229	1.731	
1.6	4.277	3.229	2.632	2.236	1.731	
1.7	4.332	3.267	2.658	2.253	1.738	
1.8	4.414	3.320	2.694	2.277	1.749	
1.9	4.517	3.385	2.736	2.306	1.762	1
2.—	4.636	3.457	2.783	2.338	1.778	
2.1	4.768	3.535	2.833	2.371	1.794	
2.2	4.911	3.617	2.885	2.406	1.811	ļ
2.3	5.062	3.702	2.938	2.441	1.828	
2.4	5.220	3.788	2.990	2.475	1.844	1
2.5	5.382	3.874	3.042	2.509	1.861	· {
2.6	5.547	3.960	3.094	2.542	1.876	
2.7	5,714	4.046	3.144	2.574	1.892	ĺ
2.8	5.883	4.129	3.193	2.605	1.906	{
2.9	6.051	4.212	3.240	2.635	1.920	
3.—	6.220	4.292	3.285	2.664	1.934	1.207

0=	2≡.1	.2			.0	1
.3	4.202	3,188	2.608	2.232	1.758	1.257
.4	4.213	3.208	2.639	2.254	1.773	1.258
.5	4.170	3.167	2.608	2.237	1.761	1.258
.6	4.346	3.236	2.639	2.254	1.773	1.253
.7	4.170	3.152	2.591	2.222	1.752	1.251
.8	4.213	3.208	2.642	2.265	1.778	1.258
.9	4.390	3.345	2.746	2.344	1.826	1.276
1	4.653	3.526	2.876	2.438	1.881	1.298
1.1	4.565 .	3.366	2.727	2.320	1.817	1.299
1.2	4.346	3.236	2.639	2.254	1.773	1.270
1.3	4.223	3.171	2.598	2.225	1.754	1.251
1.4	4.170	3.152	2.591	2.222	1.752	1.251
1.5	4.170	3.167	2.608	2.237	1.761	1.253
1.6	4.213	3.208	2.642	2.265	1.778	1.258
1.7	4.288	3.269	2.690	2.301	1.800	1.266
1.8	4.390	3.345	2.746	2.344	1.826	1.276
1.9	4.513	3.431	2.809	2.391	1.853	1.287
2.—	4.653	3.526	2.876	2.440	1.881	1.298
2.1	4.807	3.626	2.945	2.490	1.910	1.309
2.2	4.972	3.730	3.016	2.540	1.938	1.320
2.3	5.145	3.837	3.087	2.590	1.966	1.331
2.4	5.326	3.945	3.158	2.640	1.993	1.341
2.5	5.512	4.053	3.227	2.688	2.019	1.351
2.6	5.701	4.160	3.296	2.734	2.043	1.360
2.7	5.894	4.267	3.362	2,779	2.067	1.369
2.8	6.088	4.372	3.427	2.823	2.090	1.377
2.9	6.283	4.473	3.490	2.864	2.111	1.385
3.—-	6.478	4.575	3.550	2.904	2.131	1.392



Fig. 1. Biaxially compressed sandwich plate. Notations.



Fig. 3. Interaction curve  $(p_x \sim p_y)$ . Comparison of present results and those of two approximate analyses (refs. 3 and 4) for square isotropic panel with relatively weak core  $(\rho = \eta = 1, S = .4)$ .



Fig. 5. Normalized interaction curve for square isotropic panel with relatively stiff core ( $\rho = \eta = 1, S = .2$ ).

.



Fig. 2. Structural end fixtures represented by present solution.



Fig. 4. Normalized interaction curves  $(\alpha_x \sim \alpha_y)$ . Comparison of present results and those of two approximate analyses (refs. 3 and 4) for square isotropic panel with relatively weak core ( $\rho = \eta = 1, S = .4$ ).







Fig. 7. Normalized interaction curve for rectangular isotropic panel with relatively stiff core ( $\rho = .25$ ,  $\eta = 1$ , S = .2).







Fig. 8. Normalized interaction curve for rectangular isotropic panel and relatively weak core ( $\rho = 4$ ,  $\eta = 1$ , S = .1).



Fig. 10. Normalized interaction curve for square orthotropic panel with relatively weak core ( $\rho = 1, \eta = \sqrt{2}, S = .4$ ).





with relatively stiff core  $(\rho = 1, \eta = \sqrt{2}, S = .2)$ . Fig. 12. Normalized interaction curve for square orthotropic panel





by Tresca's yield criterion. Fig. 14. Domain of applicability for elastic solution as determined



with relatively weak core  $(p = 1, n = \sqrt{2^{-1}}, 5 = .4)$ . Fig. 11. Normalized interaction curve for square orthotropic panel

•



with relatively stiff core (p = 1,  $n = \sqrt{2^{-2}}$ , S = .2). Fig. 13. Normalized interaction curve for square orthotropic panel







Fig. 17. Values of buckling load parameter  $\tilde{p}_x$  for orthotropic plate in uniaxial compression (see table 5;  $\eta = .75$ ,  $\varepsilon = 0$ ).







. Fig. 18. Values of buckling load parameter  $\tilde{p}_x$  for orthotropic plate in uniaxial compression (see table 6;  $\eta = .65$ ,  $\varepsilon = 0$ ).



















# REPORT NLR-TR T.136

# A survey of symmetrical transonic potential flows around quasi-elliptical aerofoil sections

# by

# J. W. BOERSTOEL

#### Summary

The hodograph method and its application as described in reference 1 allow the construction of steady transonic potential flows around aerofoil sections. This report gives results of the application of this method as far as non-lifting symmetrical sections are concerned. The results show that with a restricted use of the possibilities of the method an already remarkable variety of practically interesting sections and corresponding pressure distributions can be obtained. The sections can be computed with a precision sufficient to meet accuracy requirements of model manufacturing.

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## List of symbols

•	
с	chord length
М	Mach number
$M_1$	free stream Mach number
M _m	maximum Mach number on aerofoil section
M _s	rate of change of Mach number along aerofoil
	section surface
R·	radius of curvature .
$R_0$	R at stagnation point
t	maximum thickness of aerofoil section
x	co-ordinate (see remark at the end of page 2)
X _{Mm}	chordwise position of suction peak

This investigation has been performed under contract with the Netherlands Aircraft Development Board (NIV).

x _n	chordwise position of leading edge
$x_t$	chordwise position of maximum thickness point
у	co-ordinate (see remark at the end of page 5)
3	thickness parameter (cf. sec. 2.1)
θ	flow angle
λ	parameter (cf. section 2.1)
λ,	maximum allowable value of $\lambda$ (cf. sec. 2.2.3)
$\lambda_{R_0=0}$	minimum allowable value of $\lambda$ to obtain blunt
	leading edge (cf. sec. 2.2.2)
τ	velocity parameter (cf. sec. 2.1)
$\tau_1$	free stream value of $\tau$

# **1** Introduction

This report gives the results of the computation of a number of non-lifting quasi-elliptical aerofoil sections in steady transonic potential flows.

The computations are based on the hodograph method of ref. 1. This method transforms the hodograph of the incompressible flow around a known contour into the hodograph of the compressible flow around an aerofoil section. As neither the shape of the section, nor its pressure distribution can be specified in advance, but depend upon parameters (the number of parameters can be made arbitrarily large), the sections and their flow fields can be determined only by explicit computation for specified values of the parameters.

The sections published in this report are constructed by transforming the hodograph of the symmetrical incompressible flow around a non-lifting elliptical cylinder (ref. 1, sec. 4.2). Three parameters were selected to be varied. The combinations of parameter values are chosen such as to obtain a survey of the possibilities most interesting from a practical point of view. Flows in the transonic region are emphasized.

2		
2		

TABLE 1 - Main aerofoil characteristics.

					•					
aerofoil section	M ₁	M _m	t/c	¢	$\frac{x_t - x_n}{c}$	$\frac{x_{M_m} - x_n}{c}$	c/ <b>R</b> 0	<i>x</i> _n	type of pressure distribution	figure
0.10-0.675-1.5	0.74536	1.294	0.1625	3.1841	0.3162	0.0497	16.1112	- 1.68755	B1	8,1
0.10-0.675-1.6	0.74536	1.474	0.1632	3.1694	0.3051	0.0306	15.0026	- 1.68755	B1-B2	8.2
0.1025-0.675-1.3	0.75567	1.226	0.1572	3.2031	0.3355	0.0827	21.6856	- 1.69058	B1C	8.3
0.1025-0.675-1.4	0.75567	1.298	0.1580	3,1878	0.3196	0.0588	18.2715	1.69058	B1-C	8.4
0.1050-0.675-1.25	0.76589	1.294	0.1528	3.2009	0.3384	0.0649	23.3954	- 1.69379	С	8.5
0.1075-0.675-1.05	0.77604	1.257	0.1467	3.2249	0.3639	0.0846	37.470	- 1.69718	С	8.6
0.12-0.7-0.0	0.82572	1.123	0.1076	3.3807	0.5000	0.3487	00	- 1.6904		8.7
0.11-0.75-0.0	0.78612	1.030	0.1108	3.4700	0.5000	0.5000		-1.7350		8.8
0.11-0.75-0.9	0.78612	1.064	0.1138	3.3771	0.3870	0.2016	165.819	-1.7950	A	8.9
0.11-0.75-1.25	0.78612	1.136	0.1163	3.3053	0.3419	0.0826	30.7975	- 1.7950	Α	8.10
0.11-0.75-1.35	0.78612	1.222	0.1170	3.2838	0.3280	0.0214	24.1684	- 1.7950	<b>B</b> 1	8.11
0.11-0.75-1.375	0.78612	1.306	0.1172	3.2783	0.3245	0.0182	22.902	- 1.79504	B1	8.12
0.11-0.75-1.4	0.78612	1.360	0.1174	3.2728	0.3209	0.0196	21.7537	- 1.7950	B2-B1	8.13
0.1125-0.75-1.325	0.79612	1.286	0.1138	3.2712	0.3201	0.0395	24.9380	-1.7993	<b>B</b> 1	8.14
0.1125-0.75-1.35	0.79612	1.331	0.1140	3.2657	0.3199	0.0313	23.5666	- 1.7793	B2-B1	8.15
0.1125-0.75-1.395	0.79612	1.535	0.1144	3.2551	0.3152	0.0188	21.4137	- 1.7993	B2	8.16
0.1150-0.75-1.20	0.80605	1.259	0.1092	3.2851	0.3322	0.0527	34.00	- 1.80375	С	8.17
0.12-0.75-0.0	0.82572	1,083	0.0953	3.4596	0.5000	0.5000	80	-1.7298		8.18
0.10-0.775-1.3	0.74536	1.013	0.1158	3.3900	0.3938	0.0132	32.7854	- 1.8107	B2	8.19
0.11-0.775-0.0	0.78612	1.003	0.1018	3.5117	0.5000	0,5000	00	- 1.7559		8.20
0.11-0.775-0.9	0.78612	1.023	0.1038	3.4203	0.3912	0.2342	263.688	- 1.8265	Α	8.21
0.11-0.775-1.3	0.78612	1.183	0.1065	3.3337	0.3357	0.0151	32.2409	-1.8265	<b>B</b> 2	8.22
0.1175-0.8-1.15	0.81592	1.086	0.0888	3.3510	0.3418	0.1098	50.027	- 1.8725	A	8.23
0.1175-0.8-1.26	0.81592	1.299	0.0897	3.3164	0.3269	0.0106	33.237	- 1.8725	B2	8.24
0.12-0.8-1.15	0.82572	1.149	0.0859	3.3405	0.3455	0.0751	49.04	- 1.87769	Α	8.25
0.12-0.8-1.19	0.82572	1.168	0.0857	3.3285	0.3388	0.0611	41.553	- 1.8777	A–B3	8.26
0.12-0.8-1.25	0.82572	1.326	0.0864	3.3022	0.3298	0.0108	33.514	- 1.8777	<b>B</b> 3	8.27
0.1225-0.8-1.15	0.83547	1.223	0.0827	3.3185	0.3406	0.0483	47.952	-1.88312	C	8.28
0.1250-0.875-1.10	0.84515	1.182	0.0552	3.4128	0.3969	0.0038	88.740	- 1.9868	B3 '	8.29

The merits of the aerofoil sections from a physical point of view (especially the stability of the transonic flow with respect to shock phenomena) will not be discussed here (cf. ref. 1, app. B). A report about the numerical work carried out for the computation of the sections is to be published (ref. 2). Non-symmetrical and lifting aerofoil sections are not included in this report; results for these will be published later.

The results are discussed in section 2. Section 3 gives information about the accuracy attainable.

# 2 Discussion of the aerofoil sections and the corresponding pressure distributions

# 2.1 Main results

As already has been remarked, the aerofoil sections of this report and the transonic flows around these sections are completely determined by three parameters. In the notation of ref. 1 (secs. 4.2, 6.2) these parameters are  $\tau_1$ ,  $\varepsilon$  and  $\lambda$ .  $\tau_1$  determines the free stream Mach number  $M_1$ :

$$M_1 = \left\{ \frac{2\tau_1}{(\gamma - 1)(1 - \tau_1)} \right\}^{\frac{1}{2}}$$

 $\gamma = \text{ratio}$  of specific heats.  $\varepsilon$  is a thickness parameter. It determines the thickness ratio  $\frac{1-\varepsilon}{1+\varepsilon}$  of the elliptical cylinder in the incompressible flow. The thickness ratio of the sections depends upon all three parameters, but mainly upon  $\varepsilon^*$ . The main effect of  $\lambda$  is to control the leading edge curvature and the rate of expansion on the front part of the section.

Values of  $\tau_1$ ,  $\varepsilon$  and  $\lambda$  are used to identify the sections. For example, section 0.11-0.75-0.9 has  $\tau_1 = 0.11$ ,  $\varepsilon = 0.75$  and  $\lambda = 0.9$ .

Table 1 lists all sections computed, together with

^{*} In tables and figures the aerofoil section data have been given in the reference system of the elliptical cylinder in the incompressible flow. The length of the elliptical cylinder is  $2(1+\epsilon)$ , the thickness  $2(1-\epsilon)$ .

some main characteristics and an index for the figures. In the figures, the pressure coefficient  $C_p$ , the section contour, and the sonic line are plotted.

An investigation of the ranges of values of  $\tau_1$ ,  $\varepsilon$  and  $\lambda$  which were of practical interest, revealed the following main results.  $\varepsilon$  can be used to fix approximately the thickness ratio. Having chosen  $\varepsilon$ , the values of  $\tau_1$ and  $\lambda$  are restricted to certain regions (cf. fig. 1). The restrictions are imposed by the interpretability of the results: when either  $\tau_1$ , or  $\lambda$ , or both are chosen too large, limit lines destroy the regularity of the section and of the potential flow around the section; on the other hand, if  $\lambda$  is too small, the leading edge is cusped, which will be in most cases less interesting from a practical point of view. These points will be returned to in section 2.2.

The shape of the pressure distribution curves can be roughly classified as follows (cf. fig. 2). The pressure distribution curve of type A is smooth. The largest values of the acceleration occur at the very front part of the section. Type B pressure distribution curves have a peaked shape and are characterized by a rapid expansion along the section surface from the leading edge up to the suction peak. According to what happens at the downstream side of the suction peak a further subdivision is possible. B1 has a gradual compression from the suction peak downstream, B2 a rapid compression just behind the suction peak, and B3 a rapid compression followed by a smooth local expansion. Type C pressure distribution curves show the phenomenon of a secondary expansion just before the suction peak.

The type of the pressure distribution curve depends upon all three parameters together. This dependence is the subject of the sections 2.2.3 and 2.2.5.

# 2.2 Aerofoil sections with a blunt leading edge

# 2.2.1 General

The investigation has been carried out systematically for three values of  $\varepsilon$ : 0.675, 0.75 and 0.8. These values correspond to thickness ratios of approximately 16%,  $11\frac{12}{6}$  and  $8\frac{1}{2}$  respectively.

At fixed  $\varepsilon$ , there is an important connection between the expansion phenomena on the front part of the section, the leading edge curvature, the effects of a limit line, and the values of  $\tau_1$  and  $\lambda$ . This will be explained in the next two sections.

# 2.2.2 The leading edge radius

The leading edge radius,  $R_0$ , is a simple function of the parameters (ref. 1, sec. 6.2) and can be computed quickly. It is one of the few aerofoil characteristics that can be specified in advance by choosing  $\tau_1$ ,  $\varepsilon$ ,  $\lambda$ suitably. The results of a computation of  $R_0$  at  $\varepsilon = 0.675$ , 0.75 and 0.8 are plotted in the figures 3.1, 3.2 and 3.3. Parts of the curves are not physically significant : these are indicated by broken lines. In this case the aerofoil sections are cusped at the leading edge.

As a consequence, a plot of  $R_0$  versus  $\lambda$  at fixed  $\tau_1$ and  $\varepsilon$  shows, that the aerofoil section is cusped at the leading edge at values of  $\lambda$  below a limiting value,  $\lambda_{R_0=0}$ . Cf. fig. 4 for an example. The lines  $\lambda_{R_0=0}$  as a function of  $\tau_1$  (or  $M_1$ ) are plotted in fig. 1 for  $\varepsilon = 0.675$ , 0.75 and 0.8. At values of ( $\tau_1$ ,  $\lambda$ ) below these lines the sections are cusped at the leading edge; at higher values they are blunt.

At fixed  $(\tau_1, \epsilon)$  the leading edge radius depends strongly upon  $\lambda$ . Cf. figure 4.

# 2.2.3 Limit lines

The aerofoil contour can be considered as the streamline dividing the flow outside the section and the analytic continuation of the flow inside the section. The inner flow usually contains limit lines. Some important practical phenomena can be interpreted in terms of the proximity of the limit line to the section contour. These are discussed now.

Limit lines are a phenomenon associated with the use of the hodograph method. They are cusped curves which bound folds of the physical plane. Usually the limit lines are situated in the inner flow, but sometimes they partially lie in the outer flow. On the limit lines the curvature of streamlines and the acceleration are infinite. An excellent discussion of the limit line and its properties can be found in ref. 3.

The figures 5.1 and 5.2 show examples of limit lines. In these cases they are partially situated in the outer flow.

When the limit line is in the inner flow, and either  $\tau_1(M_1)$ , or  $\lambda$ , or both are increased at fixed  $\varepsilon$ , the cusp of the limit line approaches the section contour from the inside. As it is not possible to give physical meaning to outer flows containing limit lines, limiting values of  $(\tau_1, \lambda)$  are attained if the cusp of the limit line is on the contour. These limiting values,  $\lambda_t$  at given  $\tau_1$  and  $\varepsilon$ , have to be estimated by detailed computations of flow fields in the neighbourhood of the cusps. The values of  $\lambda_t$  for  $\varepsilon = 0.675$ , 0.75 and 0.8 are given by the upper curves of fig. 1. Thus, at values of  $(\tau_1, \lambda)$  above and at the right-hand side of these curves, no aerofoil sections with physical meaning can be computed because of the appearance of limit lines in the outer flow.

From the fact that the acceleration is infinite on the limit line it will be clear, that locally large accelerations along the section surface may be expected if the cusp of the limit line is just below the surface (the point  $(\tau_1, \lambda)$  is then just below the upper curves of fig. 1). Several pressure distributions show this effect. These are of type B1, B2, B3 or C. The figures 6.1 and 6.2 have been prepared to illustrate how the limit line, approaching the section contour at x = -1.85 approx-

Data of aerofoil section 0.1100-0.7500-1.3750.

					·	· · · · · · · · · · · · · · · · · · ·	
τ	x	y	. <i>C</i> _p	<u>M</u>	<i>M</i> _s	θ	1/R
0.0000	- 1.79504	0.00000	1.1642	0.0000	0.0000	1.57080	6.9863
0.0100	· – 1.79177	0.03039	1.0440	0.2247	7.50	1.356	7.07
0.0200		0.04240	0.9269	0.3194	7.76	1.268	7.17
0.0300	- 1.78550	0.05121	0.8127	0.3932	8.06	1.2002	7.29
0.0400	- 1.78252	0.05829	0.7014	0.4564	8.41	1.1437	7.43
0.0500	··· — 1.77965	0.06421	0.5930	0.5130	8.80	1.0943	7.60
0.0600	- 1.77689	0.06926	0.4874	0.5649	9.26	1.0499	7.80
0.0700	-1.77427	0.07364	0.3845	0.6135	. 9.79	1.0095	8.04
0.0800	- 1.77177	0.07745	0.2844	0.6594	10.40	0.9723	8.33
0.0900	- 1.76941	0.08078	0.1870	0.7032	11.11	0.9376	8.66
0.1000	- 1.76720	0.08370	0.0922	0.7454	11.95	0.9052	9.06
0.1100	- 1.76513	0.08625	0.0000	0.7861	12,94	0.8748	9.54
0.1200	- 1.76322	0.08847	-0.0896	0.8257	14.11	0.8460	10.11
0.1300 .	-1.76146	0.09041	-0.1768	0.8644	15.51	0.8187	- 10.81
0.1400	- 1.75986	0.09208	-0.2614	0.9022	17.17	0.7927	11.64
0.1500	- 1.75840	0.09352	-0.3437	0.9393	19.2	0.7678	12.7
0.1600	- 1.75709	0.09476	-0.4235	0.9759	21.5	0.7439	13.9
0.1667	- 1.75629	0.09549	0.4754	1.0000	23.2	0.7284	14.8
0.1700	1.75590	0.09582		1.0120	24.1	0.7208	15.3
0.1800	1.75485	0.09674	-0.5762	1.0476	26.9	0.6983	16.9
0.1900	- 1.75388	0.09753	-0.6492	1.0830	29.5	0.6761	18.5
0.2000	'-1.75297	0.09824	-0.7199	1.1180	31.0	0.6540	19.8
0.2100	-1.75207	0.09892	-0.7885	1.1529	30.0	0.6314	20.0
0.2200	- 1.75106	0.09964	-0.8549	1.1875	25.8	0.6074	18.6
0.2300	- 1.74978	0.10051	-0.9192	1.2221	18.6	0.5807	15.6
0.2400	-1.74770	, 0.10181	-0.9815	1.2566	10.39	0.5476	11.8
0.2500	1.74220	0.10492	-1.0418	1.2910	2.34	0.489	7,49
0.2500	- 1.72838	0.11148	-1.0418	1.2910	- 1.076	0.4017	4,5697
0.2400	- 1.69977	0.12175	-0.9815	1.2566	- 0.8980	0.2987	2,5125
0.2300	- 1.64099	0.13628	-0.9192	1.2221	-0.4414	0.1996	1.044
0.2200	-1.55732 -	0.15027	-0.8549	1.1875	-0.3645	0.14053	0.478
0.2100	- 1.44789	0.16338	~0.7885	1.1529	- 0.27249	0.10097	0.2779
0.2000	1.30165	0.17562	-0.7199	1.1180	-0.210443	0.068200	0.18393
0.1900	-1.11730	0.18541	- 0.6492	1.0830	-0.172948	0.039374	0.134456
0.1800	~0.89108	0.19120	-0.5762	1.0476	-0:140222	0.012888	0.102660
0.1700	-0.60460	0.19099	-0.5010	1.0120	-0.11236	-0.013193	0.08238
0.1667	-0.49538	0.18904	-0.4754	1.00001	-0.107358	-0.021972	0.078596
0.1600	0.26667	0.18204	-0.4235	0.9759	-0.1052	-0.039447	0.07491
0.1500	0.06814	0.16461	-0.3437	0.9393	-0.11582	-0.064604	0.07613
0.1400	0.36217	0.14225	-0.2614	0.9022	-0.13856	- 0.087696	0.08106
0.1300	0.60800	0.11805	0.1768	0.8644	-0.171	-0.1084	0.0864
0.1200	0.81000	0.09425	- 0.0896	0.8257	-0.211	-0.1263	0.0885
0.1100	0.97800	0.07166	0.0000	0.7861	-0.261	-0.141	0.0832
0.1000	1.1174	0.0511	0.0922	0.7454	-0.3233	-0.15130	0.0601
0.0900	1.23303	0.03318	0.1870	0.7032	-0.400895	-0,155512	0.001624
0.0800	1.33002	0.01811	0.2844	0.6594	-0.49922	-0.14998	-0.13936
0.0700	1.41145	0.00661	0.3845	0,6135	-0.62687	-0.12601	-0.52725
0.0600	1.47992	0.00008	0.4874	0.5649	- 0.798	-0.0343	4.712
0.0595	1.4833	0.00000	0.4924	0.5625		0.0000	

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imately for  $\lambda$  increasing, has effect upon the whole flow field, and changes the type of the pressure distribution curve from A through intermediate shapes to B3.

# 2.2.4 The curvature of the sections

As the curvature of the streamlines is infinite on the limit line, the section will have locally large values of the curvature if the cusp of the limit line is just below the section surface. It has been explained in section 2.2.2, that the leading edge curvature then will be relatively small, because  $\lambda$  is large. The largest values of the curvature, occurring at the leading edge for low values of  $\lambda > \lambda_{R_0=0}$ , transfer to the region of the cusp, if  $\lambda$  is increased to values just below  $\lambda_{l}$ .

# 2.2.5 The choice of the parameters and the different types of pressure distribution

The question arises how the pressure distributions of the different types are related to the choice of the parameters. The collection of aerofoil sections is large enough to give a guide to the answer.

In figure 7 one of the graphs of fig. 1 has been reproduced. In this figure regions have been roughly indicated corresponding to the different types of pressure distributions.

At other values of  $\varepsilon$  than 0.75 the qualitative relation between the parameter choices and the types of pressure distribution is similar. Type B3 replaces type B1 distributions if the aerofoil is thin, i.e. if  $\varepsilon$  is large.

# 2.3 Aerofoil sections with a cusped leading edge

Four aerofoil sections are cusped at both the leading edge and the trailing edge. The double symmetry of these sections comes from the choice  $\lambda = 0$ . Sections with single symmetry having both ends cusped are obtained for  $0 < \lambda \leq \lambda_{R_0=0}$ .

The sharp bends of the pressure distribution curve of section 0.12-0.7-0.0 (fig. 8.7) indicate that the cusps of two limit lines are situated in the vicinity of the section contour. Similar effects, to a smaller extent, are visible in the graph of section 0.12-0.75-0.0 (fig. 8.18).

### 3 On the accuracy of the results

A critical point for the practical value of the results of this report is the precision which can be obtained in computing the aerofoil sections. If model manufacturing is taken as a standard for judging the quality of section data, the accuracy requirements to be satisfied in making sections is a suitable reference. For aerodynamic testing of models with about 200 mm chord length, at present one aimes at a precision of 0.01 mm  $(5 \times 10^{-5} \times \text{chord length})$  of the ordinates and of 0.001 radian of the slopes at the NLR.

It is possible to meet these requirements. With the numerical equipment now available a precision of the order of  $10^{-5} \times$  chord length for the ordinates and of the order of  $10^{-4}$  in general for the slopes can be attained. As an example the data of section 0.11-0.75-1.375 are given in table 2.  $\tau$ ,  $C_p$  and M are exactly rounded to the last decimal place specified. x, y,  $M_s$ ,  $\theta$  and 1/R are correct within some units of the last place. All aerofoil sections can be computed to a similar precision. However, for the purpose of this report – being a survey of the properties of the family of quasi-elliptical sections considered here – the data were usually computed to a somewhat lower standard of precision. Further details about the accuracy can be found in ref. 2.

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- ³ Von Mises, R., Mathematical theory of compressible fluid flow; Academic Press Inc., New York, 1958.







Fig. 2 Classification of pressure distributions to type.

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Fig. 8.29 Aerofoil section 0.1250-0.875-1.10.

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REPORT NLR-TR. T. 143

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# A digital computer programme for predicting pressure distributions on swept wings in subsonic flow according to a semi-empirical method

by

W. Loeve and A. L. Bleekrode

#### Summary

A computer programme is presented for the application of a semi-empirical method, based on work from the Royal Aircraft Establishment, for calculation of the pressure distribution on wings in subsonic attached flow.

A comparison of calculated results with results of experiments is made with the purpose to determine some of the boundaries of the region of applicability of the method.

From the results presented, conclusions are drawn about the possibilities to increase the region of applicability of the method.

The investigation has been performed under contract with the Ministry of Defense of the Netherlands.

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b _i		coefficients in the polynomial representation	s _i
		$f(u) = \sum_{i=1}^{l} b_{i} u^{i}$	
B _w	Bw	semi-empirical compressibility	S'
C	Cν	local chord	
$C_{c_{1,2}}$	Ccl, Cc2	chord at position of thickness crank	Se
C.	КſtÌ	section chord when the section	t
- <u>e</u>	** <b>[</b> -]	contour is extrapolated to zero	t _R
$C_d$		local drag coefficient	$t_T$
$C_l$		local lift coefficient	+
$C_m$		local moment coefficient with	ı,
0		respect to quarter chord point	
$C_n$	Cn r· ¬	local normal force coefficient	τ.
$C_p$	press[1,a]	pressure coefficient	V
$C_{p_i}$	Cp[a]	pressure coefficient in incom-	V ₀
C		local tangential force coefficient	$v_x$
C _i	(CP)	chord at wing root	W
$C_R$	CT	chord at wing tip	r r
$c_T$ F	F	weight factor	
$f(\mathbf{x})$		function which represents con-	
5 (*)		tour of section	x
Н	Н	parameter defined by eq. (19)	
$I_1, I_2, I_3$		integrals defined by respectively	
		eq. (43), eq. (44) and eq. (45)	
K ₁	}	spanwise interpolation factors	$x_{1}, x_{0}$
$K_2$	K2 ∫	to allow for variation of root and	x
		tip effects on velocities due to	$x_{t_{max}}$
		thickness	
$K_3$	K3	spanwise interpolation factor to	
		allow for effects of finite aspect	У
		ratio and taper on the velocity	
		at the maximum thickness posi-	_
1	t	degree of polynomial which re-	Z
ı	X	presents section contour	• •
L		L+1 is number of points in	$y_{c_{1,2}}$
12		which $f(x)$ is given	7. 70
Μ		M+1 is number of points u in	α, 20
		which $df$ is given	5
		which $\frac{1}{du}$ is given	$o_{1,2}$
$M_0$	M0	Mach number of undisturbed flow	$\delta_R$
n	n	chordwise loading parameter for	
	<b>n</b> ()	minite wing chardwise loading narowstor for	
$n_0$	10	infinite wing	-
Ν	A	number of points x in which $C_{-}$	$\delta_{ au}$
- •	-	is calculated	
p(x)		factor in modified Riegels factor eq. (59)	
s	S	semi-span	•

coefficients in the representation

$$z_s = \sum_{i=1}^{i} s_i x^i$$

spanwise distance of point of intersection of projected leading and trailing edges of the wing from the root.

wing area

- maximum section thickness
- maximum section thickness at wing root
- maximum section thickness at wing tip

coefficients in the representation

 $z_i = \sum_{i=1}^l t_i x^i$ 

local velocity velocity in undisturbed flow local perturbation velocity in

chordwise direction nose shape parameter

$$W = \frac{2x_{i_{\max}}}{t} \sqrt{\frac{\rho}{x_{i_{\max}}}}$$

non-dimensional chordwise coordinate measured from the leading edge in terms of the local chord c

- defined in table 2 (dimensional) co-ordinate defined by eq. (34) position of maximum section thickness in terms of the local chord c
- spanwise co-ordinate measured from the wing root (dimensional quantity)

vertical co-ordinate in terms of local chord c yC1, yC2

spanwise location of surface cranks 1 and 2

defined in table 2 (dimensional) effective angle of attack (degrees) change of spanwise slope of wing surface at crank 1,2

spanwise slope of wing surface defined by

$$\tan \delta_R = \frac{\mathrm{d}}{\mathrm{d}y} \left( c z_{\mathrm{max}} \right)$$

at the wing root spanwise slope of wing surface defined by

$$\tan \delta_T = \frac{-\mathrm{d}}{\mathrm{d}y} (c z_{\mathrm{max}})$$

at the wing tip

Δα	angle between the $x_0$ axis and the section chord (deg.)			aerofoil contour and the con- tour according to the algebraic
$\varDelta_s(x_{t_{\max}})$	additional velocity at the maxi- mum thickness position caused by sweep, thickness taper, plan- form taper and aspect ratio of	9	tha	representation, (dimensional) exponent in the expression for the velocity due to camber, eq. (13)
Δ ₀ d0	the wing spanwise interpolation factor to allow for the reduction of the velocities due to thickness at the	λ	la	spanwise interpolation factor to allow for variation in shape of chordwise loading across the span, eq. (16)
	maximum thickness position near the tip of a semi-infinite	А . Д.,,	Lm	angle of sweep sweep of the mid-chord line
	unswept wing	$\Lambda_{t}^{'''}$	Lax [a]	thickness sweep $\tan \Lambda_t = \tan \Lambda_t = \tan \Lambda_t$
21 ₁ 01	allow for the effects of sweep on the reduction of the velocities	$\Lambda_{t_{\max}}$	Last	sweep of the maximum thickness line
	due to thickness at the maximum thickness position near the tip	$\Lambda_0$ $\Lambda_1$	L0 L1	sweep of the leading edge sweep of the trailing edge
∆ ₃ d3	of a semi-infinite sheared wing spanwise interpolation factor to allow for the effects of a surface	Л*		interpolation of sweep at the centre and sweep at the tip of the wing, eq. (20)
	crank on the velocities at the maximum thickness position	$\mu_1, \mu_2, \mu_3$ č. n. č	3	Lagrangian multipliers cartesian co-ordinate system, $\xi$
⊿ ₄ d4	spanwise interpolation factor to allow for the effects of planform	5,1,5		axis in the direction of the un- disturbed flow
	taper on the two-dimensional velocities due to thickness at the	$ ho \sigma$	s	leading edge radius in parts of $c$ function defined by eq. (14)
⊿ ₅ d5	maximum thickness position spanwise interpolation factor to	arphisuffix		flow potential
-	allow for the effects of a plan- taper crank on the velocities	a l		analogous wing lower surface
	due to thickness at the maximum thickness position	u t		upper surface thickness
$\Delta z_0, \Delta z_1$	discrepancy between the exact	s		camber

# 1 Introduction

For flight at high-subsonic speeds special attention must be paid to the avoidance of large drag which can occur as a result of pressure losses due to viscous effects and shocks. These may lead to unacceptable low aerodynamic efficiency and unsteady effects in cruising condition. Wing-body combinations which generate flow patterns with no shocks on the surface and with separation due to viscosity confined to the wing trailing edges, belong to the class of optimal shapes with respect to aerodynamic efficiency and safety. This type of flow pattern can be realized by means of swept wings when the angle of sweep is correctly related to Mach-number, wing thickness and liftcoefficient.

The geometry which generates the desired flow pattern and fulfils the requirements which makes the aeroplane suitable for a particular task, mostly is obtained by means of an iteration process in which use is made of both measurements in windtunnels and aerodynamic calculations. In this process structural aspects play an important role. In general the number of configurations which must be measured, greatly depends on the accuracy by which the calculation method predicts details of flow phenomena. In this context the agreement of calculations with experimental results can be considered more important than the elegance or rigour of the calculation method. However, for practical applications it is a conditio sine qua non that it is known for what class of configurations this agreement exists.

Up to now potential theory forms the basis of all practical calculation methods which are applicable to wing-body combinations. It has appeared that this theory in many cases can be used as well to design, as to analyse flow patterns which belong to the class mentioned above, even when the potential equation is linearized for practical reasons. Linearization of the boundary condition leads to the relatively simple solutions of full linearized theory, however

in many cases at the cost of quantitative significance.

In ref. 1 a method is presented for prediction of the pressure distribution on swept wings with subsonic attached flow, which appears to be very attractive. With the simplicity of full linearized theory, solutions can be obtained which, in many cases, are in very good agreement with experimental results. This is partly a result of the introduction of semi-empirical parameters to take into account the effects of compressibility. Partly it is a result of taking into account the most important non-linear effects which occur in the neighbourhood of round leading edges and which are related to the mutual influence of circulation and thickness.

For application of the method it is necessary to determine the principle values of a number of singular integrals, of the type which occur frequently in linearized potential theory. The integrals are functions of the wing section contour. In ref. 1 these integrals are reduced to summation formulas which are very suitable for manual computation, however, at the cost of limitations in the choice of chordwise stations where the contour must be specified and where the pressure can be calculated. In the present report a computer programme is given for application of the method without restrictions of this type. The integrals are determined analytically in this programme by introduction of a suitable analytic representation of the aerofoil contour. This representation is determined by the method of the least squares, to make it possible to smooth given contour data.

One of the reasons to write the said computer programme is the desirability to determine the boundaries of the region of applicability of the method, by comparison of a number of calculated and measured pressure distributions. Some examples of application of the method are discussed in section 6 of the present report. In future use will be made of the flexibility of the method to introduce further refinements to increase the region of applicability.

# 2 The general expression for the velocity at any point at the surface of a swept wing

The general expression for the velocity at any point at the surface of a swept wing, is given in ref. 1 together with an explanation of the signification of the various terms. The formula is reproduced in the present report. For a better understanding of the following some features of the background of the formula will be described. It will be made clear what choices have been made in cases where ref. 1 leaves open the possibility to make use of more than one approximation and two modifications will be described.

The analysis of the flow problem, in first instance is based on the assumption of small perturbations. This validates the well known linearized differential equation for the potential  $\varphi$ :

$$(1 - M_0^2)\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial^2 \varphi}{\partial \zeta^2} = 0, \qquad (1)$$

 $M_0$  being the undisturbed Mach number and  $\xi$ ,  $\eta$ ,  $\zeta$  being a Cartesian co-ordinate system with the  $\xi$ -axis in the direction of the mainstream. The details of the subsonic flow past a swept wing are predicted by making use of the similarity of this flow with incompressible flow past an affinely related wing. The analogous wing is derived from the given wing by shrinking all dimensions transverse to the  $\xi$  axis by a factor  $\sqrt{1-M_0^2}$ . The relations between the velocity increments in the compressible flow and the incompressible increments are determined by the condition that both wings must be stream surfaces in their respective flows. In these relations a modified factor is used as far as the velocities due to thickness are concerned. The modification which is introduced by Küchemann and Weber (ref. 3) results in a greater Mach number dependence in the subcritical speed range.

The expression for the velocity at any point on a wing, in incompressible flow, is derived by means of a combination of the expressions for:

(a) the velocity on an infinite sheared wing of constant section according to full linearized theory

(b) the velocity on the centre section of an infinite swept wing of constant section according to full linearized theory

(c) an empirical spanwise interpolation between (a) and (b)

(d) a correction for the mutual interaction of thickness and circulation from an analysis of two-dimensional flow (e) a multiplicative correction for round leading edges.

The combination of (a) and (b) is based on the principle that the main three-dimensional perturbation effects are restricted to local regions. This means that the applicability of the result diminishes with decreasing aspect ratio of the analogous wing and increasing taper and sweep of this wing.

The applicability of the corrections (d) and (e) will diminish the more the shape of the forward part of a blunt section deviates from an elliptic shape.

From linearized theory no definite conclusion can be drawn with regard to the transformation of the non-linear incompressible flow solution to a solution of the compressible flow. In the method of ref. 1 a choice is made which results in a variation of non-linear effects with Mach number in the same rate as the variation of comparable linear perturbation quantities.

For each wing section in free-stream direction a system of rectangular co-ordinates x, z is used where the x-axis is along the chord. The origin has been chosen at the leading edge. The chord is defined as the longest line which joins a point on the nose of the section and the trailing edge where the thickness is zero. The co-ordinates of the section are non-dimensionalised with the section chord. When the upper and lower surfaces of the section are defined as  $z_{\mu}(x)$  and  $z_{l}(x)$  respectively, the half thickness  $z_{r}(x)$  and the camber  $z_{s}(x)$  are obtained according to

$$z_{i}(x) = \frac{z_{u}(x) - z_{i}(x)}{2}$$
(2)

and

$$z_s(x) = \frac{z_u(x) + z_l(x)}{2}.$$
 (3)

With the notation being described in the list of symbols, the velocity at any point on the surface of a streamwise section of a swept wing with subsonic attached flow is approximated by:

$$\begin{cases} \frac{V}{V_0} \right)_{x=x_v}^2 = \begin{cases} \cos \alpha_e \left[ 1 + K_{3_e} \cos A_1 \frac{1}{\pi B_w} \int_0^1 \left( \frac{dz_1}{dx} \right)_x \frac{dx}{x_v - x} - \frac{K_{2_e} \cos A_1 \cdot \frac{1}{\pi B_w} \ln \left( \frac{1 + \sin A_{4_e}}{1 - \sin A_{4_e}} \right) \cdot \left( \frac{dz_1}{dx} \right)_{x_v}}{\left\{ 1 + \left( \frac{dz_1}{dx} \right)_x \frac{dx}{B_w} \right\}^2} \right]^{-\frac{1}{2}} \\ \times \left[ 1 + \left\{ \frac{dz_1}{dx} \right)_{x_v}^2 \frac{dz_1}{dx} \right\}^2 \right]^{-\frac{1}{2}} \pm \frac{\cos \alpha_e}{(1 - M_0^2 \cos^2 A_m)^2} \frac{\cos A_m}{\cos(A_m)} \times \right] \\ \times \left\{ \frac{1 + \left\{ \frac{dz_1}{dx} \right\}_x + \frac{\pi}{\pi^2 + \sigma_x^2} \left( \frac{1 - x_v}{x_v} \right)^{a_v} \right\}^2 \right]^{-\frac{1}{2}} \pm \frac{\cos \alpha_e}{(1 - M_0^2 \cos^2 A_m)^2} \frac{\cos A_m}{\cos(A_m)} \times \\ \times \left\{ \frac{1 - \pi \sigma_u}{\pi^2 + \sigma_x^2} \left( \frac{dz_1}{dx} \right)_{x_v} + \frac{\pi}{\pi^2 + \sigma_x^2} \left( \frac{1 - x_v}{x_v} \right)^{a_v} \int_0^1 \left( \frac{dz_1}{dx} \right)_x \left( \frac{x}{1 - x} \right)^{a_v} \frac{dx}{x_v - x} \right\} \times \\ \times \left\{ 1 + \left\{ \frac{\left( \frac{dz_1}{dx} \right)_{x_v} + \frac{\pi}{\pi^2 + \sigma_x^2} \left( \frac{1 - x_v}{x_v} \right)^{a_v} \right\}^2 \right]^{-\frac{1}{2}} \pm \frac{H_u \sin \alpha_e \cos A_m}{(1 - M_0^2 \cos^2 A_m)^2} \times \\ \times \left\{ 1 + \frac{1}{\pi B_w} \cos A^* \int_0^1 \left\{ \left( \frac{dz_1}{dx} \right)_{x_v} - \frac{2z_t(x)}{1 - (1 - 2x)^2} \right\} \frac{dx}{x_v - x} \left\{ 1 + \left\{ \frac{\left( \frac{dz_1}{dx} \right)_{x_v} \pm \left\{ \frac{dz_2}{dx} \right\}_{x_v} \right\}^2 \right\}^{-\frac{1}{2}} \right\}^2 \right]^{-\frac{1}{2}} \\ \pm \cos \alpha_e \left\{ \frac{(1 - M_0^2 \cos^2 A_m)\cos^2(\lambda_u A_{m_u}) - (1 - M_0^2 \cos^2 A_m)}{(1 - M_0^2 \cos^2 A_m)} \right\}^2 \\ \times \left\{ \frac{1 - \pi \sigma_u}{\pi^2 + \sigma_x^2} \left( \frac{dz_1}{dx} \right)_{x_v} + \frac{\pi}{\pi^2 + \sigma_x^2} \left( \frac{1 - x_v}{x_v} \right)^{a_v} \int_0^1 \left( \frac{dz_2}{dx} \right)_{x_v} \frac{dx}{x_v - x} \left\{ 1 + \left\{ \frac{\left( \frac{dz_1}{dx} \right)_{x_v} \pm \left\{ \frac{dz_1}{dx} \right\}_{x_v} \right\}^2 \right\}^{-\frac{1}{2}} \right\}^2 \right\}^2 \right\}^{-\frac{1}{2}} \\ + \left\{ \cos \alpha_e \left\{ \frac{(1 - M_0^2 \cos^2 A_m)\cos^2(\lambda_u A_{m_u}) - (1 - M_0^2 \cos^2 A_m}{(1 - M_0^2 \cos^2 A_m)} \right\}^2 \right\}^2 \right\}^{-\frac{1}{2}} \\ \times \left\{ \frac{1 - \frac{1}{\pi^2 + \sigma_x^2}} \left\{ \frac{dz_1}{dx} \right\}_{x_v} + \frac{\pi}{\pi^2 + \sigma_x^2} \left\{ \frac{1 - x_v}{x_v} \right\}^{a_v} \int_0^1 \left\{ \frac{dz_2}{dx} \right\}_{x_v} \frac{dx}{(1 - M_0^2 \cos^2 A_m)} \cos^2(\lambda_u A_{m_u}) - (1 - M_0^2 \cos^2 A_m}{(1 - M_0^2 \cos^2 A_m)} \right\}^2 \right\}^2 \right\}^{-\frac{1}{2}} \\ \times \left\{ \frac{1 + \frac{1}{\pi B_w} \sin \alpha_e} \left\{ \frac{(1 - M_0^2 \cos^2 A_m)\cos^2(\lambda_u A_{m_u}) - (1 - M_0^2 \cos^2 A_m)}{(1 - M_0^2 \cos^2 A_m)} \right\}^2 \\ \times \left\{ \frac{1 + \left\{ \frac{dz_1}{dx} \right\}_{x_v} + \frac{\pi}{\pi^2 + \sigma_x^2} \left\{ \frac{dz_1}{dx} \right\}_{x_v} \frac{dz_1}{(x_v} \frac{dz_1}{(x_v} \frac{dz_1}{(x_$$

The symbol  $\pm$  refers to upper and lower surface, the positive sign being taken for the upper surface and the negative sign for the lower surface.

The significance of a number of terms in this expression will be explained below as far as this is desirable with regard to the description in the next sections of this report.

The first term in large curly brackets is approximately the velocity component in chordwise direction. The second term in large curly brackets is approximately the velocity component in spanwise direction, whereas the last main term can be regarded as a correction to the spanwise velocity.

The combination of the two basic solutions (a) and (b), mentioned before, is realized by spanwise variation of the parameters  $K_{2a}$ ,  $K_{3a}$  and  $\lambda_a$ .  $B_w$  and  $\Lambda^*$  depend on  $K_2$  and  $K_{2a}$  respectively, contrary to ref. 1 where both factors depend on  $K_2$ .  $\sigma_a$ ,  $\vartheta_a$ ,  $n_a$  and  $H_a$  depend on  $\lambda_a$ .  $\alpha_e$  is the effective angle of attack which varies along the span as a result of wing twist and variation of the effective induced angle of attack. The present authors in general determ  $\alpha_e$  by iteration in such a way that a desired normal force on the wing section is realized. The subscript a indicates that the quantities are related to the geometry of the analogous wing.

The velocity component in chordwise direction is formed by the sum of a component of the undisturbed flow, a component of the perturbation velocity due to thickness (terms with  $K_{2a}$  and  $K_{3a}$ ), a component of the perturbation velocity due to camber and a term which results from the angle of attack (the term with  $H_a$ ). In the last term the integral which is a function of  $z_t$  takes into account the interaction between angle of attack and thickness.

The term 
$$\left[1 + \left\{\frac{\left(\frac{\mathrm{d}z_{i}}{\mathrm{d}x}\right)_{x_{v}} \pm \left(\frac{\mathrm{d}z_{s}}{\mathrm{d}x}\right)_{x_{v}}}{B_{w}\cos\Lambda^{*}}\right\}^{2}\right]^{-\mu}$$

is the multiplicative correction factor mentioned on page 4. This factor will be referred to later as the Riegels factor. The exponent  $\mu$  of this factor is chosen in each case in such a way that at the leading edge finite values of the velocity are obtained. This refinement was introduced in the present report also in the term due to camber to make it possible to calculate velocities very close to the leading edge.

The interpolation factor  $K_{2a}$  allows for variation of root and tip effects on velocities due to thickness. In ref. 1  $K_{2a}$  is given by means of graphs for  $K_{2,ROOT}$  and  $K_{2,TIP}$  as function of the co-ordinate y. The values have been determined in ref. 3 by means of full linearized theory in combination with experiments

$$K_{2a} = K_{2a, \text{ROOT}} - K_{2a, \text{TIP}}$$
(5)

for each spanwise station.

 $K_3$  is the ratio of the supervelocity at the maximum thickness position of any spanwise station of a swept wing to the supervelocity at the maximum thickness of the corresponding infinite sheared wing. For  $K_3$  a formula is presented in ref. 1 which after some rearrangement leads to the following expression:

$$K_{3_{\alpha}} = 1 + \frac{\pi \Delta_s(x_{t_{\text{max}}})_a}{(1 - M_0^2)^{\frac{1}{2}} \int_0^1 \frac{\mathrm{d}z_t}{\mathrm{d}x} \frac{\mathrm{d}x}{x_u - x} \cdot \cos \Lambda_{t_{\text{max},a}}}$$
(6)

$$\begin{aligned}
\Delta_{s}(x_{i_{\max a}})_{a} &= \left[ \Delta_{0_{a}} \left( \frac{s_{a} - y_{a}}{c_{T}} \right) + \Delta_{0_{a}} \left( \frac{s_{a} + y_{a}}{c_{T}} \right) \right] \frac{t_{T}}{c_{T}} \cos A_{i_{\max a}} (1 - M_{0}^{2})^{\frac{1}{2}} + \left[ \Delta_{1_{a}} \left( \frac{s_{a} - y_{a}}{c_{T}} \right) + \Delta_{1_{a}} \left( \frac{s_{a} + y_{a}}{c_{T}} \right) \right] \\
&+ 2\Delta_{3_{a}} \left( \frac{y_{a}}{c_{R}} \right) \tan \delta_{R} + \left[ \Delta_{3_{a}} \left( \frac{s_{a} - y_{a}}{c_{T}} \right) + \Delta_{3_{a}} \left( \frac{s_{a} + y_{a}}{c_{T}} \right) \right] \tan \delta_{T} + \Delta_{3_{a}} \left( \frac{y_{a} - y_{ac_{1}}}{C_{c_{1}}} \right) \tan \delta_{1} + \Delta_{3_{a}} \left( \frac{y_{a} - y_{ac_{2}}}{C_{c_{2}}} \right) \tan \delta_{2} \\
&+ \Delta_{4_{a}} \left( \frac{y_{a}}{s_{a}'} \right) \left[ \tan A_{0_{a}} - \tan A_{1_{a}} \right] \frac{t_{R}}{c_{R}} \left( 1 - M_{0}^{2} \right)^{\frac{1}{2}} + 2\Delta_{5_{a}} \left( \frac{y_{a}}{c_{R}} \right) \left[ \tan A_{0_{a}} - \tan A_{1_{a}} \right] \frac{t_{R}}{c_{R}} \left( 1 - M_{0}^{2} \right)^{\frac{1}{2}} \\
&+ \left[ \Delta_{5_{a}} \left( \frac{s_{a} - y_{a}}{c_{T}} \right) + \Delta_{5_{a}} \left( \frac{s_{a} + y_{a}}{c_{T}} \right) \right] \left[ \tan A_{1_{a}} - \tan A_{0_{a}} \right] \frac{t_{T}}{c_{T}} \left( 1 - M_{0}^{2} \right)^{\frac{1}{2}} .
\end{aligned}$$
(7)

The notations are described in the list of symbols. The specification of the functions  $\Delta_0$  to  $\Delta_5$  is given below.

 $\Delta_0$  is introduced to take into account the reduction of the velocity near the tip of unswept wings. At the maximum thickness position of a straight wing the supervelocity in x direction due to thickness can be expressed as

$$\frac{v_x(x_{t_{\text{rmax}}})}{V_0} = \frac{1}{\pi} \int_0^1 \frac{\mathrm{d}z_t}{\mathrm{d}x} \frac{\mathrm{d}x}{x_{t_{\text{rmax}}} - x} + \Delta_0 \frac{t}{c}$$
(8)

in incompressible flow.

This means that the product  $\Delta_0 \frac{t}{c}$  is the reduction of the velocity near the tip. By means of linearized theory the

following expression for  $\Delta_0$  can be found for a semi-infinite unswept wing with constant biconvex parabolic-arc section shape:

$$\Delta_{0}(y) = \frac{-2}{\pi} \left\{ 1 - \frac{y}{c} \ln \left( \frac{\sqrt{1 + 4\left(\frac{y}{c}\right)^{2} + 1}}{\sqrt{1 + 4\left(\frac{y}{c}\right)^{2} - 1}} \right) \right\}$$
(9)

where  $\frac{y}{z}$  is the distance from the tip divided by the chord. This expression is recommended in ref. 2.

Application of eq. (9) leads to a super velocity at the tip which is half the value of this velocity at a two-dimensional biconvex parabolic arc section at the position of the maximum thickness. Due to a failure of linearized theory this value is too small (see e.g. ref. 3). In principle a semi-empirical correction can be applied to  $\Delta_0$  in accordance with earlier publications about the calculation method. For the time being eq. (9) will be used.

 $\Delta_1$  is introduced in eq. (7) to approximate the effect of sweep on the super velocity due to thickness near the centre of a swept wing. The x-component of this velocity at the maximum thickness position can be expressed as:

$$\frac{v_{x(x_{t \max})}}{V_0} = \frac{1}{\pi} \int_0^1 \frac{dz_t}{dx} \frac{dx}{x_{t_{\max}} - x} \cos \Lambda_{t_{\max}} + \Delta_1 .$$
(10)

By means of full linearized theory, applied to an infinite swept wing, the following expression can be found for  $\Delta_1$  for the analogous wing:

$$\Delta_{I_a}\left(\frac{y_a}{c}\right) = \{K_{I_a}\left(x_{t_{\max}}, \frac{y_a}{c}\right) - 1\} \frac{1}{\pi} \int_0^1 \frac{\mathrm{d}z_t}{\mathrm{d}x} \frac{\mathrm{d}x}{x_{t_{\max}} - x} \cos \Lambda_{t_{\max}a} (1 - M_0^2)^{\frac{1}{2}}$$
(11)

with

$$K_{1a}(x_{t_{\text{imax}}}, y_a) = 1 + \frac{\int_0^1 \left(\frac{dz_t}{dx}\right)_x \left\{ \frac{\frac{y_a^2 \cdot \tan A_a}{\cos A_a} - y_a \tan A_a \left[\frac{y_a^2}{\cos^2 A_a} + 2y_a \tan A_a (x_{t_{\text{max}}} - x) + (x_{t_{\text{max}}} - x)^2\right]^{\frac{1}{2}} \right\}}{\int_0^1 \left(\frac{dz_t}{dx}\right)_x \frac{dx}{x_{t_{\text{max}}} - x}}$$
(12)

Near the tip of swept wings  $\Delta_1$  is used to correct  $\Delta_0$  for the influence of sweep. This is based on the assumption that the velocity at the maximum thickness position near the tip of a semi-infinite sheared wing can be taken equal to the sum of  $\Delta_1$  and the value given by eq. (8) multiplied by the factor  $\cos \Lambda_{i_{max}}$ . The introduction of  $\Delta_1$  in this way is equivalent to the assumption that the tip of a swept back wing can be treated as the centre of a swept forward wing with regard to the influence of sweep. This means that for calculation of  $\Delta_{1a}(s_a \pm y_a)$  the angle of sweep  $\Lambda_a$ has to be taken equal to the opposite of the angle of sweep of the analogous wing.

In principle it is possible to evaluate eq. (12) for the same type of aerofoil cross section as for which  $\Delta_0$  has been calculated. However, it is felt that this may lead to unacceptable discrepancies in the calculated pressures near the centre of swept wings. For this reason eq. (12) is used in the present report to determine  $K_1$  values.

 $\Delta_3$ ,  $\Delta_4$  and  $\Delta_5$  are introduced to take into account the effects of thickness and planform taper. Values of these parameters have been determined by the authors of ref. 1 by means of full linearized theory applied to symmetrical wings under zero lift conditions. Generalized results are given in tables in ref. 1 as a function of angle of sweep  $\Lambda_{t_{max}}$  and the appropriate dimensionless spanwise co-ordinates.

In the following the interpolation functions related to camber and incidence are summarized in accordance with ref. 1.

The exponent in the expression for the velocity due to camber  $\vartheta_a$  is given by

$$\vartheta_a = \frac{1}{\pi} \tan^{-1} \frac{\pi}{\sigma_a} \tag{13}$$

with

$$\sigma_a = 2\pi \sin \lambda_a \Lambda_{m_a} \left[ 1 - \frac{1}{\pi} \ln \left\{ 10 \frac{t}{c} \left( 1 - M_0^2 \right)^{\frac{1}{2}} \right\} \right] - \ln \left( \frac{1 + \sin \lambda_a \Lambda_{m_a}}{1 - \sin \lambda_a \Lambda_{m_a}} \right).$$
(14)

The factor  $\lambda$  which is used to allow for variation in spanwise direction of chordwise loading is given by

$$\lambda_a = \lambda \left(\frac{y_a}{c(y_a)}\right)_a - \lambda \left(\frac{s_a - y_a}{c(y_a)}\right)_a \tag{15}$$

with

$$R(y_a)_a = \left\{ 1 + \left( \frac{2\pi \tan \Lambda_{m_a}}{\Lambda_{m_a}} y_a \right)^2 \right\}^{\frac{1}{2}} - \frac{2\pi \tan \Lambda_{m_a}}{\Lambda_{m_a}} y_a$$
(16)

The chordwise loading parameter  $n_a$  is

$$n_{a} = 1 - \frac{1 + \frac{2\lambda_{a}A_{m_{a}}}{\pi}}{2\left\{1 + \left(\frac{2\cos A_{m_{a}}}{AR_{a}}\right)^{2}\right\}^{\frac{1}{4\left(1 + 2\frac{A_{m_{a}}}{\pi}\right)}}}$$
(17a)

for finite swept back wings.

The corresponding parameter for infinite swept wings is

$$n_{0_{a}} = \frac{1}{2} \left\{ 1 - \frac{2\lambda_{a}\Lambda_{m_{a}}}{\pi} \right\}.$$
 (17b)

The effective aspect ratio is given by

$$AR_{a} = (1 - M_{0}^{2})^{\frac{1}{2}} \frac{(2s)^{2}}{S_{e}}.$$
 (18)

The factor  $H_a$  in eq. (4) takes into account the influence of the finiteness of the wing on the sectional lift curve slope.

$$H_a = \frac{\sin \pi n_a}{\{1 - \pi n_a (\operatorname{ctn} \pi n_a - \operatorname{ctn} \pi n_{0_a})\} \cos \lambda_a \Lambda_{m_a}}$$
(19)

The effective thickness sweep  $\Lambda^*$  is defined by

$$A^* = (1 - |K_{2_a}|)A_i \,. \tag{20}$$

In ref. 1 the factor  $K_{2a}$  is replaced by  $K_2$  in the expression for  $\Lambda^*$ . When eq. (20) is used to approximate the influence of the wing root on  $\Lambda^*$ , the present authors prefer to substitute the value of  $K_{2a}$  in the expression to introduce the effect of compressibility on the attenuation of root effects in spanwise direction.

The modified compressibility parameter  $B_w$  which is substituted for  $\{1 - M_0^2 \cos^2 \Lambda_t\}^{\frac{1}{2}}$  in the expressions for the super-velocity components due to thickness according to linearized theory, is given by:

$$B_{w} = \{1 - M_{0}^{2} [(1 - C_{p_{i}}(x_{v}) \cos^{2} \Lambda_{t}(x_{v}) - C_{p_{i}}(x_{v})(1 - |K_{2}|) \sin^{2} \Lambda_{t}(x_{v})]\}^{\frac{1}{2}}.$$
(21)

Here  $C_{p_i}$  is the local pressure coefficient calculated for  $M_0 = 0$ ,  $\alpha_e = 0$  and  $z_s = 0$  with the proviso that when  $C_{p_i} > 0$  or when  $B_w$  becomes imaginary with eq. (21),  $C_{p_i} = 0$ , so that  $B_w$  then reduces to  $\{1 - M_0^2 \cos^2 \Lambda_i\}^{\frac{1}{2}}$ .

The pressure coefficient  $C_p$  is evaluated by means of Bernouilli's relation :

$$C_{p} = \frac{2}{\gamma M_{0}^{2}} \left\{ \left( 1 + \frac{\gamma - 1}{2} M_{0}^{2} \left[ 1 - \left( \frac{V}{V_{0}} \right)_{\gamma}^{2} \right] \right)^{\gamma/(\gamma - 1)} - 1 \right\}$$
(22)

where the ratio of specific heats  $\gamma = 1.4$ .

It is not possible to deal with blunt trailing edges by means of linearized theory. In the present theory no measures have been taken to make it possible. In most practical cases, however, the section thickness is finite at the trailing edge. This difficulty is eliminated by the assumption that the velocity on a wing section with finite trailing edge thickness is equal to the velocity on a wing section which is derived from the given section by linear extrapolation of the upper and lower surface, to zero thickness. In this case  $x_v$  in equation (4) is measured in parts of the extended chord.

The expression for the pressure distribution on a two-dimensional aerofoil is a special case of the general expressions presented above.

With  $C_{p_v}(x_v)$  being the pressure distribution on the upper surface of the wing section and  $C_{p_i}(x_v)$  the pressure

distribution on the lower surface, the following expressions define the normal force, tangential force and pitching moment coefficient:

$$C_{n} = \int_{0}^{1} \{C_{p_{l}} - C_{p_{u}}\}_{x_{v}} \mathrm{d}x_{v}$$
(23)

$$C_{t} = -\int_{0}^{1} \left\{ \left( C_{p_{u}} \frac{\mathrm{d}z_{u}}{\mathrm{d}x} \right)_{x_{v}} - \left( C_{p_{1}} \frac{\mathrm{d}z_{l}}{\mathrm{d}x} \right)_{x_{v}} \right\} \mathrm{d}x_{v}$$
(24)

$$C_{m} = -\int_{0}^{1} \{C_{p_{i}}(x_{v}) - C_{p_{u}}(x_{v})\} (x_{v} - 0.25) dx_{v}.$$
⁽²⁵⁾

The pitching moment is given with respect to the quarter chord point as can be seen in eq. (25).

For two-dimensional flow the lift coefficient and drag coefficient can be calculated according to respectively

$$C_i = C_n \cos \alpha_e - C_i \sin \alpha_e \tag{26}$$

and

$$C_d = C_n \sin \alpha_e + C_t \cos \alpha_e \,. \tag{27}$$

So far, the method only deals with isolated wings in inviscid flow. In ref. 1 the method is outlined by which the effect of the presence of a fuselage can be taken into account. In first instance it can be assumed that the wing is completely reflected at the body side. When the fuselage does not extend far enough above or below the wing to generate the full reflection effect, it is possible within the scheme of the method to take into account a so-called partial reflection effect. In the same way it is possible to deal with the effect of other bodies mounted on the wing. It is possible also to simulate a circulation defect due to boundary layers by a reduction of the sectional effective incidence  $\alpha_e$ .

# 3 Representation of the wing section geometry by means of algebraic functions

The evaluation of eq. (4) involves the computation of the three integrals in this expression which are functions of the slope of the camber and the thickness distribution of the wing section. In ref. 1 the computation of the integrals is based on summation formulas given by Weber, which makes it possible to evaluate eq. (4) by means of desk computers. Summation formulas are also used in this reference to obtain the slope of the thickness and camber distribution. This, however, imposes restrictions on the distribution of the given section co-ordinates and the points  $x_v$  where the pressure can be calculated. As nowadays digital computers are available, restrictions of this kind can be eliminated.

For the computer programme, use is made of a relatively simple representation of the aerofoil contour. The representation has been chosen in such a way that it is possible to evaluate the principle values of the integrals in eq. (4) by means of exact integration. The representation of the aerofoil contour is determined according to the method of least squares with the possibility to impose constraints on fluctuations. This provides a means of smoothing the contour by relaxing the requirement that the approximated curve passes exactly through the specified contour points. Use can be made of this facility to eliminate troublesome inaccuracies in the given data which in practice very often occur. The character of the contour interpolation is chosen in such a way that good agreement can be obtained with contour shapes for which the method of ref. 1 is known to be suited. This can be elucidated as follows.

In principle the method is based on full linearized theory in which the influence of the wing is represented by means of planar distributions of singularities. As a result of this a smooth continuous distribution of the radius of curvature of the section contour is required for application of the method. Further the slope of the camber must be small everywhere because the theory fails to predict the non-linear effects of camber in combination with thickness and angle of attack. The non-linear effect of calculating the velocities at the aerofoil surface has been taken into account approximately. However, large discrepancies can be expected on blunt aerofoil noses in the real flow when the shape of the rounded nose differs too much from an elliptic shape. The choice of the analytic representation of the aerofoil contour has been based on the assumptions which apparently also form the basis of the calculation method, namely:

- the aerofoil contour is smooth
- the slope of the camber is small everywhere

- the thickness goes to zero at the leading edge as  $\sqrt{2\rho}\sqrt{x}$  ( $\rho$  = leading edge radius*)

In first approximation the thickness distribution can be represented by  $\sqrt{2\rho}\sqrt{x(1-x)}$ . By means of an algebraic polynomial this distribution has been corrected to give:

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^{*)} With regard to possible scaling of thickness it should be mentioned that the leading edge radius is proportional to the square of the thickness.

$$z_{i}(x) = \sqrt{2\rho}\sqrt{x}(1-x) + \sum_{i=1}^{l} t_{i}x^{i}.$$
(28)

Since  $z_t(x=1)$  has to be zero it follows that  $\sum_{i=1}^{l} t_i = 0$ .

For conventional aerofoil shapes the slope of  $z_t(x) - \sqrt{2\rho}\sqrt{x}(1-x)$  is finite at leading and trailing edge. When it is assumed that the camber has finite slopes at leading edge and trailing edge it is possible to represent  $z_s(x)$  by

$$z_s(x) = \sum_{i=1}^{l} s_i x^i$$
 (29)

with  $\sum_{i=1}^{l} s_i = 0.$ 

In case the upper and lower surface co-ordinates are given at different chordwise positions it is not possible to determine  $z_s(x)$  and  $z_i(x)$  directly according to eq. (2) and eq. (3). The upper and lower surface then can be approximated directly in accordance with eq. (28) and eq. (29) by

$$z_{u}(x) = \sqrt{2\rho} \sqrt{x} (1-x) + \sum_{i=1}^{l} \beta_{i} x^{i}$$
(30)

and

$$z_{l}(x) = -\sqrt{2\rho}\sqrt{x}(1-x) + \sum_{i=1}^{l} \gamma_{i} x^{i}$$
(31)

to give  $t_i = \frac{1}{2}(\beta_i - \gamma_i)$  and  $s_i = \frac{1}{2}(\beta_i + \gamma_i)$ .

The problem to find an analytic representation of the section shape now has been reduced to the determination of the coefficients  $a_i$  in the polynomial representation  $f(x) = \sum_{i=1}^{l} a_i x^i$  where f(x) is given in discrete points in the interval  $0 \le x \le 1$  and f(0) = f(1) = 0. This problem has been solved by application of the method of least squares with additional conditions. The first additional condition is:

$$\sum_{i=1}^{l} a_i = 0$$
 (32)

because f(1) = 0. By means of the method of least squares in first instance the sum of the squares of the differences between the given and the approximated values of f(x) is minimized. In that case no constraint is imposed to minimize fluctuations. Fluctuations in the camber representation near the trailing edge have a large influence on the circulation around the aerofoil section. In view of this, fluctuations are being reduced near the trailing edge by application of a second additional condition:

$$\left\{\frac{\mathrm{d}}{\mathrm{d}x}\sum_{i=1}^{l}a_{i}x^{i}\right\}_{x=1} = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{x=1}.$$
(33)

It is possible also to minimize with regard to the sum of the squares of the discrepancies between the first derivative of f(x) and the first derivative of  $\sum_{i=1}^{l} a_i x^i$ . The computer programme presents the possibility to minimize with regard to co-ordinates and first derivatives together. By means of a weight factor F, the minimization can be chosen between best co-ordinate fit and best fit of derivatives with l fixed and under the additional conditions which are represented by eqs. (32) and (33).

To obtain the best numerical accuracy the co-ordinate system is translated in x-direction. The new origin is taken at  $\bar{x}$  which is defined by

$$\bar{x} = \sum_{j=0}^{L} \frac{x_j}{N+1} \tag{34}$$

where  $x_j$  are the L+1 points where f(x) is given. With

$$u = x - \bar{x} \tag{35}$$

 $f(u_j)=f(x_j)$  is known in the points  $u_j$  (j=0, (1), ..., N) and it is assumed that  $\frac{df}{du}$  is known in the points  $u_g$ 

(g=0, (1), ..., M).f(u) is interpolated by  $f(u) = \sum_{i=0}^{l} b_i u^i$ , with the additional conditions  $f(u=-\bar{x})=0$ ;  $f(u=1-\bar{x})=0$ and  $\left(\frac{df}{du}\right)_{u=1-\bar{x}} = \left(\frac{df}{dx}\right)_{x=1}$ . The coefficients  $b_i$  are found by minimization of the following expression:

$$\sum_{j=0}^{L} \left\{ f(u_{j}) - \sum_{i=0}^{l} b_{i} u_{j}^{i} \right\}^{2} + F \sum_{g=0}^{M} \left\{ f'(u_{g}) - \sum_{i=0}^{l} i b_{i} u_{g}^{i-1} \right\}^{2} + \mu_{1} \sum_{i=0}^{l} b_{i} (-\bar{x})^{i} + \mu_{2} \sum_{i=0}^{l} b_{i} (1-\bar{x})^{i} + \mu_{3} \left\{ f'(1-\bar{x}) - \sum_{i=1}^{l} i b_{i} (1-\bar{x})^{i-1} \right\}.$$
(36)

Differentiation towards the coefficients  $b_i$  and the factors  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  leads to the equations for i=0, (1), ..., l:

$$\sum_{h=0}^{l} \sum_{j=0}^{L} b_{i} u_{j}^{k+i} + F \sum_{k=1}^{l} \sum_{g=0}^{M} b_{i} i k u_{g}^{k+i-2} + \mu_{1} (-\bar{x})^{i} + \mu_{2} (1-\bar{x})^{i} + \mu_{3} i (1-\bar{x})^{i-1} = \sum_{j=0}^{N} f(u_{j}) u_{j}^{i} + F \sum_{g=0}^{M} i f'(u_{g}) u_{g}^{i-1}$$
(37)

which together with

$$\sum_{i=0}^{l} b_i (-\bar{x})^i = 0$$
(38)

$$\sum_{i=0}^{l} b_i (1-\bar{x})^i = 0$$
(39)

and

$$\sum_{i=0}^{l} ib_i (1 - \bar{x})^{i-1} = \left(\frac{\mathrm{d}f}{\mathrm{d}u}\right)_{u=1-\hat{x}}$$
(40)

make it possible to determine the n+1 unknown quantities  $b_i$ . From equation

$$\sum_{i=0}^{i} b_i (x - \bar{x})^i = \sum_{i=1}^{i} a_i x^i$$
(41)

follows

$$a_{i} = \sum_{k=i}^{l} b_{k} \frac{k!}{i! (k-i)!} (-\bar{x})^{k-i}.$$
(42)

Herewith the representation of the aerofoil contour according to eq. (30) and eq. (31) is known.

A number of examples of the applicability of the method are presented in section 6. Normal practice is to take l=12. Discrepancies between exact contour ordinates and approximated ordinates are within  $5 \cdot 10^{-5}$  in parts of the chord for most aerofoils which have been investigated so far. Large discrepancies can occur when the method is applied to sections with rapid or discontinuous variations of radius of curvature. In this context it can be mentioned that in ref. 5 eq. (4) has been applied by means of a quadratic interpolation of given values of the slope of the symmetrical aerofoil contours. The representation according to eq. (28) was not possible in that case as a result of nearly discontinuous contour slope variations.

A description of the computer programme for determination of the contour representation is presented in section 5. The following special features can be mentioned:

- The programme provides the possibility to compute derivatives of f(x) by means of the differentiated Newton formula with divided differences when f(x) is given in discrete points. Use can be made of this facility when the derivatives are not known and all the same it is considered desirable to minimize with regard to derivatives in the above described procedure to determine the representation of f(x).
- The trailing edge thickness of most aerofoil sections on which the pressure distribution must be calculated is finite. In those cases the programme constructs a trailing edge point where  $z_t = 0$ . This point is defined as the intersection of the linear extrapolation of upper and lower surface. The programme is such that in the output data x pertains to the not-extended chord.
- The programme accepts contour co-ordinates in any rectangular axis system, with the proviso that co-ordinates of the leading edge point on the chord are given.
- The programme accepts thickness and camber co-ordinates as well as upper and lower surface co-ordinates. It is possible to make use of the programme also when it is desired to apply a definition of thickness and camber which differs from the definition represented by eq. (2) and eq. (3). For instance the NACA convention can be chosen equally legitimately within the scope of the method of ref. 1.

# 4 Evaluation of the integrals in the general expression for the velocity

The following integrals must be computed for application of eq. (4):

$$I_{1} = \frac{1}{\pi} \int_{0}^{1} \frac{dz_{i}}{dx} \frac{dx}{x_{v} - x}$$
(43)

$$I_2 = \frac{1}{\pi} \int_0^1 \left\{ \frac{\mathrm{d}z_t}{\mathrm{d}x} - \frac{2z_t}{1 - (1 - 2x)^2} \right\} \frac{\mathrm{d}x}{x_v - x} \tag{44}$$

and

$$I_3 = \int_0^1 \frac{\mathrm{d}z_s}{\mathrm{d}x} \left(\frac{x}{1-x}\right)^{\vartheta_a} \frac{\mathrm{d}x}{x_v - x} \,. \tag{45}$$

In the following the analysis is presented on which the calculation of the integrals by means of the computer programme is based.

The evaluation of  $I_1$  and  $I_2$  is based on the assumption that  $z_t$  is known as the algebraic function given in equation (28). By differentiation of this expression,  $\frac{dz_t}{dx}$  is obtained. Substitution in equation (43) leads after integration to:

$$\pi I_1(x_v) = \left(3\sqrt{2\rho} - \frac{1}{2}\sqrt{2\rho}\left(3\sqrt{x_v} - \frac{1}{\sqrt{x_v}}\right)\right) \ln \frac{1 + \sqrt{x_v}}{1 - \sqrt{x_v}} - \sum_{i=2}^{1} it_i \sum_{k=0}^{i-2} \frac{x_v^k}{i - 1 - k} - \sum_{i=1}^{1} it_i x_v^{i-1} \ln \frac{1 - x_v}{x_v}.$$
 (46)

An analogous procedure leads to

$$\pi I_2(x_v) = \pi I_1(x_v) - \frac{\sqrt{2\rho}}{\sqrt{x_v}} \ln \frac{1 + \sqrt{x_v}}{1 - \sqrt{x_v}} - \frac{1}{2} \sum_{i=3}^l t_i \left[ \sum_{k=1}^{i=2} \sum_{j=0}^{k-1} \frac{x_v^j}{k-j} + x_v^k \ln \frac{1 - x_v}{x_v} \right] - t_1 \ln \frac{1 - x_v}{x_v}.$$
(47)

The evaluation of  $I_3$  is based on the assumption that  $z_s$  is known as given in equation (29). Differentiation of equation (29) and substitution of the result in equation (45) leads to:

$$I_{3} = -\int_{0}^{1} \sum_{i=2}^{l} is_{i} \frac{(x_{v}^{i-1} - x^{i-1})}{x_{v} - x} \left(\frac{x}{1 - x}\right)^{\vartheta_{a}} dx + \int_{0}^{1} \sum_{i=1}^{l} is_{i} x_{v}^{i-1} \left(\frac{x}{1 - x}\right)^{\vartheta_{a}} \frac{dx}{x_{v} - x} = -\int_{0}^{1} \sum_{i=2}^{l} is_{i} \sum_{j=0}^{i-2} x_{v}^{i-2-j} x^{j} \left(\frac{x}{1 - x}\right)^{\vartheta_{a}} dx + \int_{0}^{1} \sum_{i=1}^{l} is_{i} x_{v}^{i-1} \left(\frac{x}{1 - x}\right)^{\vartheta_{a}} \frac{dx}{x - x}.$$
(48)

Now use can be made of

$$\int_{0}^{1} x^{m} \left(\frac{x}{1-x}\right)^{n} dx = \frac{\pi}{\sin \pi n} \prod_{l=0}^{m} \frac{n+l}{l+1}.$$

It is also known that

$$\int_0^1 \left(\frac{x}{1-x}\right)^{\vartheta_a} \frac{\mathrm{d}x}{x_v - x} = \frac{-\pi}{\sin \pi \vartheta_a} \left\{ 1 - \cos \pi \vartheta_a \left(\frac{x_v}{1-x_v}\right)^{\vartheta_a} \right\}.$$

By means of these expressions it can be found that

$$I_{3}(x_{\nu}) = \frac{\pi}{\sin(\pi\vartheta_{a})} \sum_{i=2}^{1} \sum_{j=0}^{i-2} is_{i} x_{\nu}^{i-j-2} \prod_{h=0}^{j} \frac{\vartheta_{a}+h}{h+1} + \frac{-\pi}{\sin\pi\vartheta_{a}} \left(\frac{dz_{s}}{dx}\right)_{x_{\nu}} \left\{1 - \cos\pi\vartheta_{a} \left(\frac{x_{\nu}}{1-x_{\nu}}\right)^{\vartheta_{a}}\right\}.$$
 (49)

With rearrangement of the first term:

$$I_{3}(x_{\nu}) = \frac{-\pi}{\sin\pi\vartheta_{a}} \sum_{j=0}^{l-2} \sum_{i=j+2}^{l} is_{i}x_{\nu}^{i-j-2} \prod_{h=0}^{j} \frac{\vartheta_{a}+h}{h+1} + \frac{-\pi}{\sin\pi\vartheta_{a}} \left(\frac{\mathrm{d}z_{s}}{\mathrm{d}x}\right)_{x_{\nu}} \left\{1 - \cos\pi\vartheta_{a} \left(\frac{x_{\nu}}{1-x_{\nu}}\right)^{\vartheta_{a}}\right\}.$$
(50)

Apart from the integrals which have been discussed above, it appears necessary to evaluate the integral in expression (12) for the factor  $K_1$ . In this expression the following variables can be substituted for convenience:

$$p = \frac{y_a}{\cos A_a} \tag{51}$$

$$g = \left[ p^2 + 2p \sin \Lambda_a (x_{t_{\max}} - x) + (x_{t_{\max}} - x)^2 \right]^{\frac{1}{2}}.$$
 (52)

This makes it possible to rewrite eq. (12) as:

$$K_{1} = \frac{1 - p \sin A_{a}}{\pi I_{1}(x_{v} = x_{t_{max}})} \int_{0}^{1} \frac{dz_{t}}{dx} \frac{dx}{(p+g)g}.$$
 (53)

For computation of the integral in equation (53) the singular behaviour for  $x \rightarrow 0$  is eliminated through subtraction. With

$$g_1 = [p^2 + 2p \sin \Lambda_a x_{i_{\max}} + x_{i_{\max}}^2]^{\frac{1}{2}}$$
(54)

equation (53) becomes after subtraction of  $\frac{p \sin \Lambda_a}{I_1(x_v = x_{t_{max}})} \int_0^1 \frac{dz_t}{dx} \frac{dx}{(p+g_1)g_1}$ 

$$K_{1} = \frac{1+p \sin \Lambda_{a}}{\pi I_{1}(x=x_{t_{\max}})} \int_{0}^{1} \frac{\mathrm{d}z_{t}}{\mathrm{d}x} \frac{(p+g+g_{1})(g-g_{1})\mathrm{d}x}{(p+g)(p+g_{1})g-g_{1}}$$
(55)

because  $\int_{0}^{1} \frac{dz_{t}}{dx} \frac{dx}{(p+g_{1})g_{1}} = 0$ . The integral in equation (55) is evaluated by means of the trapezoidal rule, the value of the integrand for x=0 being zero.  $x_{i_{\text{max}}}$  is determined for each aerofoil section by means of Newton's iteration process and the required values of  $\frac{dz_{t}}{dx}$  again are obtained by differentiation of equation (28).

Now all significant details concerning the determination of the pressure distribution on wings have been described. A detailed description of the computer programme is presented in section 5.

# 5 Description of the Algol programme

#### 5.1 General remarks

#### 5.1.1 Layout of the programme

In section 2 the expression for the velocity at any point on the surface of a swept wing is presented. By means of Bernouilli's equation the local pressure coefficient is calculated in terms of this velocity. The Algol programme is based on the analytical evaluation of the various integrals in the expression for the velocity, which is presented in section 4.

The programme is divided into three main parts. Part I covers the evaluation of the aerofoil section contour in algebraic functions and the computation of  $\frac{dz_i}{dx}$ ,  $\frac{dz_s}{dx}$ ,  $I_1$  and  $I_2$  according to sections 3 and 4. Part II deals with the computation of the remaining integrals described in section 4 and the pressure distribution in two-dimensional flow. With part III these integrals are being calculated together with the pressure distribution on streamwise sections

in three-dimensional flow. The analysis of the programme will be given in the present section, with the aid of flow diagrams and an appendix

which contains the Algol programme. In the list of symbols, the symbols that occur both in the Algol programme and in the text have been described.

5.1.2 Procedures programmed in internal computer code

For the understanding of the programme it will be of some help to know the meaning of the following procedures in internal machine code, which have been used:

FIXP(n,m,x):	punches the fixed point number x with n figures before and m figures after the point.
FLOP(n, m, x):	punches the floating point number $x$ with a mantisse of $n$ figures and an exponent
	of <i>m</i> figures
IMPROD(j, l, n, P[j], Q[j])	forms the inner product of the vectors $P$ and $Q$ of order $n$ .
print(x):	prints the number x in floating point representation.
PUNLCR:	punches new line carriage return
PUTEXT ( $\leq$ text $\geq$ ):	punches the text between the strings.
readn (a, b, c):	reads $a, b, c \dots$
RUNOUT:	gives 10 inches blank
$SUM(i, a, b, f_i)$ :	forms $\sum_{i=a}^{b} f_i$
TAPEND:	punches "?" followed by 10 inches blank.

In the following the three parts of the programme will be described.

5.2 Part I

The general flow diagram is given in fig. 1.

# 5.2.1 Procedures

In this part of the programme use is made of the following procedures.

- The real procedure dif(h, s) is a recursive procedure which evaluates dif =  $\frac{dz}{dx}$  at x = X from the given section or-

dinates, following the method mentioned in section 3 based on divided differences. 2h+1 is the number of points of the contour which are used.

- the real procedure DET(A, n, p) makes DET equal to the determinant of the *n*th order matrix A, following Crout's method with row interchanges. The triangular decomposition  $L \times U$  of A which is formed when this method is applied, replaces A.
- the procedure SOL(LU, r1, n, p) replaces the vector r1 by the solution  $\xi$  of the linear system  $L \times U \times \xi = r1$ .
- the real procedure DETSOL(A,r1,n) replaces the vector r1 by the solution  $\xi$  of the linear system  $A \times \xi = r1$ .
- the procedure klkw (l,n) forms the polynomial  $\sum_{i=1}^{i} a_i x^i$  according to the version of the method of the least squares described in section 3.

# 5.2.2 Input and output data

The flow diagram of the input is given in fig. 2. the tables 1, 2, 3 and 4 summarize the input data in the order in which they are assimilated in the programme. Data concerning the section contour can be given in any rectangular axis system and in any consistent dimension system. Only  $x_v$  in table 4 must be given in parts of the section chord c as indicated.

The limitations in the choice of given data, which are indicated by means of notes on the tables, have made it possible to simplify the computer programme for the time being.

The output of the programme is presented in fig. 5. The first eight columns contain data concerning the contour representation. Deviations from the given contour are presented together with first and second derivatives of the contour representation. In practice these data are used to verify the applicability of the method in each case. The data can be of use to find errors in the given section co-ordinates.

The last group of seven columns on the output sheet forms part of the input of the programmes part II and part III. An other part of the input of these programmes is the group of numbers just above these seven columns. These numbers refer to the degree of the polynomial which has been used for the aerofoil representation and the coefficients of the polynomial which represents the camber distribution.

# 5.3 Part II

The general flow diagram of this part is presented in fig. 1.

# 5.3.1 Procedures

One procedure has been used in this programme: real procedure int, which evaluates  $C_w C_b C_t$  and  $C_d$  by means of the trapezoidal rule.

# 5.3.2 Input and output data

The flow diagram of the input of this programme is given in fig. 3. Table 5 provides the input data which are required in addition to the results obtained by means of programme part I and which are described in section 5.2.2.

The output consists of Mach number, angle of attack, the values of the coefficients of normal force, pitching moment, lift, tangential force and drag followed by a column in which the values of  $x_v, C_{p_v}(x_v)$  and  $C_{p_v}(x_v)$  are presented.

# 5.4 Part III

The general flow diagram again is presented in fig. 1.

# 5.4.1 Procedures

In this part of the programme use is made of the procedures:

- procedure determ K, which evaluates the value of  $K_{2a}$  and  $K_{3a}$  according to equations (5) and (6) respectively. Values of  $K_{2,ROOT}$  and  $K_{2,TIP}$  which are given in ref. 1 for a number of y values, are part of the input of the programme. From these values  $K_2$  is determined by means of linear interpolation. For the determination of  $K_3$ use is made of eq. (9) and (12) as far as  $\Delta_0$  and  $\Delta_1$  are concerned. Values of  $\Delta_3$ ,  $\Delta_4$  and  $\Delta_5$  are obtained by linear interpolation of values from ref. 1 which form part of the input of the programme. - procedure int, which calculates the coefficients of normal force and tangential force by means of the trapezoidal rule.

# 5.4.2 Input and output data

The flow diagram of the input of this part is given in fig. 4, whereas the wing data which are required are presented in table 6. In accordance with the description in section 5.2.2 the last part of the output of programme part I is used also as input of this programme.

The output of the programme is presented in fig. 6. It consists of calculated aerodynamic data, wing data and aerodynamic factors by which the aerodynamic data are determined.

# 6 Application of the computer programme

It is realized that the reader will be interested in conclusions about the applicability of the method. The character of the method is such that general conclusions can only be drawn when a large number of applications have been compared with experimental results and results of more exact methods as far as these are available. Up to now this has not been possible in a satisfying way, mainly for lack of sufficient accurate experimental results. The following must be seen as an attempt to show the usefulness of the method to predict pressure distributions of wings and aerofoils in compressible flow. Attention will be paid to the items:

- the analytical representation of aerofoil contours

- the calculation of pressure distributions on two-dimensional aerofoils in incompressible and compressible flow.
- the calculation of pressure distributions on isolated wings in incompressible and compressible flow.
- the calculation of pressure distributions on wings in the presence of a body.

The examples which will be presented next, form a collection which is representative for the experience which the NLR has had with the method so far. Examples of application of the method by means of Weber's summation formulas can be found in RAE reports which are mentioned in ref. 1. These include a limited experimental verification of individual terms in equation (4).

#### 6.1 Two-dimensional flows

To calculate the pressure distribution on two-dimensional aerofoils, equation (4) can be reduced to the formula which has been presented by Küchemann and Weber for this purpose. This formula has been applied by them in combination with Weber's summation formula's for the integration and differentiation of functions which depend on camber and thickness. They have shown that good agreement exists with exact pressure distributions for symmetrical Joukowski sections without circulation in incompressible flow. This is confirmed by application of the present analytic representation the aerofoil contour. An example is given in table 7 where the calculated pressure distribution on a 10.37% thick symmetrical Joukowski section has been presented together with exact results. In the table the discrepancies of the analytic contour representation from the exact contour are presented as  $\Delta z$  in parts of the chord. In equation (28) l=12 has been chosen in this case. It can be seen that the deviation  $\Delta z$  does not exceed  $2.10^{-5}$ .

Less good results have been obtained by Küchemann and Weber for cambered Joukowski sections. This tendency also exists when the present method of evaluation of equation (4) is being applied. An example is presented in table 8. It concerns a 12% thick Karman-Trefftz section with a 3% camber. Results of the present method are given in the table together with exact results; both for incompressible flow. Again l=12 has been chosen in the analytic representation of the aerofoil contour. Herewith the maximum deviation  $\Delta z$  slightly exceeds  $3.10^{-5}$  in parts of the chord in this case. It can be noted that the discrepancies between exact  $C_p$  values and  $C_p$  values according to equation (4) in table 8 are of the same order of magnitude as discrepancies which occur when calculations are being compared with measurements under lifting conditions. This is a result of different boundary layer development on upper and lower surface under these conditions in viscous flow as will be illustrated next.

A good deal of experimental verification of the method by Küchemann and Weber has been based on wings with RAE thickness distributions. Of one of these, namely the RAE 101 section, reliable experimental pressure distributions at low Mach numbers can be found in ref. 9. For this section a number of pressure distributions have been calculated by the present method. First results for incompressible flow have been compared with results presented in ref. 8, which have been obtained by means of the Goldstein III approximation. In fig. 7 of the present report it can be seen that good agreement exists at  $C_n=0.4$ . At lower and higher  $C_n$  values, up to  $C_n=0.8$ , the same has been found. As far as the analytic representation of the aerofoil contour is concerned it can be noted that  $\Delta z$  does not exceed  $5.10^{-5}$  in parts of the chord with l=12. At zero degrees angle of attack experimental  $C_p$  values from ref. 9 coincide with calculated results within the experimental accuracy at  $M_0=0.18$ . Also for the RAE 102 thickness

distribution good agreement between results of eq. (4) and results of Goldstein III in ref. 8 has been found.

Under lifting conditions the comparison of experiments and calculations is troubled by reduction of circulation during measurements, mainly due to different development of the boundary layer on upper and lower surface. It is possible in principle to correct the calculation approximately for this effect along the lines which have been described in ref. 1. It is necessary, however, then to know the boundary layer displacement thickness. Especially as far as three-dimensional wings are concerned the boundary layer thickness is not known in most practical cases. Comparison of calculated and measured pressure distributions at the same value of sectional lift is possible by equation (4), by treating the effective angle of attack  $\alpha_e$  as an empirical parameter. This procedure is recommended in ref. 1 as a crude way of allowing for boundary layer effects. For the two-dimensional RAE 101 section it is shown in fig. 8 in the present report that this is not very satisfactory. It can be seen that neglecting the effects of boundary layers on circulation at 4 degrees angle of attack, causes rather large differences between calculated and measured  $C_p$  values in fig. 8. However, correction of angle of attack  $\alpha_e$  in the calculation, to obtain the same lift coefficient as during measurements, does not reduce the absolute values of  $C_p$  differences.

The two examples of symmetrical aerofoils which have been discussed above have one feature in common. The nose of the sections has a shape which is very similar to an elliptic shape. From the derivation of equation (4) it becomes clear that for this shape, results can be obtained with the present method which are in good agreement with exact results in incompressible flow. A characteristic dimensionless parameter for the shape of the nose must depend on the radius of curvature of the nose  $\rho$ , the position of maximum thickness  $x_{t_{max}}$  and the maximum thickness t. These can be combined to give

$$W = \frac{2x_{t_{\text{max}}}}{t} \sqrt{\frac{\rho}{x_{t_{\text{max}}}}}$$
(56)

which is equal 1 for elliptic shapes. The parameter W is also equal to 1 for the sections which have been described up to now. One more example will be given of a section for which W = 1.

The last example of an aerofoil section for which W = 1 will be the section NACA 65 A 006. In ref. 6 it has been shown that for this section results of a method based on a distribution of singularities on the surface differ from results of the high order Theodorsen method which has been applied in ref. 7. Both methods must be regarded as exact in incompressible flow. The present method has been applied to calculate the pressure distribution on the section approximately under zero lift condition. The result can be found in fig. 9 where  $C_p$  has been plotted against the co-ordinate x. In fig. 9 it can be seen that the result of ref. 6 has been reproduced by the present method. As a reference also the result of ref. 7 has been given in fig. 9. The shape of the aerofoil section, as given in ref. 7, appeared to have a point of infexion near x = 0.05 in parts of the chord. This made it necessary to take l=15 in the representation of the aerofoil shape to keep  $\Delta z < 5.10^{-5}$  in parts of the chord.

Equation (4) has been applied by NLR to a number of aerofoil sections for which W > 1. It has appeared that in those cases errors can occur as a result of reduced applicability of the corrections for thickness effects near round leading edges which are incorporated in equation (4). This will be illustrated for a 10% thick symmetrical section of the NACA 4-digit type for which W=1.8. The thickness distribution of the section is

$$z_t = 0.20000 \sqrt{x} - 0.30959 + 0.50904 + 0.53280 + 0.53280 = 0.4$$
  
$$z_t = -0.00701 + 0.14757 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.001680 + 0.0$$

and

$$z_t = z_{t_{x=0,7}} - 1.0563 \ (x - 0.7) \text{ for } 0.7 < x \le 1.00828$$
 (57)

With l=12 in the analytic representation of this distribution according to equation (28),  $\Delta z$  does not exceed  $5 \cdot 10^{-5}$  in parts of the section chord.

At angle of attack zero, the pressure distribution in incompressible flow has been calculated by NLR on the section defined by eq. (57) by means of a method based on a distribution of singularities on the surface. The results of this calculation can be regarded as exact. Comparison with results obtained by application of equation (4) under the same condition ( $\alpha = 0$ ,  $M_0 = 0$ ), in table 9, makes clear that just ahead of the minimum pressure maximal discrepancies in the order of  $\Delta C_p = 0.08$  occur. By means of equation (4) compressible flow solutions are being derived from incompressible flow solutions by multiplication factors which depend on Mach number. From this it is clear that errors in the solution for incompressible flow will become apparent in the same way in the solution for compressible flows. This is illustrated in fig. 10 and fig. 11 where measured and calculated pressures are presented for this section as function of the thickness co-ordinate, for  $M_0 = 0.5$  and 0.725 respectively. The small differences between pressures measured on upper and lower surface are caused by a small angle of attack during experiments.

It can be demonstrated that improvement of the basic solution for incompressible flow leads to a much better

agreement of the measured and the calculated pressure distribution. According to equation (4) the velocity  $V(x_v)$ on a symmetrical non-lifting two-dimensional section can be approximated by

$$\frac{V(x_{\nu})}{V_{0}} = \frac{1 + \frac{1}{\pi} \int_{0}^{1} \frac{dz_{t}}{dx} \frac{dx}{x_{\nu} - x}}{\sqrt{1 + \left(\frac{dz_{t}}{dx}\right)_{x_{\nu}}^{2}}}$$
(58)

in incompressible flow. This expression can be formally generalized to

$$\frac{V(x_{\nu})}{V_{0}} = \frac{1 + \frac{1}{\pi} \int_{0}^{1} \frac{\mathrm{d}z_{t}}{\mathrm{d}x} \frac{\mathrm{d}x}{x_{\nu} - x}}{\sqrt{1 + p^{2}(x_{\nu})}}.$$
(59)

In this expression  $p(x_v)$  can be calculated by substitution of the exact values  $V(x_v)$ . This has been done for the section given by eq. (57). The values of  $p(x_v)$  have been substituted in the equation for  $V(x_v)$  in compressible twodimensional flow to give:

$$\frac{V(x_{\nu})}{V_{0}} = \frac{1 + \frac{1}{\pi B_{\nu}} \int_{0}^{1} \frac{dz_{t}}{dx} \frac{dx}{x_{\nu} - x}}{\sqrt{1 + \left(\frac{p}{B_{\nu}}\right)^{2}}}$$
(60)

with  $B_w$  according to the limit of expression (21) for two-dimensional flow

$$B_{w} = \left[1 - M_{0}^{2} \left\{1 - C_{p_{1}}(\mathbf{x}_{v})\right\}\right]^{\frac{1}{2}}$$

Here  $C_{p_i}(x_v)$  is the pressure coefficient in incompressible flow according to the exact method. By means of eq. (22)  $C_p(x_v)$  has been evaluated for  $M_0 = 0.5$  and  $M_0 = 0.725$ . The results are given with the indication "improved method" in fig. 10 and fig. 11 respectively. Within the scheme of the method of ref. 1 this type of improvement can be applied directly to calculate pressure distributions on three-dimensional non-lifting wings.

However, in fact not the Riegels factor in eq. (58) has to be improved but the integral is not correct in the original expression for incompressible flow. This illustrates that the improvement has an arbitrary character. The procedure can only be justified by the fact that it increases the region of applicability of the existing method in a simple way. For the symmetrical aerofoil given by eq. (57) under lifting conditions, it can be shown that the mere substitution of p(x) in the Riegels factor does not give correct results as may be expected from the derivation of the original expressions by Weber. Results of application of the formula

$$\frac{V(x_{v})}{V_{0}} = \frac{1}{\sqrt{1+p^{2}(x_{v})}} \left\{ \cos \alpha_{e} \left[ 1 + \frac{1}{\pi} \int_{0}^{1} \frac{\mathrm{d}z_{t}}{\mathrm{d}x} \frac{\mathrm{d}x}{x_{v} - x} \right] + \sin \alpha_{e} \sqrt{\frac{1-x_{v}}{x_{v}}} \left[ 1 + \frac{1}{\pi} \int_{0}^{1} \left( \frac{\mathrm{d}z_{t}}{\mathrm{d}x} - \frac{2z_{t}}{1 - (1 - 2x)^{2}} \right) \frac{\mathrm{d}x}{x_{v} - x} \right] \right\} (61)$$

are presented in table 9 for  $\alpha_e = 6^\circ$ , together with exact results and results of eq. (4). Near the nose, differences between results of eq. (61) and exact results are about equal on upper and lower surface.

With regard to realization of improvements for general lifting aerofoils it is considered worthwhile to study the applicability of higher order terms which result from existing perturbation methods. Also improvement of calculated compressibility effects can be expected in principle from some of these methods.

As mentioned before, direct comparison of calculations and experiments under lifting conditions is troubled by boundary layer effects. As no suitable correction method for these effects is available at the moment, the authors will confine detailed comparisons with experiments to non-lifting conditions for the time being.

# 6.2 Three-dimensional flows

By means of eq. (4) the pressure distribution has been calculated on the symmetrical Warren 12 wing under zero lift conditions in incompressible flow. The features of the wing are: aspect ratio 2.828, taper ratio 1.3; leading edge sweep back 53.5 deg.; trailing edge sweep back 32.9 deg. and streamwise section : 6% RAE 102. The results from eq. (4) have been presented in fig. 12 in comparison with "exact" results from Douglas according to the method of ref. 6. These results have been obtained through the National Physical Laboratory, Teddington.

Apparently three-dimensional effects are such for this wing that the semi-empirical method fails to predict the

pressure distribution exactly. It is likely that the errors will increase with Mach number as a result of decreasing effective aspect ratio and increasing effective sweep and taper ratio with increasing Mach number. This cannot be verified because at Mach numbers which are different from zero, errors due to three-dimensional effects cannot be separated from errors due to the approximation of compressibility effects. The errors due to compressibility are different in the method of ref. 6 compared with the present method. Two-dimensional experimental evidence suggests that application of the semi-empirical factor  $B_w$  gives better correspondance with experimental results than the compressibility effects which are predicted by the Goethert rule which has been applied in ref. 6.

A second example of application of eq. (4) to a three-dimensional wing can be found in fig. 13. The wing is symmetrical with 6% thick RAE 101 sections in streamwise direction and aspect ratio 2.84. The sweep of the midwing is 55°. The wing was mounted symmetrically on a circular cylindrical body during the experiments of which results have been plotted in fig. 13. In the calculations the body side was treated as a full reflection plane. At Mach 0.52 the effective aspect ratio of the wing is about 2.5 and the effective angle of sweep of the mid wing is about 60°. These values are about the same as the corresponding values of the Warren 12 wing in incompressible flow. An effective sweep of 60° and an effective aspect ratio of 2.5 have been mentioned in ref. 1 as boundaries of the applicability of the method. The discrepancies between calculated and measured pressures appear to be of the same order of magnitude as the accuracy of the measurements at Mach 0.52. In ref. 10 it is shown that discrepancies between calculated and measured pressures for this wing. The discrepancies at station 4 at Mach 0.96 can be reduced to about zero with a choice of the thickness interpolation factors  $K_{2a}$  and  $K_{3a}$  which is different from the choice which follows from the rules given in ref. 1. This is demonstrated in ref. 10 based on an *ad hoc* iteration with respect to  $K_{2a}$  and  $K_{3a}$ .

In view of the results presented in fig. 12 and the possibility to improve the results in fig. 13 as described, it seems to be possible to increase the range of applicability of the method, by improvement of the calculation of the linearized thickness effects which form part of eq. (4). When these effects are being calculated directly for each three-dimensional wing, the semi-empirical interpolation between the basic solutions as mentioned in section 2 can be avoided. At least as far as the disturbance velocity in x-direction is concerned this improvement is recommended.

Finally in fig. 14 results of application of eq. (4) have been given for a cambered wing in combination with a body, at Mach 0.45 under lifting conditions ( $C_n \approx 0.2$ ). The body side has been treated as a reflection plane in the calculations. Because it is a low-wing configuration this causes at the wing root slight discrepancies at the lower side of the wing between calculated and measured pressures. The calculation has been executed in such a way that the local normal force at each station corresponds to the local normal force from experiment. This only concerns the choice of  $\alpha_e$  in equation (4). The minimum pressures are predicted remarkably well under these circumstances by theory. However, this happens to be a result of the cancellation of an error due to boundary layer effects, as demonstrated in fig. 8, by a counteracting error due to non-linear thickness effects near the leading edge as demonstrated in fig. 10 and 11. Discrepancies between theory and experiment will increase only slightly when the calculation is based on the local  $C_n$  from linear lifting surface theory at this relatively small angle of attack.

# 7 Conclusions

- 1 The applicability of the computer programme for calculation of the pressure distribution on aerofoils and wings in subsonic attached flow has been demonstrated.
- 2 Comparison of calculated results with results obtained by means of experiments or other calculation methods has shown some of the boundaries of the family of shapes for which the semi-empirical method of ref. 1 is a reliable means for predicting pressure distributions on wings and aerofoils.
- 3 Some examples have shown that the region of applicability of the method may be increased by making use of results of full linearized theory for wings and exact results for symmetrical aerofoil sections in incompressible flow.
- 4 Correction of calculated data for the influence of the boundary layer is approximately possible within the scope of the method. Results have made clear that improvement with respect to this is desirable in lifting cases.

#### 8 Acknowledgement

The authors wish to thank members of the Royal Aircraft Establishment and the Aircraft Research Association for their willingness to discuss various questions about the method of ref. 1. In this context the names of Mr. J. A. Bagley, Mr. A. B. Haines and Mr. K. W. Newby certainly should be mentioned.

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#### Appendix

Computer programme part I, part II and part III.

In section 5 of the report the programme is discussed.

```
begin
                comment
                                                 Part L
                                                X, Z, xne, zne, xst, zst, al, F, C, a0, xzw, sn, cs, kl, kl1, kl2, A1, A2, pi;
n, n1, n2, N, l, i, j, t, k, g, h;
extma, boole, thickness, der;
                real
                Integer
                boolean
               m[0:1];
                begin
               rea1
                \begin{array}{c} \mbox{real procedure dif(i, s); value i, s; integer 1, s;} \\ \hline \mbox{begin integer t;} \\ \hline \mbox{for t:= 1 step 1 until i do} \\ \hline \mbox{d[1,t]:= (d[1,t+1] - d[1,t])/(d[0,t+s] - d[0,t]);} \\ \hline \mbox{if i>1 then dif:= d[1,i] - (d[0,1] - x) \times dif(i-1,s+1) } \end{array} 
                end dif;
               real procedure DET(A, n, p); value n; integer n; array A; integer array p;
begin integer 1, j, k; real d, r, s; array v[1:n];
for I:=1 step 1 until n do v[i]:=sqrt(INPR(D(j, 1, n, A[i, j], A[i, j]));
d:=1; for k:=1 step T until n do
begin r:=-1; for i:=k step 1 until n do
begin A[i, k]:=A[1, k] - INPR(D(j, 1, k-1, A[1, j], A[j, k]);
s:=abs(A[1, k])/v[1];
if a > r then begin r:=s; p[k]:=i end
                                                                  if s > r then begin r = s; p[k] = i end
                                                 end LOWER;
                                                 v[p[k]]:=v[k];
                                                 \begin{array}{l} for \ j = 1 \ \text{step } 1 \ \text{until } n \ \text{do} \\ \hline \underline{begin} \quad r \coloneqq A[k, \ J]; \ A[k, \ J] \coloneqq \text{if } j < k \ \text{then } A[p[k], \ j] \ \underline{else} \\ & (A[p[k], \ J] - INPROD(\overline{l}, \ l, \ k - \overline{l}, \ A[k, \ 1], \ A[1, \ J]))/A[k, \ k]; \\ & \text{if } p[k] \ \frac{1}{2} \ k \ \text{then } A[p[k], \ J] \coloneqq -r \end{array}
                                                 end UPPER;
                                                d = A[k, k] \times d
                                end LU;
                                DET := d
               end DET;
                              re SOL(UU, b, n, p); value n; integer n; array IU, b; integer array p;

integer 1, k; real r;

for k:=1 step f until n do

begin ri=b[k];

b[k]:=(b[p[k]] - INFROD(1, 1, k-1, IU[k, 1], b[1]))/IU[k, k];

if p[k] = k then b[p[k]]:=-r
                procedure
               begin
                               end;
for k:=n step -1 until 1 do
5[k]:=b[k] - INFRID(1, k+1, n, LU[k, 1], b[1])
               end SOL;
               real procedure DETSOL(A, b, n); value n; integer n; array A, b;

<u>begin</u> integer array p[1:n];

<u>DETSOL:=DET(A, n, p);</u> SOL(A, b, n, p)
               end DETSOL;
                                               klkwr(N,m); value N,m; integer N, m;
max; integer 1,j,k; real array u[0:N], v[0:m], M[1:1+4, 1:1+4],r1[1:1+4];
                procedure
               begin
                             real
                                \begin{array}{l} \mbox{for } i:=0 \mbox{ step } 1 \mbox{ until } N \mbox{ do } u[i]:=x[t,i] \mbox{--}xzw; \\ \mbox{for } j:=0 \mbox{ step } 1 \mbox{ until } 1 \mbox{ do } u[i]:=x[t,i] \mbox{--}xzw; \\ \mbox{begin} \mbox{ } r1[J+1]:=SUM[1,0,N,f[t,i] \mbox{ \times } u[i] \mbox{/--} j); \\ \end{array} 
                                               end;
               for 1 = 2, 3, 4 do for j = 2, 3, 4 do M[1+1,1+j] = 0;
r1[1+2] = r1[1+3] = 0; r1[1+4] = f1[t,0];
                              end;
```

DETSOL(M, rl, l+4); max = 0; for 1:= 1 step 1 until N -1 do <u>begin</u> kl:= sbs(f[t,1] - SUM(j,1,1+1,rl[j] × u[1]/(j-1))); <u>if</u> max - kl<0 then max = kl end: NLCR; print(max); for i = 1 step 1 until 1 do  $c[t,i] = SUM(k,1,1, r1[k+1] \times fac[k]/fac[i]/fac[k-i] \times (-xzv)A(k-1))$ end klkor; for t:=0, 1 do begin N:=  $(T-t) \times n1 + t \times n2;$ for i:=0 step 1 until N-1 do
readn( x[t,1],z[t,1]); if der then goto notang; ab[t]:=(z[t, N-1]-z[t, N-2])/(x[t, N-1] - x[t, N-2]);notang: x[t,0]:=xne; z[t,0]:=zneend; ncextrap: end else begin x[0,n1]:=x[1, n2]:=xst:=(ab[0] × x[0, n1-1] - ab[1] × x[1, n2-1] - z[0, n1-1] + z[1, n2-1])/(ab[0] - ab[1]); z[0,n1]:=z[1, n2]:=zst:= ab[0] × (xst - x[0, n1-1]) + z[0, n1-1]; al:=arctan((zst - zne)/(xst - xne)); cs:=cos(a1); sn:=sin(a1); K[0]:=K[1]:=(xst - xne) × cs + (zst - zne) × sn end; a0:=a0xsqrt(C/K[0]);for t:=0, 1 do
begin boole:= 7(thickness A t = 1); AA:  $N := (1-t) \times n1 + t \times n2;$ for i=0 step 1 until N do  $\begin{array}{l} \hline 10^{-1} 1 \approx 0 & step + until k do \\ \hline begin & X \approx [t, 1] = xne; \\ x[t, 1] \approx (X \times cs + Z \times sn)/K[t]; \\ z[t, 1] \approx (-X \times sn + Z \times cs)/K[t]; X \approx x[t, 1]; \\ kl := if boole then (1-2 \times t) \times a0 \times sqrt(X) \times (1-X) & else 0; \\ f[t, 1] \approx z[t, 1] = kl \\ \end{array}$ end; if extra then for k= 1 step 1 until m[t] do 
 If extra then for k:= ; step ; until in 0; up

 begin
 g:= redd; X:= x1[t,k]:= x[t,g];

 for i:= i step i until h do

 begin
 d[0,h+1-i]:= x[t,g-1]; d[1,h+1-i]:= z[t,g-i];

 d[0,h+1]:= x[t,g+i]; d[1,h+i]:= z[t,g+i];
 end: If N < 5 then for i = 1 step 1 until 1 do c[t,i] = 0 else klkwr(N, m[t]);</pre> end;  $\begin{array}{c} for \ i:=1 \ step \ 1 \ until \ n1 \ do \\ \hline begin \\ \hline begin \\ \hline FUNICR; \ for \ t:=0, \ 1 \ do \\ \hline begin \\ \hline X:=x[t, 1]; \ FLOP(6, 3, k[t] \times k); \ FLOP(6, 3, k[t] \times (f[t, 1] - SUM(j,1,1,c[t,j] \times X \not hj))); \\ kl:=1 \ f \ thickness \ \wedge t = 1 \ then \ 0 \ else \\ \hline a0 \times (.5 - t)/sqrt(X); \\ FLOP(5, 3, kl \times (1-3 \times X) + \ SUM(j,1, 1, j \times k(j-1))); \\ FLOP(6, 3, -.5 \times kl \times (1/X + 3) + \ SUM(j, 2, 1, j \times (j - 1) \times c[t, j] \times X \not h(j - 2))); \end{array}$ end re: end; RUNOUT; PUNICR; PUNICR; FLOP(6, 3, a1 × 57.2957795); FLOP(9, 3, K[0]); PUNICR; PUNICR; stop; end; 
$$\begin{split} & \text{N} \coloneqq \text{read}; \text{ for } i \coloneqq 1 \text{ step } 1 \text{ until } \mathbb{N} \text{ do } \mathbb{X}[1] \coloneqq \text{read}; \\ & \text{for } i \coloneqq 1 \text{ step } 1 \text{ until } \mathbb{N} \text{ do} \\ & \underline{\text{begin }} \text{ PUNICR; } \text{ for } t \coloneqq 0, \text{ Tdo} \\ & \underline{\text{begin }} \text{ Kimes } (\mathsf{X} \models \mathsf{C}/\mathsf{K}[1] \times [1]; \text{ klime } a 0 \times \operatorname{sqrt}(\mathsf{X}); \\ & \mathbb{K}[1] \coloneqq \text{ if } \text{ thickness } \wedge t = 1 \text{ then } 0 \text{ else } (1 - 2 \times t) \times \mathbb{k}[1 \times (1 - X); \\ & \mathbb{Z}[t, t] \coloneqq (\mathbb{k}[1] + \operatorname{SUM}(1, t], t], c[t, t]) \times \mathbb{K}[t]/\mathsf{C}; \end{split}$$
 $\begin{array}{c} \text{if thickness then} \\ \overline{z[0, 1]} := z[0, 1] + z[1, 1]; \\ z[1, 1] := 2 \times z[1, 1] - z[0, 1] \end{array}$ end; begin end; end FUNICR; FUNICR; for i =1 step 1 until 1 do For the constant of the state of the constant begin :03 end;

PUNICR; RUNOUT; RUNOUT; PUNICR; FIXP(2, 0, 1); PUNICR; for 1:=1 step 1 until 1 do FLOP(12, 3, c[1, 1]); PUNICR; TAPEND; real array begin sı, s2, SJ, SJ; `b[0:1]; repeat: kl:=.5 x a0/sgrt(xst);  $\begin{aligned} & \texttt{kl} := (\texttt{kl} \times (1-3 \times \texttt{xst}) + \texttt{SUM}(\texttt{n}, 1, \texttt{l}, \texttt{n} \times \texttt{c[0, n]} \times \texttt{xst} \land (\texttt{n-1}))) / \\ & (-.5 \times \texttt{kl} \times (1/\texttt{xst} + 3) + \texttt{SUM}(\texttt{n}, 2, \texttt{l}, \texttt{n} \times (\texttt{n-1}) \times \texttt{c[0, n]} \times \texttt{xst} \land (\texttt{n-2}))); \end{aligned}$ xst:=xst - kl1; if abs(kl1) > .00001 then goto repeat; xl0):=xstxk[0]/X[1]; x[N+1]:=T; for t:=0, 1 do z[t, 0]:= z[t, N+1]:=0; PUNICR; FUNICR; FIXP(2,0,N); PUNICR; for t:=0 step 1 until N+1 do begin xst:=x[t]; X:=K[1];xst/K[0]; kl:=art(X); kl1:=.5 x a0 x ln((1 + kl)/(1 - kl)); kl2:=ln((1 - X)/X); FUNICR; FLDP(9, 3, xst); FLDP(9, 3, z[0, t]); FLDP(9, 3, z[1, t]); for i:=1 step 1 until 1 do b[1]:=X x b[i-1]; AT:=kl1/kl; A2:=5 x (a0 - kl x kl1) + A1; kl1:=SUM(n. 1. 1. n x c[0. n] x b[n-1]); xst:=xst - kl1;  $k11:=SUM(n, 1, 1, n \times c[0, n] \times b[n-1]);$  $52 = .5 \times a0 \times (1 - 5 \times X)/k1 + k11;$  $\begin{array}{l} & 5 = 50M(n, 1, 1, n \times c[1, n] \times xstA(n-1)); \\ & 55 = 50M(n, 2, 1, n \times c[0, n] \times xstA(n-1)); \\ & 51 = (A2 - 50M(n, 2, 1, n \times c[0, n] \times 50M(1, 0, n-2, b[1]/(n-1-1))) - kl1 \times kl2)/pi; \\ & 53 = 51 - (A1 + .5 \times (50M(n, 3, 1, c[0, n] \times 50M(1, 1, n-2, s0M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2)) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) - c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) + c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) + c[0, 1] \times kl2) ) + c[0, 1] \times kl2) ) / pi; \\ & 50M(1, 0, 1-1, b[1]/(1-1)) + b[1] \times kl2) ) + c[0, 1] \times kl2) ) + c[0, 1] \times kl2) ) + c[0, 1] \times kl2)$ FLOP(9, 3, s1); FLOP(9, 3, s2); FLOP(9, 3, s3); FLOP(9, 3, s5); TA PEND end: end; RUNOUT; end begin comment Part II: pi, rad, kl, kl1, kl2, kl3, kl4, cs, sn, B, C, D, E, xnu, A1, A2, MD, MO2, M1, Bw, alfa, alfa1, alfa2, Cn, Cn1, Cn2, Cl, Cl1, Cl2, CD, CT; rcal integer array 1, j, n, N, a, A, k; prod, c, d[0:15]; pi:= 3.14159265359; rad:= pi/180; prod[0]:= .5; for i:= 1 step 1 until 15 do prod[1]:= (i + .5) × prod[1 - 1]/(i + 1); N = read; for i = 1 step 1 until N do d[i] = read; A = read; real array x, zb, zo, S1, S2, S3, S4, S5, p, q, A11, A12, A21, A22[ 0: A+1 ]; begin  $\begin{array}{rcl} \mbox{real procedure} & \mbox{int;} \\ \hline \mbox{begin} & \mbox{real} & \mbox{u; CT:= u:= 0;} \\ \hline \mbox{for a:= 1 step 1 until A do} \\ \hline \mbox{begin} & \mbox{p[a]:= 1 - (cs \times AT1[a] + sn \times A12[a])} \\ \hline \mbox{q[a]:= 1 - (cs \times A21[a] + sn \times A22[a])} \\ \hline \mbox{if way } > .00001 then \end{array}$  $\begin{array}{l} f (M) > .0001 \text{ then} \\ \hline \text{Degin} \quad p[a] \coloneqq D \times (1 - (1 + C \times p[a]) \land 3.5); \\ g[a] \coloneqq D \times (1 - (1 + C \times q[a]) \land 3.5) \end{array}$ end end; ena;  $\mathbf{kl} := (1 - x[A-1])/(x[A] - x[A-1]);$   $p[A+1] := (p[A] - p[A-1]) \times kl + p[A-1];$   $q[A+1] := (q[A] - q[A-1]) \times kl + q[A-1];$   $p[0] := q[0] := .5 \times (x[1]/(x[2] - x[1]) \times (p[1]+q[1]-p[2]-q[2]) + p[1] + q[1]);$  $\begin{array}{rcl} p_{1} & q_{1} & p_{2} & p_{3} & p_{3}$ Int:= u end int;  $\begin{array}{l} for a := 0 & step 1 & until A+1 & do & readn(x[a], zb[a], zo[a], S1[a], S2[a], S3[a], S5[a]); \\ \hline x[0] := 0; & B := abs(d[N]); \\ for a := 1 & step 1 & until A & do \\ \hline begin & xnu := x[a]; \\ \hline begin & xnu := x[a]; \\ \hline \end{array}$  $S^{4}[a] := if B < 1_{y} - 100 \text{ then } 0 \text{ else} = -\text{sqrt}((1-xnu)/xnu) \times (SUM(n, 0, N-2, SUM(1, n+2, N, i \times d[1] \times xnu)(1-n-2)) \times (xrod[n]) + S5[a])$ end; new M: MD = read; MD2 = MD × MD; M1 = sqrt(1 - MD2); PUNICR; for a:= 1 step 1 until A do  $\overline{begin}$  kl = (1 + ST[a])  $\frac{2}{2}$  (1 + S2[a]  $\frac{2}{2}$ ); xnu = x[a]; if kl< 1 then By = MI else begin kl1 = 1 - MD2 xkl; oegin if kl1> 0 then Bw = eqrt(kl1) else begin FIXP(3, 0, a); Bw = M end end: end; kIT:=  $\operatorname{sqrt}(1 + ((S2[a] + S5[a])/Bw) \land 2); kI3 := 1 + S1[a]/Bw;$ A11[a]:= (kI3 + S4[a]/M1)/kl1;kI4 :=  $(1 + S5[a]/Bw) \times \operatorname{sqrt}((1 - xnu)/xnu)/M1;$ A12[a]:= kI4/kI1;kl1 :=  $\operatorname{sqrt}(1 + ((S2[a] - S5[a])/Bw) \land 2);$ A21[a]:= (kI3 - S4[a]/M1)/kl1;A22[a] := -k14/k11 e<u>nđ</u>;

C:= .2 × MO2; D:= - 1/.7/MO2; cs:= 1; sn:= 0; B:= Clit= int; new alfa: k:= read; alfa := read; if k= 0 then alfa2:= rad × alfa else begin Clin Cni:= B; alfa1:= 0; alfa2:= .05; if ka 1 then Cn = alfa else Cl = alfa enā; alfa:= if k= 1 then  $(alfa2 - alfa1) \times (Cn - Cn1)/(Cn2 - Cn1) + alfa1$  $(alfa2 - alfa1) \times (C1 - C11)/(C12 - C11) + alfa1;$ else Cn1 := Cn2; Cl1 := Cl2; alfa1 := alfa2; alfa2 := alfa; goto improve; ready: PUNICR; PUNICR; PUTEXT( $\ddagger$  Weber - 2 dim $\ddagger$ ); PUNICR; PUNICR; PUTEXT( $\ddagger$  MO alfa $\ddagger$ ); PUNICR; FIXP(1, 6, MO); FIXP(2, 6, alfa2/rad); PUNICR; PUTEXT( $\ddagger$  Cn CT Cl CD Cm $\ddagger$ ); PUNICR; PUTEXT({ Cn CT C1 CD Cm;}); PUNICR; FUNICR; FIXP(2, 6, Cn2); FIXP(2, 6, CC); FIXP(2, 6, C12); FIXP(2, 6, CD); Cn:= 0; for a:= 1 step 1 until A +1 do Cn:= Cn +  $.5 \times (x[a] - x[a-1]) \times ((p[a-1] - q[a-1]) \times (x[a-1] - .25) + (p[a] - q[a]) \times (x[a] - .25));$ FIXP(2, 6, Cn); FUNICR; FUNICR; FUTEXT( $\xi$  xnu Cp upper Cp lower}); FUNICR; for a:= 1 step 1 until A do begin FUNICR; FIXP(1, 5, x[a]); FIXP(3, 3, p[a]); FIXP(3, 3, q[a]) end; TAPEND; FUNICR; k:= read; If k= 3 then goto new k; 11 k= 2 then stop; goto new alfa end programme; end comment Part-III; begin comment Part-111; real pi, rad, xst, tT, tR, CT, CR, Lm, Last, L1, L0, Cy; s, s1, y, Se, delT, delR, sn, cs, csnu, tentc, tl1, tl0, Cc1, Cc2, yC1, yC2, del1, del2, xk, kl, kl1, xnu, A1, A2, so1, B, C, D, E, S1, B2, B3, t1, t2, t3, t4, t5, t6, g, g1, g2, g3, g4, g5, K2, K3, M0, M02, M1, Lma, La, ma, n0a, Ha, Aa, sa, tma, Cn, Cn1, Cn2, al,al1, al2; integer 1, A, a, N, M, n, k, s1, j, cranks; real array prod, xg, yg, zg, d[0:12], de3[0:5, 0:13], de4[0:6, 0:8], de5[0:4, 0:9], Cp,x,zu,zl, Lax, S1, S2, S3, S5[0:45], term1, term2, term3[1:2,1:44], press[1:2,0:45]; procedure real determ K clab, p, ps1, d4; array d0, d1, d5, e[1:3], d3[1:5]; real real procedure delta(M, N, e, de); real e; integer M, N; array de; begin real a1, a2, a3, la; integer 1, j; [:=j:=2;:la:=abs(g4/rad); begin AB: BC: delta :=a3 end delta; g4 :=arctan(sin(Iast)/cos(Iast)/M); clab :=cos(g4); e[1]:=M1 × (s-y)/CT; e[2]:=M1 × y/CR; e[3]:=M1 × (s+y)/CT; for i := 1 step 1 until 5 do begin p:=e[i]/clab; psl:=p × sin(g4) × (-1) / i; g1:=sqrt(xst × (xst + 2 × psl) + p × p); g5:=g2:=0;  $\begin{array}{l} \mbox{for a:=1 step 1 until A+1 do} \\ \hline \mbox{begin} & g:=xst - x[a]; g:=sirt(g \times (g + 2 \times ps1) + p \times p); \\ g:=s2[a] \times (p + g + g1) \times (g - g1)/(p + g)/(p + g1)/g/g1; \\ g:=g2 + (g5 + g3) \times (x[a] - x[a-1]); g5 :=g3 \end{array}$ end; d1[1]:=.5 × M1 × psl × g2 × clab/pi; g1:=aqrt(1 + 4 × e[1] ↓ 2); d0[1]:=-2 × (1 - e[1] × ln((g1 + 1)/(g1 - 1)))/pi; d3[1]:=delta(5, 13, e[1], de3); d5[1]:=delta(4, 9, e[1], de5); db :==delta(6, 8, y/s1, de4); and: end;  $\begin{array}{l} K3 \approx 1 + (((d0[1] + d0[3]) \times M1 \times clab - (d5[1] + d5[3]) \times xk) \times tT/CT + d1[1] + d1[2] + d1[3] + \\ d3[4] \times sin(del1)/cos(del1) + d3[5] \times sin(del2)/cos(del2) + 2 \times d3[2] \times sin(delR)/cos(delR) + \\ (d3[1] + d3[3]) \times sin(delT)/cos(delT) + (d4 \times abs(xk) + 2 \times d5[2] \times xk) \times tR/CR)/s1[0]/M1/clab; \end{array}$ i=1; AA: if e[2] < xg[i] then  $K^{2}:=(yg[i] - yg[i-1]) \times (e[2] - xg[i-1])/(xg[i] - xg[i-1]) + yg[i-1]$ else begin 1:=i+1; goto AA end; if e[1] < xg[1] i;=1; BB:  $\frac{1}{1} = \frac{1}{2} - \frac{1}{2g[i] - 2g[i-1]} \times (e[1] - 2g[i-1]) / (2g[i] - 2g[i-1]) - 2g[i-1]$ else begin 1:=1+1; goto BB end; end determ K;

procedure int; begin T:=1; begin 1:-., plusmin:for a:=1 step 1 until A do begin press[i, a]:=1 - cs × (cs × term1[i, a] + sn × term2[i, a]) - sn × sn × term3[i, a]; if MO2 > .000001 then press[i, a]:=A1 × (1-(1 + A2 × press[i, a]) ↓ 3.5)  $\overline{\text{press}}[i, A+1] := (\text{press}[i, A] - \text{press}[i, A-1]) \times (1 - x[A-1])/(x[A] - x[A-1]) + \text{press}[i, A-1];$ 1:=i+i; if i=2 then goto plusmin; press[1, 0]:=press[2, 0]:=.5 × (press[1, 1] + press[2, 1]); D:= .5×SUM(a,1,A+1,(press[2, a-1] + press[2, a] + press[1, a-1] - press[1, a]) × (x[a] - x[a-1])) end int; pi := 3.14159265359; rad := .01745329252;  $\begin{array}{c} for \ i:=0 \ step \ i \ until \ 12 \ do \ readn(xg[i], \ yg[i], \ zg[i]); \\ \hline for \ i:=0 \ step \ i \ until \ 5 \ do \ for \ n:=0 \ step \ i \ until \ 15 \ do \ de5[i, \ n]:=read; \\ \hline for \ i:=0 \ step \ i \ until \ 6 \ do \ for \ n:=0 \ step \ i \ until \ 8 \ do \ de4[i, \ n]:=read; \\ \hline for \ i:=0 \ step \ i \ until \ 4 \ do \ for \ n:=0 \ step \ i \ until \ 9 \ do \ de5[i, \ n]:=read; \\ \hline for \ i:=0 \ step \ i \ until \ 4 \ do \ for \ n:=0 \ step \ i \ until \ 9 \ do \ de5[i, \ n]:=read; \\ \hline for \ i:=0 \ step \ i \ until \ 4 \ do \ for \ n:=0 \ step \ i \ until \ 9 \ do \ de5[i, \ n]:=read; \\ \hline for \ i:=0 \ step \ i \ until \ 4 \ do \ for \ n:=0 \ step \ i \ until \ 9 \ do \ de5[i, \ n]:=read; \\ \hline for \ i:=0 \ step \ i \ until \ 4 \ do \ for \ n:=0 \ step \ i \ until \ 9 \ do \ de5[i, \ n]:=read; \\ \hline for \ i:=0 \ step \ i \ until \ 4 \ do \ for \ n:=0 \ step \ i \ until \ 9 \ do \ de5[i, \ n]:=read; \\ \hline for \ i:=0 \ step \ i \ until \ 4 \ do \ for \ n:=0 \ step \ i \ until \ 9 \ do \ de5[i] \ step \$ stop; N:=read; for 1:= 1 step 1 until N do d[1]:=read; A:= read; for a = 0 step 1 until A+1 do readn(x[a], zu[a], z1[a], S1[a], S2[a], S3[a], S5[a]); xst:= x[0]; stop;  $\begin{array}{l} \texttt{read}n(\texttt{tT}, \texttt{tR}, \texttt{CT}, \texttt{CR}, \texttt{Im}, \texttt{Iast}, \texttt{L1}, \texttt{L0}, \texttt{Cy}, \texttt{s}, \texttt{s1}, \texttt{y}, \texttt{Se}, \texttt{delT}, \texttt{delR}, \texttt{tentc}, \texttt{cranks});\\ \texttt{if} \ \texttt{cranks} > 0 \ \texttt{then} \ \texttt{read}n(\texttt{del1}, \texttt{del2}, \texttt{yC1}, \texttt{yC2}, \texttt{Cc1}, \texttt{Cc2}) \ \texttt{else} \ \texttt{del1} := \texttt{del2} := \texttt{yC1} := \texttt{yC2} := \texttt{Cc2} :=$ CE: end; E:=s5: MD:=read; MD2:=MD × MD; M1:=sqrt(1 - MD2); A1:=-1/.7/MD2; A2:=.2×MD2; determ K; ream: g3 := 1 - abs(K2); kl:=sin(Im)/cos(Lm)/M1; Lma:=arctan(kl); B:= $2 \times pi \times kl \times y/Cy/Lma \times M1$ ; C :=B  $\times (s - y)/y$ ; La:=sqrt(1 + B  $\times$  B) - B - sqrt(1 + C  $\times$  C) + C; B :=La  $\times Lma$ ; C := $2 \times B/pi$ ; B := .25/(1 + 2 mabs(Lma)/pi);  $\begin{array}{l} B := la \times Ina; \ C := 2 \times B/pi; \ B := .25/(1 + 2xabs(Ina)/pi); \\ Aa := la \times Ina; \ C := 2 \times B/pi; \ B := .25/(1 + 2xabs(Ina)/pi); \\ Aa := la \times s \times s \times M/Se; \ kl := cos(B); \ g := cos(Ina); \ sn := sin(B); \\ na := 1 - .5 \times (1 + C)/((1 + (2 \times g/Aa) \land 2) \land D); \ no := .5 \times (1 - C); \ D := pi \times na; \\ Ha := sin(D)/(1 - D \times (cos(D)/sin(D) - cos(pi \times nOa)/sin(pi \times nOa)))/kl; \\ as := 2 \times pi \times sn \times (1 - ln(M \times tentc)/pi) - ln((1 + sn)/(1 - sn)); \\ tha := arctan(pi/sn)/pi; \ if ths < 0 \ then \ tha := 1 + tha; \\ prod[0] := tha; \ for \ i := 1 \ step \ 1 \ until \ 12 \ do \ prod[i] := (i + tha) \times prod[i - 1]/(i + 1); \\ g := g/M; \ g i := g/kl; \ g 2 := Ha \times g; \\ g^{t} := sqrt(1 - (M \times g1) \land 2); \ g 5 := Ha \times kl \times g^{t}; \\ t := gi \times g 1 + g^{t} \times g^{t}; \ t 5 := g 1 \times g^{2} + g^{t} \times g^{5}; \ t 5 := g^{2} \times g^{2} + g^{5} \times g^{5}; \end{array}$ for a = 1 step 1 until A do  $\begin{array}{l} C = \sin/\operatorname{sqlrt}(1 - kl1); \ csnu = \cos(g3 \times lax[a]); \\ \text{if } X \equiv N(-O) = 16 \ \text{then } csnu = \cos(g3 \times lax[a]); \\ B = 1 + (k3 \times S1[a] - k2 \times ln((1 + C)/(1 - C)) \times S2[a]/pi/sqrt(1 + (S2[a]/B) \land 2)) \times cs/B; \\ B = (1 + S3[a]/B/csnu) \times (1/xnu - 1) \land na; B = xn \times S1[a]/B; \\ t1 = B1 \times B1 + (g3 \times B3) \land 2; \ t2 = 2 \times (B1 \times g1 + B3 \times g3 \times g^4); \\ t^{1} = g^{2} \times B1 + g^{3} \times g5 \times B3; \ i := si := 1; \\ sol = 11 \ abs(d[N]) < 1_{D} - 100 \ \text{then } 0 \ else \\ pi/(pi \times pi + sa \times sa) \times (-sa \times S3[a] - pi/sin(pi \times tha) \times (((1 - xnu)/xnu) \land tha \times ((SUM(n,0,N-2,SUM(k,n+2,N, k \times d[k] \times xnu)(k-n-2)) \times prod[n]) + S5[a]) - \cos(pi \times tha) \times S5[a])); \end{array}$  $\begin{array}{l} D:=1 + ((S2[a] + si \times S5[a])/B/csnu) & 2; \ kl1:=si \times D & (:5 - na); \\ C:=B2\times kl1; kl1:=kl1\times soi \times D & (na - tha); \\ term![i, a]:=(t1 + (1 - K2 \times K2) \times (D - 1) \times sn \times sn + kl1 \times (t2 + t5 \times kl1))/D; \\ term2[i, a]:=2 \times C \times (t4 + t5 \times kl1)/D; \ term3[i, a]:=t6 \times C \times C/D; \end{array}$ olm: i := 2; si :=-si; if si <0 then goto pin end a - cycle; cs=1; sn=0;int; B=D; realf: k=read; al:=read; if k=0 then al2:= rad x al else begin Cn1:= B; al1:= 0; al2:= .05; if k= 1 then Cn:= al end; improval: cs:=cos(al2); sn:=sin(al2); int; Cn2:=D; if  $k = 0 \lor k = 1 \land abs(Cn - Cn2) < .00001$  then goto completed; al:=(al2 - al1) × (Cn - Cn1)/(Cn2 - Cn1) + al1; Cn1:= Cn2; al1:=al2; al2:=al; goto improval;

 $\begin{array}{l} \mbox{completed:} \\ \mbox{completed:} \\ \mbox{completed:} \\ \mbox{read:} \mbox{FIXP}(h, 5, v); \mbox{FIXP}(h, 5, a-v); \mbox{FIXP}(h, 5, lm/rad); \mbox{FIXP}(h, 5, last/rad); \\ \mbox{FIXP}(h, 5, vet); \\\mbox{FIXP}(h, 5, vet); \\\mbox{FIXP}($ 

Input data for the interpolation of k2,  $\Delta 3$ ,  $\Delta 4$  and  $\Delta 5$ .

ENDY:

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220.0-	LE0.0 -	-0.028	£10.0-	£00.0	0+	07+	
120.0-	210.0-	110.0	200.0	200.0	0+	09+	
010.0+	200.0+	\$00°0+	£00.0+	100.0 +	0+	0 <u>5</u> +	
££0.0+	420.0+	110.0+	600'0+	+0.003	0+	0++	
190.0+	<b>44</b> 0.0+	260.0+	910.0+	900.0+	0 +	0Z+	
280.0+	190.0+	540.0+	40.022	800.0+	0+	0+	
8.0+	£0+	9.0+	<b>†</b> 0+	4.0.2	0+	0+	
0+	0+	0+	200 ^{.0} +	200 [.] 0+	910.0+	09+	
0 +	200.0+	200 [.] 0+	210.0+	+0.025	6£0 [.] 0+	0 <b>5</b> +	
400.0 +	£10.0+	120.0+	+0.033	81-0.0+	940.0+	0++0	
750.0+	60.0+	990'0+	060.0+	+0.124	991.0+	07+	
940.0+	060'0+	+0.114	221.0+	012.0+	+0.293	0+	
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	Data	Symbol	Description
		1	degree of polynomial in the contour representation
		n,	number of points at which upper surface or thickness is specified
		n ₂	number of points at which lower surface or camber is specified $(n_1 \ge n_2 \ge l+1)$
		Xne	x coordinate of section leading edge point axis system of
		Zne	z coordinate of section leading edge point $\int$ table 2
		a _o	$\sqrt{2\rho}$ ; $\rho$ being radius of curvature of thickness distribution at leading edge ( $\rho$ in parts of c)
		extra	value 0 for best polynomial fit irrespective of derivatives
		· .	value 1, best polynomial fit including requirement with regard to derivatives
		thick-	value 0, values on table 2 are coordinates of upper and lower
		ness	surface
			value 1, values on table 2 are coordinates of thickness and camber
		der	value 0 derivatives at trailing edge are determined by programme
			value 1 derivatives at trailing edge are given with respect to the chordline
only if		ab0	derivative of upper surface or thickness at tr. e.
der = 1		ab1	derivative of lower surface or camber at tr. e.
only if		mU	number of points in which derivative of upper surface or thickness
extra = 1			is used (trailing edge point excluded)
•		m1	as m0 but with respect to lower surface or camber
		h	2h+1 points used in procedure dif
		F	weight factor defined in section 3
		j	j=0 programme determines derivatives
\			j=1 derivatives are given on table 3
		с	length of local chord

Notes:  $l \leq 15$ 

ab0 and ab1 both  $\neq 0$ 

# Table 1, input data for least squares approximation (part I).

TABL	E	2
------	---	---

Data	Symbol		Description	
	x _{0,}	z _{0;}	<ul> <li>if thickness = 1 x₀, z₀ refer to thickness distribution; i=0, (1) n₁-1</li> <li>if thickness = 0 x₀, z₀ refer to upper surface</li> </ul>	
	XLi	z _t	<ul> <li>if thickness = 1 x₁, z₁ refer to camber distribution; i = 0, (1) n₂ - 1</li> <li>if thickness = 0 x₁, z₁ refer to lower surface</li> </ul>	

Note:  $n_1 \ge n_2 \ge 2$ .

if thickness value 1,  $x_{1_{n_2-1}} = c$  and  $z_{1_{n_2-1}} = 0$ ;  $z_{0_{n_1-1}} \neq 0$ if thickness value 0,  $z_{0_{n_1-1}} \neq z_{1_{n_2-1}}$ .

Table 2, input data for least squarer approximation (part 1)

TA	BLE	3
----	-----	---

IABLE 3
(only if derivatives are used for determination of contour approximation)

Data	Data S		Description	
	g	z'0,,	if $j=0$ , only g is given if $j=1$ , g and $z'_{0_p}$ are given $(z'_{0_p}$ is the derivative of $z_0$ at $x_g$ ; g is the ordinal number of the x value in table 2)	
	<b>g</b> .	z' _{1,9}	if $j=0$ , only g is given if $j=1$ , g and $z'_{1_g}$ are given $(z'_{1_g}$ is the derivative of $z_1$ at $x_g$ ; g is the ordinal number of the x value in table 2)	

Note: derivative at trailing edge excluded

Table 3, input data for least squares approximation (part 1)

TADIC	1	
TABLE	1	

TABLE 4					
Data	Symbol	Description			
	N	number of points where the pressure on both surfaces have to be calculated			
	x,	points on the chord where the pressure has to be calculated $v = 1(1) \dots N$ (in parts of c)			

Note:  $0 < x_v < 1$ 

Table 4, input data for least squares approximation (part I).

### TABLE 5

Data	Symbol	ol Description		
	Mo	Mach		
	k ₁	choice parameter (see flow diagram fig. 3)		
	magnitude	value of $\alpha_e$ , $C_n$ or $C_l$ ( $\alpha_e$ in degrees)		
	k₂ ↓	choice parameter (see flow diagram fig. 3)		

Table 5, Input data for  $C_p$  calculation on two dimensional aerofoil (part II).

TABLE	6
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Data	Symbol	Description
	tT, <i>t</i> _T	maximum section thickness at tip (dimensional)
	$tR, t_R$	maximum section thickness at root (dimensional)
	СТ, <i>С</i> _т	tip chord (dimensional)
	$CR, C_R$	root chord (dimensional)
	$Lm, A_m$	mean sween (degrees)
	Last, $A_{t_{max}}$	sweep at max thickness position (degrees)
	L1, $A_1$	sweep of trailing edge (degrees)
	L0, A ₀	sweep of leading edge (degrees)
	Cy, c(y)	local chord (dimensional)
	\$	semi-span (dimensional)
	s1, <i>s</i> ′	extended semi-span (dimensional)
	y	distance to root (dimensional)
	Se	wing area (dimensional)
	del T, $\delta_i$	surface slope at tip in span direction (radians)
	del R, $\delta_{r}$	surface slope at root in span direction (radians)
	tentc, 10 $t/c$	local value of 10 $t/c$
	cranks	choice parameter value 1 or 0
only if	del 1, $\delta_1$	strength of thickness crank 1 (radians)
cranks	del 2, $\delta_2$	strength of thickness crank 2 (radians)
value 1	$yc1, y-y_{c_1}$	distance from crank 1 (dimensional)
	yc2, $y - y_{c_2}$	distance from crank 2 (dimensional)
	$CC1, C_{e_1}$	chord at crank 1 (dimensional)
	CC2, $C_{e_2}$	chord at crank 2 (dimensional)
	мо	machnumber
	$k_1$	choice parameter (see flow diagram fig. 4)
	magnitude	value $\alpha_e$ or $C_n$ ( $\alpha_e$ in degrees)
↓	$k_2 \downarrow$	choice parameter (see flow diagram fig. 4)

Table 6 Input data for  $C_p$  calculation on three dimensional wing (part III).

Comparison of calculated pressure distributions on a symmetrical non-lifting 10.37% thick Joukowski section M0=0

x	$z_{t} = 10^{2}$	. ∆z·10 ⁵	C _p , exact	$C_p$ , present method
+ .00001	.047	~ .0	+ .998	+ .998
+ .00008	.140	+ .1	+ .981	+ .981
+ .00022	.233	+ .1	+ .949	+ .949
+ .00042	.326	+ .2	+ .903	+ .903
+ .00086	.465	+ .3	+ .814	+ .815
+ .00146	.604	+ .4	+ .710	+ .710
+ .00221	.742	+ .5	+ .598	+ .598
+ .00311	.880	+ .6	+ .485	+ .487
+ .00417	1.017	+ .6	+ .378	+ .380
+ .00538	1.154	+ .6	+ .279	+ .281
+ .00657	1.289	+ .6	+ .188	+ .190
+ .00827	1.424	+ .5	+ .108	+ .110
+ .00994	1.557	+ .4	+ .036	+ .038
+ .01177	1.690	+ .3	026	025
+ .01375	1.821	+ .1	081	080
+ .01588	1.951	0	- 129	128
+ .01816	2.079	1	171	170
+ .02059	2.206	- ,3	208	207
+ .03079	2.657	· — .5	305	306
+ .04790	3.228	2	379	379
+ .06868	3.740	-1.5	413	412
+ .09272	4.184	+1.2	424	423
+ .12011	4.553	+ .9	422	420
+ .15073	4.840	+ .1	409	409
+ .18422	5.043	7	390	390
+ .22042	5.158	7	366	366
+ .25907	5.187	<b>+</b> .0	339	338
+ .29991	5.131	+ .6	309	308
+ .34266	4.994	+ .5	277	276
+ .38698	4.784	1	244	244
+ .43254	4,508	7	210	210
+ .47899	4.177	4	176	176
+ .52594	3.801	+ .3	143	142
+ .57299	3.393	+ .7	110	109
+ .61973	2.967	+ .3	078	077
+ .66570	2.535	5	047	047
+ .71047	2.111	8	018	018
+ .75357	1.707	<b>4</b> .0	+ .010	+ .010
+ .79456	1.334	+ .9	+ .035	+ .036
+ .83295	1.002	+ .7	+ .059	+ .059
+ .86833	.716	5	+ .080	+ .079
+ .90024	.480	-1.2	+ .098	+ .097
+ .92829	.297	· .4	+ .114	+ .114
+ .95212	.164	+ .2	+ .127	+ .129
+ .97679	.056	+ .2	+ .141	+ .143
+ .98962	.017	+ .3	+ .148	+ .148

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- X _u	$z_u \cdot 10^2$	$\Delta z_{\mu} \cdot 10^{5}$	$C_{p_u}$ exact	$C_{p_u}$ pr. meth	<i>x</i> ₁	$z_l \cdot 10^2$	$\Delta z_i \cdot 10^5$	$C_{p_l}$ exact	C _{pi} pr. meth
+ .00078	+ .387	+ 2.5	+0.787	+ .823	+ .00059	282	- 1.9	+ 0.866	+ .848
+ .00635	+1.091	2.0	+0.254	+ .299	+ .00584	925	+3.2	+0.296	+ .262
+ .01705	1.828	~ 1.1	-0.022	+ .020	+ .01640	1.547	+ 1.6	+0.017	007
+ .03262	2.585	+ .9	-0.173	134	+ .03206	-2.142	-1.5	-0.108	122
+ .05280	3.342	+1.1	-0,270	235 ·	+ .05259	-2.700	- 1.6	-0.173	185
+ .07731	4.082	,2	-0.339	310	+ .07776	3.208	+ .3	-0.210	223
+ .10587	4.785	9	-0.391	365	+ .10729		+1.4	-0.231	243
+ .13817	5.432	4	-0.428	404	+ .14089	-4.033	+ .5	-0.240	251
+ .17390	6.008	+ .5	0.455	433	+ .17823	-4.333	9	-0.242	251
+ .21271	6.500	+ .6	-0.473	452	+ .21895	-4.550	9	-0.237	245
+ .25426	6.884	0. –	-0.482	462	+ .26268	-4.678	+ .2	-0.228	235
+ .29816	7,164	5	-0.483	465	+ .30900	-4.718	+ .9	-0.213	221
+ .34403	7.326	4	-0.477	459	+ .45748	-4.671	+ .4	-0.196	202
+ .39148	7.369	+ .2	- 0.464	446	+ .40763	-4.541	6	~0.175	181
+ .44007	7.291	+ .5	-0.444	427	+ .45898	4.334	7	0.152	157
+ .48938	7.095	+ .1	-0.419	403	+ .51101	-4.059	+ .1	-0.127	~ .132
+ .53898	6.788	4	-0.388	373	+ .56319	-3.727	+ .8	-0.100	105
+ .58842	6.379	4	0.352	338	+ .61497	3.350	+ .4	-0.072	~ .076
+ .63723	5.881	+ .1	-0.313	298	+ .66578	- 2.943	6	-0.043	046
+ .68495	5.310	+ .5	-0.270	255	+ .71505	- 2.519	8	-0.013	016
+ .73111	4.684	+ .3	-0.224	- ,209	+ .76221	2.095	+ .1	+0.017	+ .014
+ .77523	4.023	3	-0.175	161	+ .80665	- 1,684	+ 1.0	+0.048	+ .045
+ .81682	3.349	~ .5	-0.125	111	+ .84783	- 1.299	+ .5	+0.078	+ .077
+ .85540	2.685	2	0.073	058	+ .88516	953	9	+0.109	+ .109
+ .89048	2.053	+ .6	-0.019	004	+ .91810	654	~ 1.4	+0.142	+ .143
+ .92156	1.475	-+ .7	+0.036	. + .052	+ .94615	410	+ .2	+0.175	+ .177
+ .94817	.972	2	+ 0.092	+ .109	+ .96882	224	+ 2.6	+0.213	+ .214
+ .96982	.561	1.6	+0.153	+ .170	+ .98565	096	+ 3.1	+ 0.259	+ .261
+ .98603	.255	-2.0	+0.221	+ .240	+ .99622	023	+ .2	+0.328	+ .341
+ .99629	.066	2	+0.311	+ .340					

TABLE 8 Comparison of calculated pressure distributions on a cambered Karman-Trefftz section (camber 3% and thickness 12%) M0=0.

TABLE 9 Comparison of pressure coefficients on a symmetrical NACA 4 digit type of aerofoil in incompressible flow M0=0

	$\alpha = 0^{\circ}$				α	=6°		
X	C _p exact	$C_p$ eq. (4)	C _{pu} exact	$C_{p_u}$ eq. (4)	C _{pu} eq. (61)	$C_{p_l}$ exact	$C_{p_1}$ eq. (4)	$\frac{C_{p_1}}{eq. (61)}$
0.00005	+ 0.987	+0.981	-1.319	- 1.835	-1.012	-0.663	-0.985	-0.409
0.00010	+0.972	+0.964	-1.474	-1.921	- 1.251	-0.543	-0.773	-0.366
0.00050	+0.871	+0.844	-2.101	-2.439	- 1.836	-0.078	-0.140	+ 0.060
0.00100	+0.747	+0.711	- 2.583	- 2.842	- 2.358	+0.231	+0.207	+0.307
0.00400	+0.226	+ + 0.156	- 3.506	- 3.710	3.322	+0.868	+0.882	+0.892
0.00900	-0.182	-0.256	- 3.506	- 3.654	-3.381	+0.997	+0.995	+0.995
0.02000	- 0.421	-0.468	-2.847	2.904	-2.780	+0.828	+0.812	+0.818
0.04000	-0.419	-0.448	- 2.042	- 2.048	- 1.986	+0.595	+0.581	+0.589
0.08000	-0.321	-0.341	-1.317	1.332	- 1.298	+0.416	+0.397	+0.406
0.12000	~0.243	-0.275	- 1.011	1.016	-0.965	+0.330	+0.317	+0.334
0.18000	~0.222	-0.228	-0.784	-0.781	-0.773	+0.252	+0.245	+ 0.248
0.25000	-0.221	-0.224	-0.668	-0.666	-0.661	+0.179	+0.172	+0.175
0.31000	-0.239	-0.243	-0.625	-0.624	-0.620	+0.117	+0.111	+0.113
0.37000	-0.261	-0.259	-0.599	-0.589	-0.591	+0.061	+ 0.059	+0.060
0.43000	-0.367	-0.262	~0.562	-0.552	-0.547	+0.022	+0.023	+0.027
0.49000	-0.252	-0.249	~0.506	-0.499	0.499	+0.003	+ 0.004	+ 0.004
0.57000	-0.213	-0.210	~0.419	-0.412	-0.415	+0.	+0.001	~0.002
0.64000	-0.164	-0.157	-0.331	-0.320	-0.327	+0.015	+0.019	+0.013
0.70000	-0.106	-0.103	-0.241	-0.236	0.239	+0.045	+0.045	+0.042
0.76000	-0.051	- 0.050	-0.158	-0.156	-0.157	+0.075	+0.072	+0.071
0.82000	- 0.009	-0.006	-0.092	-0.088	0.091	+0.094	+0.094	+0.090
0.88000	+ 0.033	+0.034	+0.026	-0.025	-0.026	+0.115	+0.112	+0.111
0.94000	+ 0.086	· +0.089	+0.096	+0.055	+0.053	+0.179	+0.142	+0.140

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Fig. 2 Flow diagram of the input of part I.

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 $x_{0_i} \quad \Delta z_{0_i} = \left(\frac{\mathrm{d} z_0}{\mathrm{d} x_0}\right)_i \quad \left(\frac{\mathrm{d}^2 z_0}{\mathrm{d} x_0^2}\right)_i \quad x_{1_i} = -\Delta z_{1_i} \quad \left(\frac{\mathrm{d} z_1}{\mathrm{d} x_1}\right)_i \quad \left(\frac{\mathrm{d}^2 z_1}{\mathrm{d} x_1^2}\right)_i$ 

 $i = 1(1) \dots n_1 - 1$  for index 0  $i=1(1)...n_2-1$  for index 1

$$\begin{aligned} \Delta \alpha & c_e \\ t_i & i = 1(1) \dots l \\ l \\ s_i & i = 1(1) \dots l \end{aligned}$$

$$v = 1(1) \dots N$$

$$x_{\mathbf{v}} = z_{\mathbf{u}}(x_{\mathbf{v}}) = z_{\mathbf{l}}(x_{\mathbf{v}}) = I_{1}(x_{\mathbf{v}}) = \left(\frac{\mathrm{d}z_{\mathbf{t}}}{\mathrm{d}x}\right)_{x_{\mathbf{v}}} = I_{2}(x_{\mathbf{v}}) \left(\frac{\mathrm{d}z_{\mathbf{s}}}{\mathrm{d}x}\right)_{x_{\mathbf{v}}}$$

Fig. 5 Output of programme part I (for notation see list of symbols).

#### Notes

N

If thickness and camber have been given on input table 1 then in the first columns index 0 refers to the thickness distribution and index 1 refers to the camber distribution.

If upper and lower surface coordinates have been given on table 1 then index 0 and 1 refer to upper and lower surface respectively. Of the last row of data presented above for  $x_y = 1$  only the values of  $z_{io} z_i$  and  $\frac{dz_i}{dx}$  are used in further computation. The values of the

integrals in that row are meaningless when  $c = c_e$ .

The first row of the last block refers to the maximum thickness position except for the values of  $z_u$  and  $z_l$  which belong to  $x_v = 0$ .

у	s – y	c(y)	C _R	
Cτ	<b>s</b> '	Å,	$A_{t_{\max}}$	
x _{fmax}				
$\Lambda_1$	Δ	K _{2a}	K _{3a}	
$\lambda_a$	Ha	n _a	n ₀ ,	·
$M_0$	α _e	C _n	C _m	C,
<i>x</i> _v	$C_{p_{\mu}}(x_{\nu})$		$C_{p_l}(x_v)$	$\nu=1(1)\ldotsN$







EXPERIMENT REF. 9  $C_n = 0.425$  = 4.04CALCULATION  $C_n = 9$ 

C_n = 0.425 CALCULA

CALCUL Mg=0.18

0

Fig. 7 Calculated pressure distribution on a 10% thick RAE 101 section.



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c,











 $\alpha = 0, M_0 = 0.$ Fig. 9 Calculated pressure distributions on section NACA 65 A 006,



digit type section. Fig. 11 Calculated and measured pressures on a 10% thick NACA 4



12 wing in incompressible flow  $(\alpha = 0)$ . Fig. 12 Calculated pressure distributed on the symmetrical Warren



Fig. 13 Measured and calculated pressure distributions on a symmetrical wing mounted on a circular body,  $\alpha = 0$  (ref. 10).



Fig. 14 Calculated and measured pressure distribution on a wing in presence of a body (low wing configuration).

# REPORT NLR-TR. W. 31

1

# The propagation of elastic waves in circular cylindrical shells of sandwich-type

by

J. van der Vooren

# Summary

The differential equations are derived, accounting for the effects of shear and rotatory inertia. Numerical solutions are given for a shell which is representative for an actual aircraft.

For low-, respectively high frequencies, the theory is analytically evaluated as far as possible in order to obtain simplified expressions for the calculation of propagation speeds. Their validity is checked numerically.

This investigation has been performed under contract with the Netherlands Aircraft Development Board (NIV).

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A	ppendix B: Derivation of inertia forces and			shell in the xz- and the $\varphi z$ -plane respec-
	moments	27		tively

$\gamma_{\varphi z}(z)$	transverse shear angle of the shell in the $\varphi z$ -plane for arbitrary z-value	m _c m	mass of the core per unit area total mass of the shell per unit area, see
С	thickness of sandwich shell minus the thickness of one face	I	eq. (B.6) = $\frac{1}{12}m_cc^2 + \frac{1}{2}(m_f + m_g)c^2$ , see eq. (B.7)
h a	thickness of one face radius of the circular cylindrical shell with respect to the neutral plane	$\begin{array}{c} X, Y, Z \\ \mathbf{m}_x, \mathbf{m}_{\varphi} \end{array}$	inertia forces and moments, defined in fig. 4
v	Poisson's ratio of the material of the faces	ω ω ₀	= $2\pi f$ , angular frequency frequency at which there occurs a two-
Ε	Youngs modulus of the material of the faces	ŕ	dimensional resonance
$G_{xz}, G_{\varphi z}$	shear moduli of the honeycomb core in the $xz$ - and the $\varphi z$ -plane respectively	λ C =	half wave length propagation speed of elastic waves
A	$=\frac{2Eh}{1-v^2}$ , normal stiffness of the sand-	$C_L$	propagation speed of nearly longitudinal waves in the cylindrical shell in case the
В	wich per unit length = $\frac{Ec^2 h}{2(1-v^2)}$ , bending stiffness of the		contraction in the direction of the wave front is not restrained. This motion only exists at very low frequencies
$S_{xz}, S_{\varphi z}$	sandwich per unit length = $G_{xz}c$ , $G_{\varphi z}c$ , shear stiffness of the core in the xz- and $\varphi z$ -plane respectively	$\bar{C}_L$	propagation speed of longitudinal waves in a flat panel in case the contraction in the direction of the wave front is fully
$\left.\begin{array}{c}Q_{x}, Q_{\varphi}\\N_{x}, N_{\varphi}\\T_{x\varphi}, T_{\varphi x}\end{array}\right\}$	forces and moments caused by elastic	n	restrained number of waves in circumferential direction
$ \begin{array}{c} M_x, M_\varphi \\ M_{x\varphi}, M_{\varphi x} \end{array} $	enects, defined in fig. 4	$[a_{ij}]$	square matrix with elements $a_{ij}$ , defined in table 1
m _f m _g	mass per unit area of one face mass per unit area of one glue layer between face and core	$egin{aligned}  a_{ij}  \ O \ O(q) \end{aligned}$	determinant value of square matrix $[a_{ij}]$ zero column matrix quantity of order q.

#### **1** Introduction

In studies concerning the sound transmitting properties of circular cylindrical shells the radial motions which propagate in the direction of the shell axis are of primary interest (ref. 4). So-called coincidence occurs when the velocity of the sound waves along the cylinder equals the propagation speed of free travelling elastic flexural waves in its wall. The transmission loss then shows a strong dip and the level of the transmitted sound can be almost as high as that of the incident sound. In order to know whether or not coincidences are to be expected it is essential that plots of the propagation speed versus frequency are available for the various partial waves of different circumferential shape.

The present paper discusses the latter subject in case of sandwich-type shells.

### 2 Distortions

For sandwiches where the stiffnesses of the core in the plane of the shell are very small with respect to those of the faces, Libove and Batdorf (ref. 1) state that the assumption of plane cross sections remaining plane in the deformed state, as far as the core is concerned, is practically correct. Because the condition mentioned holds for honeycomb cores, the axial- and circumferential displacements  $u_z$  and  $v_z$  (figs 2, 3) in the core may be given by the expressions

$$u_z = u + \beta_{xz} z \tag{2.1}$$

and

$$v_z = v + \beta_{\varphi z} z . \tag{2.2}$$

In (2.1), (2.2),  $\beta_{xz}$  and  $\beta_{\varphi z}$  respectively represent the change in slope of the normal to the faces in the xz- and the  $\varphi z$ -plane and u, v (figs 1, 2, 3) are the displacements in the neutral plane (z=0). Since the faces are thin compared with the thickness of the shell they are assumed to act as membranes and the validity of eqs (2.1), (2.2) may be extended over them.





Fig. 1. Co-ordinate system and positive directions of displacements.

Fig. 2. Cross-section of shell in  $\varphi$ -z plane.



Fig. 3. Cross-section of shell in x-z plane.

Deflections w in radial direction will be taken independent of z. The shear angle of the core in the xz-plane may now be found from the relation

$$\gamma_{xz} = \frac{\partial u_z}{\partial z} + \frac{\partial w}{\partial x} = \beta_{xz} + \frac{\partial w}{\partial x}$$
(2.3)

and is apparently constant throughout the thickness.

In the  $\varphi z$ -plane the shear angle reads

$$\gamma_{\varphi z}(z) = \frac{\partial v_z}{\partial z} + \frac{1}{a+z} \frac{\partial w}{\partial \varphi} - \frac{v_z}{a+z}$$
$$= \frac{\beta_{\varphi z}}{\frac{1}{a+z}} + \frac{\partial w}{a \partial \varphi} - \frac{v}{a}}{1 + \frac{z}{a}}.$$
(2.4)

For large values of the cylinder radius "a", which is the case for the present analysis, eq. (2.4) may readily be simplified into

$$\gamma_{\varphi z}(z) = \left(\beta_{\varphi z} + \frac{\partial w}{a \partial \varphi} - \frac{v}{a}\right) \left(1 - \frac{z}{a}\right),\tag{2.5}$$

which differs only little from the average value

$$\gamma_{\varphi z} = \beta_{\varphi z} + \frac{\partial w}{a \partial \varphi} - \frac{v}{a} \,. \tag{2.6}$$

Elimination of  $\beta_{xz}$  and  $\beta_{\varphi z}$  from (2.1), (2.2), using the expressions (2.3), (2.6), then yields

$$u_z = u + \left(\gamma_{xz} - \frac{\partial w}{\partial x}\right)z, \qquad (2.7)$$

$$v_z = v + \left(\gamma_{\varphi z} - \frac{\partial w}{a \partial \varphi} + \frac{v}{a}\right) z .$$
(2.8)

#### 3. Loading of a shell element

In fig. 4 the loading of an infinitesimal shell element is shown.

The forces and moments caused by elastic effects are defined per unit length, while mass forces (stemming from the effect of inertia) will be given per unit area, both with reference to the neutral plane.



Fig. 4. Loading of an infinitesimal shell element.

# 3.1 Forces and moments caused by elastic effects

Starting from the assumptions mentioned in section 2, appendix A gives the derivation of expressions for the loading caused by elastic effects. An additional assumption is that the core carries transverse shear stresses only. Resultant normal forces  $N_x$ ,  $N_{\varphi}$  and shear forces in the  $\varphi x$ -plane are obtained in the form.

$$N_{x} = A \left( \frac{\partial u}{\partial x} + v \frac{\partial v}{a \partial \varphi} + v \frac{w}{a} - \frac{c^{2}}{4a} \frac{\partial^{2} w}{\partial x^{2}} + \frac{c^{2}}{4a} \frac{\partial \gamma_{xz}}{\partial x} \right),$$
(3.1)

$$N_{\varphi} = A \left( v \frac{\partial u}{\partial x} + \frac{\partial v}{a \partial \varphi} + \frac{w}{a} + \frac{c^2}{4a} \frac{\partial^2 w}{a^2 \partial \varphi^2} - \frac{c^2}{4a} \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} \right), \tag{3.2}$$

$$T_{x\varphi} = A \left( \frac{1-\nu}{2} \frac{\partial u}{a \partial \varphi} + \frac{1-\nu}{2} \frac{\partial v}{\partial x} - \frac{1-\nu}{2} \frac{c^2}{4a} \frac{\partial^2 w}{a \partial x \partial \varphi} + \frac{1-\nu}{2} \frac{c^2}{4a} \frac{\partial \gamma_{\varphi z}}{\partial x} \right), \tag{3.3}$$

$$T_{\varphi x} = A \left( \frac{1-v}{2} \frac{\partial u}{a \partial \varphi} + \frac{1-v}{2} \frac{\partial v}{\partial x} + \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial^2 w}{a \partial x \partial \varphi} - \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial \gamma_{xz}}{a \partial \varphi} \right),$$
(3.4)

where

$$A = \frac{2Eh}{1 - v^2}.$$
 (3.5)

Transverse shear forces may under the present assumptions be assigned to transverse shear of the core only, thus

$$Q_x = S_{xz} \gamma_{xz} \tag{3.6}$$

and

$$Q_{\varphi} = S_{\varphi z} \gamma_{\varphi z} , \qquad (3.7)$$

where the shear stiffnesses  $S_{xz}$  and  $S_{\varphi z}$  may be related to the corresponding shear moduli  $G_{xz}$  and  $G_{\varphi z}$  by the simple expressions

$$S_{xz} = G_{xz} c , \qquad (3.8)$$

$$S_{\varphi z} = G_{\varphi z} c . \tag{3.9}$$

The resultant bending- and twisting moments read

$$M_{x} = B \left\{ -\frac{\partial^{2} w}{\partial x^{2}} - v \frac{\partial^{2} w}{a^{2} \partial \varphi^{2}} + \frac{\partial \gamma_{xz}}{\partial x} + \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} + \frac{1}{a} \frac{\partial u}{\partial x} + \frac{v}{a} \frac{\partial v}{a \partial \varphi} \right\},$$
(3.10)

$$M_{\varphi} = B \left\{ -v \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{a^2 \partial \varphi^2} + v \frac{\partial \gamma_{xz}}{\partial x} + \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} - \frac{w}{a^2} \right\},$$
(3.11)

$$M_{x\varphi} = B\left\{-\frac{1-\nu}{2}2\frac{\partial^2 w}{a\partial x\partial \varphi} + \frac{1-\nu}{2}\frac{\partial \gamma_{xz}}{a\partial \varphi} + \frac{1-\nu}{2}\frac{\partial \gamma_{\varphi z}}{\partial x} + \frac{1-\nu}{2}\frac{2}{a}\frac{\partial v}{\partial x}\right\},\tag{3.12}$$

$$M_{\varphi x} = B \left\{ -\frac{1-\nu}{2} 2 \frac{\partial^2 w}{a \partial x \partial \varphi} + \frac{1-\nu}{2} \frac{\partial y_{xz}}{a \partial \varphi} + \frac{1-\nu}{2} \frac{\partial y_{\varphi z}}{\partial x} - \frac{1-\nu}{2} \frac{1}{a} \frac{\partial u}{a \partial \varphi} + \frac{1-\nu}{2} \frac{1}{a} \frac{\partial v}{\partial x} \right\},$$
(3.13)

with

$$B = \frac{Ec^2 h}{2(1-v^2)}.$$
(3.14)

#### 3.2 Inertia forces and moments

Appendix B gives the derivation of expressions for the inertia loading. Accounting for the assumptions mentioned in section 2, the inertia forces in x-,  $\varphi$ - and z-direction read respectively

$$X = -m\ddot{u} - \frac{I}{a} \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right), \tag{3.15}$$

$$Y = -m\ddot{v} - \frac{I}{a} \left( \ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{a \partial \varphi} + \frac{\ddot{v}}{a} \right), \tag{3.16}$$

$$Z = -m\ddot{w}, \qquad (3.17)$$

where m (see eq. (B.6)) is the mass of the shell and I is a quantity defined in eq. (B.7) which equals the moment of inertia with respect to rotation. Both quantities are defined per unit area.

Rotatory inertia effects are accounted for by the moments

$$\mathbf{m}_{x} = I\left(\ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{a \, \partial \varphi} + 2 \, \frac{\ddot{v}}{a}\right) \tag{3.18}$$

and

$$\mathbf{m}_{\varphi} = -I\left(\ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} + \frac{\ddot{u}}{a}\right). \tag{3.19}$$

# **4** Equilibrium conditions

The equilibrium of an infinitesimal shell element (see fig. 4) yields the conditions

$$\frac{\partial N_x}{\partial x} + \frac{\partial T_{\varphi x}}{a \partial \varphi} + X = 0, \qquad (4.1)$$

$$\frac{\partial N_{\varphi}}{\partial \partial \varphi} + \frac{\partial T_{x\varphi}}{\partial x} + \frac{Q_{\varphi}}{a} + Y = 0, \qquad (4.2)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_{\varphi}}{a \partial \varphi} - \frac{N_{\varphi}}{a} + Z = 0, \qquad (4.3)$$

$$\frac{\partial M_{\varphi}}{\partial \partial \varphi} + \frac{\partial M_{x\varphi}}{\partial x} - Q_{\varphi} - \mathbf{m}_{x} = 0 , \qquad (4.4)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{\varphi x}}{a \, \partial \varphi} - Q_x + \mathbf{m}_{\varphi} = 0 , \qquad (4.5)$$

$$T_{x\phi} - T_{\phi x} - \frac{M_{\phi x}}{a} = 0.$$
 (4.6)

From these six equations the last requirement (4.6) is always satisfied and therefore of no further importance, see appendix A.

# 5 Propagation of axisymmetric waves

When the displacements are axisymmetric all derivatives with respect to  $\varphi$  vanish. Furthermore it may be shown that no coupling exists between the set of distortions u, w,  $\gamma_{xz}$  and the set of distortions v,  $\gamma_{\varphi z}$ . The latter set is of little interest from an acoustical point of view because radial deflections do not occur, but for sake of completeness they will be briefly discussed in appendix C. In this sections, however, attention will only be paid to waves associated with u, w and  $\gamma_{xz}$ .

The expressions for the loading caused by elastic effects and the inertia loading (section 3) then reduce to

$$N_x = A \left\{ \frac{\partial u}{\partial x} + v \frac{w}{a} - \frac{c^2}{4a} \frac{\partial^2 w}{\partial x^2} + \frac{c^2}{4a} \frac{\partial \gamma_{xz}}{\partial x} \right\},$$
(5.1)

$$N_{\varphi} = A \left\{ v \, \frac{\partial u}{\partial x} + \frac{w}{a} \right\},\tag{5.2}$$

$$T_{x\varphi} = T_{\varphi x} = 0, \qquad (5.3)$$

$$Q_x = S_{xz} \gamma_{xz} , \qquad (5.4)$$

$$Q_{\varphi} = 0 , \qquad (5.5)$$

$$M_{x} = B\left\{-\frac{\partial^{2} w}{\partial x^{2}} + \frac{\partial \gamma_{xz}}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial x}\right\},$$
(5.6)

$$M_{\varphi} = B \left\{ -v \frac{\partial^2 w}{\partial x^2} + v \frac{\partial \gamma_{xz}}{\partial x} - \frac{w}{a^2} \right\},$$
(5.7)

$$M_{x\phi} = M_{\phi x} = 0 , \qquad (5.8)$$

$$X = -m\ddot{u} - \frac{I}{a} \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right), \tag{5.9}$$

$$Y = 0, \qquad (5.10)$$

$$Z = -m\ddot{w}, \qquad (5.11)$$

$$\mathbf{m}_{\mathbf{x}} = 0 , \qquad (5.12)$$

$$\mathbf{m}_{\varphi} = -I\left(\ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} + \frac{\ddot{u}}{a}\right),\tag{5.13}$$

while the equilibrium conditions (section 4) become

$$\frac{\partial N_x}{\partial x} + X = 0, \qquad (5.14)$$

$$\frac{\partial Q_x}{\partial x} - \frac{N_{\varphi}}{a} + Z = 0, \qquad (5.15)$$

$$\frac{\partial M_x}{\partial x} - Q_x + \mathbf{m}_{\varphi} = 0.$$
(5.16)

Differential equations for the displacements u, w and the transverse shear angle  $\gamma_{xx}$  may now be obtained by substitution of (5.1), (5.2), (5.4), (5.6), (5.9), (5.11) and (5.13) into (5.14)-(5.16). They read

$$\left(\frac{\partial^2}{\partial x^2} - \frac{m}{A}\frac{\partial^2}{\partial t^2}\right)u + \left(\frac{v}{a}\frac{\partial}{\partial x} - \frac{c^2}{4a}\frac{\partial^3}{\partial x^3} + \frac{1}{a}\frac{I}{A}\frac{\partial^3}{\partial x\partial t^2}\right)w + \left(\frac{c^2}{4a}\frac{\partial^2}{\partial x^2} - \frac{1}{a}\frac{I}{A}\frac{\partial^2}{\partial t^2}\right)\gamma_{xz} = 0, \quad (5.17)$$

$$\left(-\frac{v}{a}\frac{\partial}{\partial x}\right)u + \left(-\frac{1}{a^2} - \frac{m}{A}\frac{\partial^2}{\partial t^2}\right)w + \left(\frac{S_{xz}}{A}\frac{\partial}{\partial x}\right)\gamma_{xz} = 0, \qquad (5.18)$$

$$\left(\frac{1}{a}\frac{\partial^2}{\partial x^2} - \frac{1}{a}\frac{I}{B}\frac{\partial^2}{\partial t^2}\right)u + \left(-\frac{\partial^3}{\partial x^3} + \frac{I}{B}\frac{\partial^3}{\partial x\partial t^2}\right)w + \left(\frac{\partial^2}{\partial x^2} - \frac{S_{xz}}{B} - \frac{I}{B}\frac{\partial^2}{\partial t^2}\right)\gamma_{xz} = 0.$$
(5.19)

By assuming distortions in the form

$$u = U \cos\left(\frac{\omega}{C_E} x - \omega t\right),$$

$$w = W \sin\left(\frac{\omega}{C_E} x - \omega t\right),$$

$$\gamma_{xz} = \Gamma_{xz} \cos\left(\frac{\omega}{C_E} x - \omega t\right),$$
(5.20)

where  $C_E$  is the propagation speed of elastic waves and  $\omega$  is the angular frequency, the differential equations yield the linear system

$$\left\{-\left(\frac{\omega}{C_E}\right)^2 + \frac{m}{A}\omega^2\right\}U + \left\{\frac{\nu}{a}\left(\frac{\omega}{C_E}\right) + \frac{c^2}{4a}\left(\frac{\omega}{C_E}\right)^3 - \frac{1}{a}\frac{I}{A}\left(\frac{\omega}{C_E}\right)\omega^2\right\}W + \left\{-\frac{c^2}{4a}\left(\frac{\omega}{C_E}\right)^2 + \frac{1}{a}\frac{I}{A}\omega^2\right\}\Gamma_{xz} = 0, \quad (5.21)$$

$$\left\{\frac{v}{a}\left(\frac{\omega}{C_{E}}\right)\right\}U + \left\{-\frac{1}{a^{2}} + \frac{m}{A}\omega^{2}\right\}W + \left\{-\frac{S_{xz}}{A}\left(\frac{\omega}{C_{E}}\right)\right\}\Gamma_{xz} = 0, \qquad (5.22)$$

$$\left\{-\frac{1}{a}\left(\frac{\omega}{C_{E}}\right)^{2}+\frac{1}{a}\frac{I}{B}\omega^{2}\right\}U+\left\{\left(\frac{\omega}{C_{E}}\right)^{3}-\frac{I}{B}\left(\frac{\omega}{C_{E}}\right)\omega^{2}\right\}W+\left\{-\left(\frac{\omega}{C_{E}}\right)^{2}-\frac{S_{xz}}{B}+\frac{I}{B}\omega^{2}\right\}\Gamma_{xz}=0.$$
(5.23)

The above system only has a non-zero solution if the determinant of the coefficient-matrix (the unknowns being U, W and  $\Gamma_{xz}$ ) is zero.

In order to explain the behaviour of axisymmetric waves this requirement is evaluated into

$$\frac{1}{a^{2}} \frac{S_{xz}}{B} \left( \frac{1-v^{2}}{C_{E}^{2}} - \frac{m}{A} \right) + \left\{ \frac{1}{a^{2}} \frac{1}{C_{E}^{2}} \left( \frac{1-v^{2}-v\frac{S_{xz}}{A} - v\frac{c^{2}}{4}\frac{S_{xz}}{B}}{C_{E}^{2}} - \frac{m}{A} \right) - \frac{1}{a^{2}} \frac{I}{B} \left( \frac{1-v^{2}}{C_{E}^{2}} - \frac{m}{A} \right) - \frac{m}{A} \frac{S_{xz}}{B} \left( \frac{1}{C_{E}^{2}} - \frac{m}{A} \right) \right\} \omega^{2} + \frac{S_{xz}}{A} \left( \frac{1}{C_{E}^{2}} - \frac{I}{B} \right) \left( \frac{1}{C_{E}^{2}} - \frac{m}{S_{xz}} \right) \left( \frac{1}{C_{E}^{2}} - \frac{m}{A} \right) \omega^{4} = 0.$$
(5.24)

To make the above expression manageable from an analytical point of view, the coefficients of each power of  $\frac{1}{C_E}$  and (or)  $\omega$  were simplified during the derivation by neglecting all terms that are small of order  $\frac{c^2}{a^2}$ . When  $C_E$  tends to infinity the characteristic equation (5.24) reduces to

$$\frac{m^2}{A^2} \frac{I}{B} \left( \omega^2 - \frac{A}{ma^2} \right) \left( \omega^2 - \frac{S_{xz}}{I} \right) = 0$$
(5.25)

and hence  $C_E$  has asymptotic values at

$$\omega_{04} = \frac{1}{a} \sqrt{\frac{A}{m}}$$
(5.26)

and

$$\omega_{05} = \sqrt{\frac{S_{xz}}{I}}.$$
(5.27)

To (5.26) belongs an eigenvector U=0, W,  $\Gamma_{xz}=0$ . This mode represents radial expansion independent of the x(length)-direction of the cylinder.

To (5.27) belongs an eigenvector U=0, W=0,  $\Gamma_{xz}$ . This mode represents a shear vibration in x-direction.

It be remarked that  $C_E = \infty$  means an infinite wavelength, i.e. all points in x-direction are in phase, which indicates a two-dimensional resonance.

#### 5.1 Low-frequency behaviour

When the frequency is low, i.e. if the fourth power term in (5.24) may be neglected, the characteristic equation is approximated by

$$\frac{1}{a^2} \frac{S_{xz}}{B} \left( \frac{1 - v^2}{C_E^2} - \frac{m}{A} \right) + \left\{ \frac{1}{a^2} \frac{1}{C_E^2} \left( \frac{1 - v^2 - v \frac{S_{xz}}{A} - v \frac{c^2}{4} \frac{S_{xz}}{B}}{C_E^2} - \frac{m}{A} \right) - \frac{1}{a^2} \frac{I}{B} \left( \frac{1 - v^2}{C_E^2} - \frac{m}{A} \right) - \frac{m}{A} \frac{S_{xz}}{B} \left( \frac{1}{C_E^2} - \frac{m}{A} \right) \right\} \omega^2 = 0.$$
(5.28)

Rearrangement of this equation in powers of  $\frac{1}{C_E}$  gives

$$\frac{1-\nu^2-\nu\frac{S_{xz}}{A}-\nu\frac{c^2}{4}\frac{S_{xz}}{B}}{a^2}\omega^2\frac{1}{C_E^4} + (1-\nu^2)\left\{\frac{1}{a^2}\frac{S_{xz}}{B} - \left(\frac{1}{a^2}\frac{m}{A(1-\nu^2)} + \frac{1}{a^2}\frac{I}{B} + \frac{m}{A(1-\nu^2)}\frac{S_{xz}}{B}\right)\omega^2\right\}\frac{1}{C_E^2} + \frac{m}{A}\left\{\frac{1}{a^2}\frac{S_{xz}}{B} - \left(\frac{1}{a^2}\frac{I}{B} + \frac{m}{A}\frac{S_{xz}}{B}\right)\omega^2\right\} = 0.$$
(5.29)

Since the coefficient of  $\frac{1}{C_E^4}$  is small (of order  $\omega^2$ ) the only solution for  $C_E^2$  that is positive for low frequencies may be approximated by

$$C_E^2 = \frac{(1-v^2)A}{m} \frac{\frac{1}{a^2} \frac{S_{xz}}{B} - \left(\frac{1}{a^2} \frac{m}{A(1-v^2)} + \frac{1}{a^2} \frac{I}{B} + \frac{m}{A(1-v^2)} \frac{S_{xz}}{B}\right)\omega^2}{\frac{1}{a^2} \frac{S_{xz}}{B} - \left(\frac{1}{a^2} \frac{I}{B} + \frac{m}{A} \frac{S_{xz}}{B}\right)\omega^2}.$$
(5.30)

When  $\omega \rightarrow 0$ ,  $C_E$  tends to

$$C_E = C_L = \sqrt{\frac{A(1 - v^2)}{m}} \,. \tag{5.31}$$

The displacements are mainly in axial direction (u). Hence the wave motion is very nearly longitudinal. Moreover, it can be verified that the contraction in the direction of the wave front is not restrained. Obviously, the same result for  $C_E$  could directly be obtained from eq. (5.28) by putting  $\omega = 0$ .

From the eqs (3.5), (3.14), (B.6) and (B.7) it follows that  $B = O(Ac^2)$ ,  $I = O(mc^2)$ . Hence, the quantities  $\frac{1}{a^2} \frac{m}{A(1-v^2)}$  and  $\frac{1}{a^2} \frac{I}{B}$  in the numerator of (5.30) are of the same order.

Furthermore, if

$$\frac{S_{xz}}{B} = O\left(\frac{S_{xz}}{A} \frac{1}{c^2}\right) \gg \frac{1}{a^2},$$
(5.32)

which certainly is the case for the sandwich presently investigated, the expression (5.30) for  $C_E^2$  can be further simplified into

$$C_E^2 = \frac{(1-v^2)A}{m} \frac{\frac{1}{a^2} - \frac{m}{A(1-v^2)}\omega^2}{\frac{1}{a^2} - \frac{m}{A}\omega^2}$$
(5.33)

and apparently the low frequency behaviour of axisymmetric waves is practically independent of the shear stiffness  $S_{xz}$ .

Both in (5.30) and (5.33) the coefficient of  $\omega^2$  in the numerator is only little greater than the corresponding one in the denominator. Therefore, the value of  $C_E$  will decrease only very slowly with increasing frequency as long as

 $\omega^2 \ll \frac{1}{a} \sqrt{\frac{A(1-v^2)}{m}}$ . However, since in (5.33)

$$C_E^2 \sim 0$$
 for  $\omega = \frac{1}{a} \sqrt{\frac{A(1-v^2)}{m}}$ 

and

$$C_E^2 \sim \frac{1}{0} \text{ for } \omega = \frac{1}{a} \sqrt{\frac{A}{m}}$$

which frequencies differ only little, the propagation speed  $C_E$  will show a sudden drop at approximately (see eq. (5.31))

$$\omega = \frac{1}{a} \sqrt{\frac{A(1-v^2)}{m}} = \frac{C_L}{a}.$$
 (5.34)

In ref. 4 the environment of this frequency is called the transition region.

#### 5.2 High-frequency behaviour

⁻ In case of high frequencies, i.e. if the first term of the characteristic equation (5.24) is negligible, there may be put

$$\begin{cases} \frac{1}{a^2} \frac{1}{C_E^2} \left( \frac{1 - v^2 - v \frac{S_{xz}}{A} - v \frac{c^2}{4} \frac{S_{xz}}{B}}{C_E^2} - \frac{m}{A} \right) - \frac{1}{a^2} \frac{I}{B} \left( \frac{1 - v^2}{C_E^2} - \frac{m}{A} \right) - \frac{m}{A} \frac{S_{xz}}{B} \left( \frac{1}{C_E^2} - \frac{m}{A} \right) \right\} + \frac{S_{xz}}{A} \left( \frac{1}{C_E^2} - \frac{I}{B} \right) \left( \frac{1}{C_E^2} - \frac{m}{S_{xz}} \right) \left( \frac{1}{C_E^2} - \frac{m}{A} \right) \omega^2 = 0.$$
(5.35)

When the expression (5.32) holds and the wavelength, which is proportional to  $C_E/\omega$ , is small with respect to the cylinder radius "a", the expression (5.35) may further be simplified into

$$\left(\frac{1}{C_E^2} - \frac{m}{A}\right) \left\{ \left(\frac{1}{C_E^2} - \frac{I}{B}\right) \left(\frac{1}{C_E^2} - \frac{m}{S_{xz}}\right) \omega^2 - \frac{m}{B} \right\} = 0.$$
(5.36)

Of course this expression is not valid when  $C_E$  tends to infinity because then the wavelength would also become infinite.

One solution for  $C_E$  now follows directly, i.e.

$$C_{E_4} = \overline{C}_L = \sqrt{\frac{m}{A}}, \qquad (5.37)$$

which equals the propagation speed of longitudinal waves in a flat panel in case the contraction in the direction of the wave front is fully restrained.

The other two solutions are to be obtained from

$$\left(\frac{1}{C_E^2} - \frac{I}{B}\right) \left(\frac{1}{C_E^2} - \frac{m}{S_{xz}}\right) \omega^2 - \frac{m}{B} = 0, \qquad (5.38)$$

which is the characteristic equation for the flat panel when the displacements u in the neutral plane are taken zero. After rearrangement in powers of  $1/C_E$ ,

$$\frac{1}{C_E^4}\omega^2 - \frac{1}{C_E^2} \left(\frac{m}{S_{xz}} + \frac{I}{B}\right) \omega^2 + \left(\frac{m}{S_{xz}}\frac{I}{B}\omega^2 - \frac{m}{B}\right) = 0,$$
(5.39)

the further solutions for  $C_E$  may be expressed as

$$C_{E_1} = \left\{ \frac{1}{2} \left( \frac{m}{S_{xz}} + \frac{I}{B} \right) + \frac{1}{2} \sqrt{\left( \frac{m}{S_{xz}} + \frac{I}{B} \right)^2 - 4 \left( \frac{m}{S_{xz}} \frac{I}{B} - \frac{m}{B\omega^2} \right)} \right\}^{-\frac{1}{2}}$$
(5.40)

and

$$C_{E_{5}} = \left\{ \frac{1}{2} \left( \frac{m}{S_{xz}} + \frac{I}{B} \right) - \frac{1}{2} \sqrt{\left( \frac{m}{S_{xz}} + \frac{I}{B} \right)^{2} - 4 \left( \frac{m}{S_{xz}} \frac{I}{B} - \frac{m}{B\omega^{2}} \right)} \right\}^{-\frac{1}{2}}.$$
 (5.41)

¹ In this region of values of  $\omega$  the neglected terms in  $\omega^4$  (compare (5.24) and (5.28)) play their role. They ensure that  $C_E^2$  remains positive.

The solution for  $C_{E_1}$  in (5.40) is always real, which can be seen by rewriting in the form

$$C_{E_1} = \left\{ \frac{1}{2} \left( \frac{m}{S_{xz}} + \frac{I}{B} \right) + \frac{1}{2} \sqrt{\left( \frac{m}{S_{xz}} - \frac{I}{B} \right)^2 + \frac{4m}{B\omega^2}} \right\}^{-\frac{1}{2}}$$
(5.42)

When  $\omega \rightarrow \infty$ ,  $C_{E_1}$  tends to

$$C_{E_1} = \sqrt{\frac{S_{xz}}{m}} \tag{5.43}$$

and the motion of the shell is due to transverse shear  $\gamma_{xz}$ .

Furthermore it may be observed that  $C_{E_5}$ , according to (5.41), becomes infinite at

$$\omega_{05}^* = \sqrt{\frac{S_{xz}}{I}} \tag{5.44}$$

and is only real for frequencies above this value. Notwithstanding the fact that the expression (5.41) as a high-frequency approximation for the cylindrical shell is no longer valid in the vicinity of  $\omega_{05}$  (large values of  $C_E$  imply large values of the wavelength) the coincidence of  $\omega_{05}^*$  and the asymptote  $\omega_{05}$  of (5.27) indicates to which branch of  $C_E$  this asymptote belongs.

Rewriting of (5.41) in the form

$$C_{E_{5}} = \left\{ \frac{1}{2} \left( \frac{m}{S_{xz}} + \frac{I}{B} \right) - \frac{1}{2} \sqrt{\left( \frac{m}{S_{xz}} - \frac{I}{B} \right)^{2} + \frac{4m}{B\omega^{2}}} \right\}^{\frac{1}{2}}$$
(5.45)

finally shows that  $C_{E_5}$  has the limiting value

$$C_{E_5} = \sqrt{\frac{B}{I}} \tag{5.46}$$

when  $\omega \rightarrow \infty$ . In this case the motion of the shell is again due to transverse shear  $\gamma_{xz}$  only, compare eq. (5.43). This same phenomenon occurs with non-axisymmetric waves and is explained in the next section.

#### 6 Propagation of non-axisymmetric waves

In this case, substitution of the expressions for the elastic- and inertia-loading (sections 3.1 and 3.2) into the equilibrium conditions (4.1), (4.5) gives the differential equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2} - \frac{m}{A}\frac{\partial^2}{\partial t^2}\right)u + \left(\frac{1+\nu}{2}\frac{1}{a}\frac{\partial^2}{\partial x\partial \varphi}\right)v + \left(\frac{\nu}{a}\frac{\partial}{\partial x} - \frac{c^2}{4a}\frac{\partial^3}{\partial x^3} + \frac{1-\nu}{2}\frac{c^2}{4a^3}\frac{\partial^3}{\partial x\partial \varphi^2} + \frac{1}{a}\frac{I}{A}\frac{\partial^3}{\partial x\partial t^2}\right)w + \left(\frac{c^2}{4a}\frac{\partial^2}{\partial x^2} - \frac{1-\nu}{2}\frac{c^2}{4a^3}\frac{\partial^2}{\partial \varphi^2} - \frac{1}{a}\frac{I}{A}\frac{\partial^2}{\partial t^2}\right)\gamma_{xz} = 0, \qquad (6.1)$$

$$\left(\frac{1+\nu}{2}\frac{1}{a}\frac{\partial^2}{\partial x\,\partial\varphi}\right)u + \left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} - \frac{m+I/a^2}{A}\frac{\partial^2}{\partial t^2}\right)v + \\ + \left(\frac{1}{a^2}\frac{\partial}{\partial\varphi} + \frac{c^2}{4a^4}\frac{\partial^3}{\partial\varphi^3} - \frac{1-\nu}{2}\frac{c^2}{4a^2}\frac{\partial^3}{\partial x^2\partial\varphi} + \frac{1}{a^2}\frac{I}{A}\frac{\partial^3}{\partial\varphi\partial t^2}\right)w + \\ + \left(-\frac{c^2}{4a^3}\frac{\partial^2}{\partial\varphi^2} + \frac{1-\nu}{2}\frac{c^2}{4a}\frac{\partial^2}{\partial x^2} + \frac{1}{a}\frac{S_{\varphi z}}{A} - \frac{1}{a}\frac{I}{A}\frac{\partial^2}{\partial t^2}\right)\gamma_{\varphi z} = 0,$$

$$(6.2)$$

$$\left(-\frac{v}{a}\frac{\partial}{\partial x}\right)u + \left(-\frac{1}{a^2}\frac{\partial}{\partial \varphi}\right)v + \left(-\frac{1}{a^2} - \frac{c^2}{4a^4}\frac{\partial^2}{\partial \varphi^2} - \frac{m}{A}\frac{\partial^2}{\partial t^2}\right)w + \left\{\frac{1}{a}\left(\frac{S_{\varphi z}}{A} + \frac{c^2}{4a^2}\right)\frac{\partial}{\partial \varphi}\right\}\gamma_{\varphi z} + \left(\frac{S_{xz}}{A}\frac{\partial}{\partial x}\right)\gamma_{xz} = 0,$$
(6.3)

$$\left(\frac{1-\nu}{2}\frac{2}{a}\frac{\partial^{2}}{\partial x^{2}}-\frac{2}{a}\frac{I}{B}\frac{\partial^{2}}{\partial t^{2}}\right)\nu + \left(-\frac{1}{a}\frac{\partial^{3}}{\partial x^{2}\partial \varphi}-\frac{1}{a^{3}}\frac{\partial^{3}}{\partial \varphi^{3}}-\frac{1}{a^{3}}\frac{\partial}{\partial \varphi}+\frac{1}{a}\frac{I}{B}\frac{\partial^{3}}{\partial \varphi\partial t^{2}}\right)w + \left(\frac{1}{a^{2}}\frac{\partial^{2}}{\partial \varphi^{2}}+\frac{1-\nu}{2}\frac{\partial^{2}}{\partial x^{2}}-\frac{S_{\varphi z}}{B}-\frac{I}{B}\frac{\partial^{2}}{\partial t^{2}}\right)\gamma_{\varphi z} + \left(\frac{1+\nu}{2}\frac{1}{a}\frac{\partial^{2}}{\partial x\partial \varphi}\right)\gamma_{xz}=0, \quad (6.4)$$

$$\left(\frac{1}{a}\frac{\partial^2}{\partial x^2} - \frac{1-\nu}{2}\frac{1}{a^3}\frac{\partial^2}{\partial \varphi^2} - \frac{1}{a}\frac{I}{B}\frac{\partial^2}{\partial t^2}\right)u + \left(\frac{1+\nu}{2}\frac{1}{a^2}\frac{\partial^2}{\partial x\partial\varphi}\right)v + \left(-\frac{\partial^3}{\partial x^3} - \frac{1}{a^2}\frac{\partial^3}{\partial x\partial\varphi^2} + \frac{I}{B}\frac{\partial^3}{\partial x\partial t^2}\right)w + \left(\frac{1+\nu}{2}\frac{1}{a}\frac{\partial^2}{\partial x\partial\varphi}\right)\gamma_{\varphi z} + \left(\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2} - \frac{S_{xz}}{B} - \frac{I}{B}\frac{\partial^2}{\partial t^2}\right)\gamma_{xz} = 0.$$
 (6.5)

When the distortions are assumed in the form

$$u = U \cos n\varphi \cos\left(\frac{\omega}{C_E} x - \omega t\right),$$

$$v = V \sin n\varphi \cos\left(\frac{\omega}{C_E} x - \omega t\right),$$

$$w = W \cos n\varphi \sin\left(\frac{\omega}{C_E} x - \omega t\right),$$

$$\gamma_{\varphi z} = \Gamma_{\varphi z} \sin n\varphi \sin\left(\frac{\omega}{C_E} x - \omega t\right),$$

$$\gamma_{xz} = \Gamma_{xz} \cos n\varphi \cos\left(\frac{\omega}{C_E} x - \omega t\right),$$
(6.6)

where n is an integer (n = 1, 2, 3, ...), the differential equations yield the system of linear equations

$$[a_{ij}]Z = 0 \tag{6.7}$$

where Z is a column matrix with elements U, V, W,  $\Gamma_{\varphi z}$  and  $\Gamma_{xz}$ , O is a zero column matrix and the square matrix  $[a_{ij}]$  is defined in table 1.

Since this system is homogeneous it only has a non-zero solution if the determinant of the coefficient matrix  $|a_{ij}|$ , vanishes.

Hence, the propagation speed  $C_E$  may for each value of n be solved from the requirement

$$|a_{ij}| = 0$$
. (6.8)

Analogous to the case of axisymmetric waves the frequencies  $\omega$  where  $C_E$  approaches infinity will first be determined. Because at these asymptotes the wavelength is infinite, all points in x-direction are in phase, and thus these asymptotes represent the two-dimensional resonance frequencies.

Substitution of  $1/C_E = 0$  in (6.8) then gives the equation (see also table 1)

$$\begin{vmatrix} \left\{ -\frac{1-v}{2} \left(\frac{n}{a}\right)^2 + \frac{m}{A} \omega^2 \right\} & 0 & 0 & \left\{ \frac{1-v}{2} \left(\frac{n}{a}\right)^2 + \frac{1}{a} \frac{1}{A} \omega^2 \right\} \\ 0 & \left\{ -\left(\frac{n}{a}\right)^2 + \frac{m+I/a^2}{A} \omega^2 \right\} & \left\{ -\frac{1}{a} \left(\frac{n}{a}\right) + \frac{c^2}{4a} \left(\frac{n}{a}\right)^3 + \frac{1}{a} \frac{I}{A} \left(\frac{n}{a}\right) \omega^2 \right\} & \left\{ \frac{c^2}{4a} \left(\frac{n}{a}\right)^2 + \frac{1}{a} \frac{S_{\sigma z}}{A} + \frac{1}{a} \frac{1}{A} \omega^2 \right\} & 0 \\ 0 & \left\{ -\frac{1}{a} \left(\frac{n}{a}\right) \right\} & \left\{ -\frac{1}{a^2} + \frac{c^2}{4a^2} \left(\frac{n}{a}\right)^2 + \frac{m}{A} \omega^2 \right\} & \left\{ \left(\frac{S_{\sigma z}}{A} + \frac{c^2}{4a^2}\right) \left(\frac{n}{a}\right) \right\} & 0 \\ 0 & \left\{ \frac{2}{a} \frac{I}{B} \omega^2 \right\} & \left\{ -\left(\frac{n}{a}\right)^3 + \frac{1}{a^2} \left(\frac{n}{a}\right) + \frac{I}{B} \left(\frac{n}{a}\right) \omega^2 \right\} & \left\{ -\left(\frac{n}{a}\right)^2 - \frac{S_{\sigma z}}{B} + \frac{I}{B} \omega^2 \right\} & 0 \\ \left\{ \frac{1-v}{2} \frac{1}{a} \left(\frac{n}{a}\right)^2 + \frac{1}{a} \frac{I}{B} \omega^2 \right\} & 0 & 0 & \left\{ -\frac{1-v}{2} \left(\frac{n}{a}\right)^2 - \frac{S_{z z}}{B} + \frac{I}{B} \omega^2 \right\} \\ \end{bmatrix}$$

$$(6.9)$$

which can be separated into two requirements, viz.

$$\left\{ -\frac{1-\nu}{2} \left(\frac{n}{a}\right)^2 + \frac{m}{A} \omega^2 \right\} \qquad \left\{ \frac{1-\nu}{2} \frac{c^2}{4a} \left(\frac{n}{a}\right)^2 + \frac{1}{a} \frac{I}{A} \omega^2 \right\} = 0 \qquad (6.10)$$

$$\left\{ \frac{1-\nu}{2} \frac{1}{a} \left(\frac{n}{a}\right)^2 + \frac{1}{a} \frac{I}{B} \omega^2 \right\} \qquad \left\{ -\frac{1-\nu}{2} \left(\frac{n}{a}\right)^2 - \frac{S_{xz}}{B} + \frac{I}{B} \omega^2 \right\}$$

and

TABLE 1

The square matrix  $[a_{ij}]$  introduced in eq. (6.7).

L _x	$-\frac{c^2}{4a}\left(\frac{\omega}{C_E}\right)^2 + \frac{1-v}{2}\frac{c^2}{4a}\left(\frac{n}{a}\right)^2 + \frac{1}{a}\frac{1}{A}\omega^2$	o	$-\frac{S_{zz}}{A}\left(rac{\omega}{C_E} ight)$	$\frac{1+\nu}{2} \left(\frac{\omega}{C_{\rm E}}\right) \left(\frac{n}{a}\right)$	$-\left(\frac{\omega}{C_E}\right)^2 - \frac{1-\nu}{2}\left(\frac{n}{a}\right)^2 - \frac{S_{xx}}{B} + \frac{I}{B}\omega^2$
Γ _{α2}	O	$\frac{c^2}{4a} \left(\frac{n}{a}\right)^2 - \frac{1-v}{2} \frac{c^2}{4a} \left(\frac{\omega}{C_E}\right)^2 + \frac{1}{a} \frac{S_{\sigma z}}{A} + \frac{1}{a} \frac{1}{A} \frac{1}{a} \frac{1}{A} \omega^2$	$\left(\frac{S_{az}}{A} + \frac{c^2}{4a^2}\right) \left(\frac{n}{a}\right)$	$-\left(\frac{n}{a}\right)^2 - \frac{1-\nu}{2}\left(\frac{\omega}{C_E}\right)^2 - \frac{S_{\sigma z}}{B} + \frac{I}{B}\omega^2$	$\frac{1+\nu}{2}\left(\frac{\omega}{C_E}\right)\left(\frac{n}{a}\right)$
M	$\frac{\frac{v}{a}\left(\frac{\omega}{C_{E}}\right)}{\frac{v}{a}\left(\frac{\omega}{C_{E}}\right)} + \frac{c^{2}}{4a}\left(\frac{\omega}{C_{E}}\right)^{3} - \frac{1-v}{2}\frac{c^{2}}{4a}\left(\frac{\omega}{C_{E}}\right)\left(\frac{n}{a}\right)^{2} + \frac{1}{a}\frac{I}{A}\left(\frac{\omega}{C_{E}}\right)\omega^{2}$	$-\frac{1}{a} \left(\frac{n}{a}\right) + \frac{c^2}{4a} \left(\frac{n}{a}\right)^3 - \frac{1-v}{2} \frac{c^2}{4a} \left(\frac{\omega}{C_E}\right)^2 \frac{n}{a} + \frac{1}{a} \frac{1}{A} \left(\frac{n}{a}\right) \omega^2$ $+ \frac{1}{a} \frac{1}{A} \left(\frac{n}{a}\right) \omega^2$	$-\frac{1}{a^2} + \frac{c^2}{4a^2} \left(\frac{n}{a}\right)^2 + \frac{m}{A}\omega^2$	$-\left(\frac{\omega}{C_E}\right)^2 \left(\frac{n}{a}\right) - \left(\frac{n}{a}\right)^3 + \frac{1}{a^2} \left(\frac{n}{a}\right) + \frac{I}{B} \left(\frac{n}{a}\right) \omega^2$	$\left(\frac{\omega}{C_E}\right)^3 + \left(\frac{\omega}{C_E}\right) \left(\frac{m}{a}\right)^2 - \frac{I}{B} \left(\frac{\omega}{C_E}\right) \omega^2$
	$\frac{1+\nu}{2}\left(\frac{\omega}{C_{\rm E}}\right)\left(\frac{n}{a}\right)$	$-\left(\frac{n}{a}\right)^2 - \frac{1-v}{2}\left(\frac{\omega}{C_E}\right)^2 + \frac{m+I/a^2}{A}\omega^2$	$-\frac{1}{a}\left(\frac{n}{a}\right)$	$-\frac{1-\nu}{2}\frac{2}{a}\left(\frac{\omega}{C_{\rm E}}\right)^2+\frac{2}{a}\frac{1}{B}\omega^2$	$\frac{1+\nu}{2}\frac{1}{a}\left(\frac{\omega}{C_{\rm E}}\right)\left(\frac{n}{a}\right)$
D	$-\left(\frac{\omega}{C_{\rm E}}\right)^2 - \frac{1-v}{2}\left(\frac{\pi}{a}\right)^2 + \frac{m}{A}\omega^2$	$\frac{1+\nu}{2} \left(\frac{\omega}{C_E}\right) \left(\frac{n}{a}\right)$	$\frac{v}{a}\left(\frac{\omega}{C_{\rm E}}\right)$	0	$-\frac{1}{a}\left(\frac{\omega}{C_{E}}\right)^{2}+\frac{1-\nu}{2}\frac{1}{a}\left(\frac{n}{a}\right)^{2}+\frac{1}{a}\frac{1}{B}\omega^{2}$

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$$\left\{ -\left(\frac{n}{a}\right)^{2} + \frac{m + I/a^{2}}{A}\omega^{2} \right\} \quad \left\{ -\frac{1}{a}\left(\frac{n}{a}\right) + \frac{c^{2}}{4a}\left(\frac{n}{a}\right)^{3} + \frac{1}{a}\frac{I}{A}\left(\frac{n}{a}\right)\omega^{2} \right\} \quad \left\{ \frac{c^{2}}{4a}\left(\frac{n}{a}\right)^{2} + \frac{1}{a}\frac{S_{\varphi z}}{A} + \frac{1}{a}\frac{I}{A}\omega^{2} \right\} \\ \left\{ -\frac{1}{a}\left(\frac{n}{a}\right) \right\} \quad \left\{ -\frac{1}{a^{2}} + \frac{c^{2}}{4a^{2}}\left(\frac{n}{a}\right)^{2} + \frac{m}{A}\omega^{2} \right\} \quad \left\{ \left(\frac{S_{\varphi z}}{A} + \frac{c^{2}}{4a^{2}}\right)\left(\frac{n}{a}\right) \right\} \\ \left\{ \frac{2}{a}\frac{I}{B}\omega^{2} \right\} \quad \left\{ -\left(\frac{n}{a}\right)^{3} + \frac{1}{a^{2}}\left(\frac{n}{a}\right) + \frac{I}{B}\left(\frac{n}{a}\right)\omega^{2} \right\} \quad \left\{ -\left(\frac{n}{a}\right)^{2} - \frac{S_{\varphi z}}{B} + \frac{I}{B}\omega^{2} \right\}$$

$$(6.11)$$

When in the coefficients of each power of  $\omega$  all terms that are small of order  $c^2/a^2$  are neglected, the expression (6.10) may be rewritten as

$$\left\{-\frac{1-\nu}{2}\left(\frac{n}{a}\right)^{2}+\frac{m}{A}\omega^{2}\right\}\left\{-\frac{1-\nu}{2}\left(\frac{n}{a}\right)^{2}-\frac{S_{xz}}{B}+\frac{I}{B}\omega^{2}\right\}=0.$$
(6.12)

This yields the asymptotes

$$\omega_{n2} = \frac{n}{a} \sqrt{\frac{A(1-\nu)}{2m}} \tag{6.13}$$

and

$$\omega_{n5} = \sqrt{\frac{S_{xz}}{I} + \frac{1 - v}{2} \left(\frac{n}{a}\right)^2 \frac{B}{I}}.$$
(6.14)

At  $\omega_{n2}$  the motion of the shell mainly consists of axial displacements u and at  $\omega_{n5}$  of transverse shear  $\gamma_{x2}$ .

When n = 0,  $\omega_{n5}$  yields the value  $\omega_{05}$  of eq. (5.27).

Generally, the further asymptotes following from (6.11) must be solved numerically due to the complex character of the determinant involved. However, for a few special cases some results may be obtained from analytical means. Using the same approximations as in the derivation of (6.12) the requirement (6.11) may be rewritten as

$$\begin{cases} \frac{n^{2}+1}{a^{2}} - \frac{m}{A}\omega^{2} \\ \left\{ \left( \frac{n^{2}}{a^{2}} \frac{m}{A} + \frac{S_{\varphi z}}{B} \frac{m}{A} - \frac{2(n^{2}-1)}{a^{2}} \frac{S_{\varphi z}}{A} \frac{I}{B} \right) \omega^{2} - \frac{m}{A} \frac{I}{B} \omega^{4} \\ + \left\{ \frac{n^{2}-1}{a^{2}} - \frac{m}{A} \omega^{2} \right\} \left\{ -\frac{n^{2}(n^{2}-1)}{a^{4}} \frac{S_{\varphi z}}{A} + \frac{3n^{2}}{a^{2}} \frac{S_{\varphi z}}{A} \frac{I}{B} \omega^{2} \\ \right\} = 0, \quad (6.15)$$

which, for large values of n, may be approximated by

$$\left\{\frac{n^2}{a^2} - \frac{m}{A}\omega^2\right\} \left\{-\frac{n^4}{a^4}\frac{S_{\varphi z}}{A} + \left(\frac{n^2}{a^2}\frac{m}{A} + \frac{S_{\varphi z}}{B}\frac{m}{A} + \frac{n^2}{a^2}\frac{S_{\varphi z}}{A}\frac{I}{B}\right)\omega^2 - \frac{m}{A}\frac{I}{B}\omega^4\right\} = 0.$$
(6.16)

Hence,

$$\omega_{n4} = \frac{n}{a} \sqrt{\frac{A}{m}}, \quad n \ge 1 , \qquad (6.17)$$

while  $\omega_{n3}$  and  $\omega_{n1} (n \ge 1)$  may be solved by equating the second term within brackets of (6.16) to zero. Another case that can be solved occurs when n = 1. The cubic equation (6.15) then reduces to

$$\left\{\frac{2}{a^2} - \frac{m}{A}\omega^2\right\}\left\{\left(\frac{1}{a^2}\frac{m}{A} + \frac{S_{\varphi z}}{B}\frac{m}{A}\right) - \frac{m}{A}\frac{I}{B}\omega^2\right\} - \frac{3}{a^2}\frac{S_{\varphi z}}{A}\frac{m}{A}\frac{I}{B}\omega^2 = 0$$
(6.18)

and since the last term is negligible (compare with  $\frac{S_{\varphi z}}{B} \frac{m^2}{A^2} \omega^2$ ) the solutions become with fair accuracy

$$\omega_{14} = \frac{1}{a} \sqrt{\frac{2A}{m}}, \qquad n = 1.$$

$$\omega_{13} = \sqrt{\frac{S_{\varphi z} + B/a^2}{I}}, \qquad n = 1.$$
(6.19)

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TABLE 2	14

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(6.20)

$\omega^2 \dots \omega^8$	$\omega^2 \dots \omega^6$	$\omega^4 \dots \omega^8$	$(\omega^2 \dots \omega^8)$
+a21a55a13a32a44	$+ a_{21} a_{55} a_{34} a_{43} a_{12}$	$-a_{21}a_{55}a_{13}a_{34}a_{42}$	$-a_{21}a_{55}a_{12}a_{33}a_{44}$
$\omega^2 \dots \omega^8$	$\omega^2 \dots \omega^8$	$\omega^2 \dots \omega^{10}$	$\omega^4 \dots \omega^{10}$
$+a_{51}a_{24}a_{35}a_{43}a_{12}$	$+a_{51}a_{24}a_{13}a_{32}a_{45}$	$+a_{51}a_{24}a_{15}a_{33}a_{42}$	$-a_{51}a_{24}a_{13}a_{35}a_{42}$
ω ⁰ ω ⁸	$\omega^2 \dots \omega^8$	$\omega^4 \dots \omega^6$	$\omega^4 \dots \omega^6$
$-a_{51}a_{24}a_{32}a_{43}a_{15}$	$-a_{51}a_{24}a_{12}a_{33}a_{45}$	-a ₂₁ a ₅₄ a ₃₅ a ₄₃ a ₁₂	$-a_{21}a_{54}a_{13}a_{32}a_{45}$
$\omega^4 \cdots \omega^8$	ω ^ę ω ⁸	$\omega^2 \dots \omega^6$	$\omega^4 \dots \omega^6$
$-a_{21}a_{54}a_{15}a_{33}a_{42}$	$+a_{21}a_{54}a_{13}a_{35}a_{42}$	$+a_{21}a_{54}a_{32}a_{43}a_{15}$	$+a_{21}a_{54}a_{12}a_{33}a_{45}$
$\omega^2 \dots \omega^8$	$\omega^2 \dots \omega^8$	ω ⁰ ω ⁸	$\omega^2 \dots \omega^6$
$-a_{51}a_{23}a_{44}a_{35}a_{12}$	$-a_{51}a_{23}a_{15}a_{34}a_{42}$	$+a_{51}a_{23}a_{32}a_{44}a_{15}$	$+a_{51}a_{23}a_{12}a_{34}a_{45}$
$\omega^4 \dots \omega^8$	$\omega^4 \dots \omega^8$	$\omega^2 \dots \omega^8$	$\omega^4 \dots \omega^6$
$+a_{21}a_{53}a_{44}a_{35}a_{12}$	+ a21 a53 a15 a34 a42	-a21a33a32a44a15	-a21a53a12a34a45
$\omega^2 \dots \omega^{10}$	$\omega^{0} \dots \omega^{8}$	$\omega^0 \dots \omega^{10}$	$\omega^2 \dots \omega^8$
$+a_{51}a_{12}a_{44}a_{35}a_{13}$	$+a_{51}a_{22}a_{43}a_{34}a_{15}$	-a ₅₁ a ₂₂ a ₃₃ a ₄₄ a ₁₅	$-a_{51}a_{22}a_{13}a_{34}a_{45}$
$\omega^4 \dots \omega^8$	$\omega^2 \dots \omega^6$	$\omega^2 \dots \omega^8$	$\omega^4 \dots \omega^6$
$-a_{21}a_{52}a_{44}a_{35}a_{13}$	$-a_{21}a_{52}a_{43}a_{34}a_{15}$	$+a_{21}a_{52}a_{33}a_{44}a_{15}$	$+a_{21}a_{52}a_{13}a_{34}a_{45}$
$\omega^2 \dots \omega^{10}$	$\omega^4 \cdots \omega^8$	$\omega^2 \dots \omega^8$	$\omega^2 \dots \omega^8$
+ a ₃₁ a ₁₅ a ₄₄ a ₅₃ a ₂₂	$+a_{31}a_{15}a_{23}a_{42}a_{54}$	$+a_{31}a_{15}a_{24}a_{43}a_{52}$	$-a_{31}a_{15}a_{23}a_{44}a_{52}$
$\omega^4 \dots \omega^{10}$	$\omega^2 \cdots \omega^8$	$\omega^2 \dots \omega^{10}$	$\omega^4 \cdots \omega^8$
$-a_{31}a_{15}a_{42}a_{53}a_{24}$	$-a_{31}a_{15}a_{22}a_{43}a_{54}$	$-a_{11}a_{35}a_{44}a_{53}a_{22}$	$-a_{11}a_{35}a_{23}a_{42}a_{54}$
ω ² ω ⁸	$\omega^2 \dots \omega^8$	$\omega^4 \dots \omega^{10}$	$\omega^2 \dots \omega^8$
$-a_{11}a_{35}a_{24}a_{43}a_{52}$	$+a_{11}a_{35}a_{23}a_{44}a_{52}$	$+a_{11}a_{35}a_{42}a_{53}a_{24}$	+a11 a35 a22 a43 a54
$\omega^2 \dots \omega^8$	$\omega^2 \dots \omega^8$	$\omega^1 \dots \omega^6$	$\omega^{0} \dots \omega^{8}$
$+a_{11}a_{34}a_{41}a_{23}a_{55}$	$+a_{11}a_{34}a_{45}a_{53}a_{22}$	-a11 a34 a23 a45 a52	-a11 a34 a22 a43 a55
$\omega^4$ $\omega^{10}$	ω ⁴ · · · ω ⁸	ω ⁴ ω ⁸	ω ² ω ¹⁰
$+a_{31}a_{13}a_{24}a_{42}a_{55}$	$+a_{31}a_{13}a_{54}a_{45}a_{22}$	$-a_{31}a_{13}a_{24}a_{45}a_{52}$	$-a_{31}a_{13}a_{22}a_{44}a_{55}$
$\omega^2 \dots \omega^{10}$	$\omega^2 \dots \omega^8$	$\omega^2 \dots \omega^8$	$\omega^0 \dots \omega^{10}$
$-a_{11}a_{33}a_{24}a_{42}a_{55}$	$-a_{11}a_{33}a_{54}a_{45}a_{22}$	$+ a_{11} a_{33} a_{24} a_{45} a_{52}$	$+a_{11}a_{33}a_{22}a_{44}a_{55}$
$\omega^2 \dots \omega^8$	$\omega^4 \dots \omega^6$	$\omega^4 \dots \omega^8$	$\omega^2 \dots \omega^8$
$-a_{31}a_{12}a_{24}a_{43}a_{55}$	$-a_{31}a_{12}a_{54}a_{45}a_{23}$	$+a_{31}a_{12}a_{24}a_{45}a_{53}$	$+a_{31}a_{12}a_{23}a_{44}a_{55}$
$\omega^0 \dots \omega^8$	$\omega^2 \dots \omega^6$	$\omega^2 \dots \omega^8$ .	$\omega^0 \dots \omega^8$
$+a_{11}a_{32}a_{24}a_{43}a_{55}$	$+ a_{11}a_{32}a_{54}a_{45}a_{23}$	$-a_{11}a_{32}a_{24}a_{45}a_{53}$	$-a_{11}a_{32}a_{23}a_{44}a_{55}$

When each element of the matrix in table 1 is represented by a symbol  $a_{ij}$ , the characteristic equation (6.8) becomes

$$|a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} = 0$$
(6.20)

and evaluation gives the result as shown in table 2.

By inserting the lowest-, respectively the highest frequency power, of each element  $a_{ij}$  the range of frequency powers for each term in table 2 is easily found (the results are mentioned in the said table 2). This knowledge will be useful when the behaviour for very low-, respectively very high frequencies is considered.

#### 6.1 Low-frequency behaviour

For extremely low frequencies the characteristic equation (6.8) (see also table 1) reduces to the term of lowest frequency power, i.e.  $\omega^0$ . Using table 2 it appears that in that case it is sufficient to account for the terms

$$-a_{11}a_{32}a_{23}a_{44}a_{55} + a_{11}a_{32}a_{24}a_{43}a_{55} + a_{11}a_{33}a_{22}a_{44}a_{55}$$
  
$$-a_{11}a_{34}a_{22}a_{43}a_{55} - a_{51}a_{22}a_{33}a_{44}a_{15} + a_{51}a_{22}a_{43}a_{34}a_{15}$$
  
$$+a_{51}a_{23}a_{32}a_{44}a_{15} - a_{51}a_{24}a_{32}a_{43}a_{15}.$$

When from table 1 the parts with lowest frequency of each  $a_{ij}$  are now substituted there is found

$$-\frac{1-v}{2}\frac{S_{\varphi z}}{A}\frac{n^4}{a^4}\left(\frac{1-v}{2}\frac{n^2}{a^2}+\frac{S_{xz}}{B}\right)\left(\frac{n^2-1}{a^2}\right)=0^{\circ}.$$
(6.21)

During the derivation of this equation terms small of order  $c^2/a^2$  were again neglected.

Since this requirement can only be satisfied for n=1, it is obvious that elastic waves as assumed in (6.6) cannot exist for such extremely low frequencies in case n > 1.

A more accurate low frequency approximation for the characteristic equation than (6.21), which should include a few higher frequency powers, can hardly be obtained in an analytically manageable form because this would require the evaluation of all terms in table 2.

In case n = 1, however, a low-frequency approximation can be suggested along quite different lines. Ref. 4 mentions the fact that, when the effect of transverse shear is neglected, the cylinder behaves as a uniform beam when the frequency is low.

When this statement is extended to the present configuration, where transverse shear is certainly of importance, the following analysis becomes possible. By writing the distortions in the form

$$u = \bar{u} \cos \varphi ,$$
  

$$v = \bar{v} \sin \varphi ,$$
  

$$w = \bar{w} \cos \varphi ,$$
  

$$\gamma_{\varphi z} = \bar{\gamma}_{\varphi z} \sin \varphi ,$$
  

$$\gamma_{xz} = \bar{\gamma}_{xz} \cos \varphi ,$$
  
(6.22)

the forces and moments  $N_x$ ,  $T_x$ ,  $Q_x$ ,  $M_x$  and  $M_{x\varphi}$  become from (3.1), (3.3), (3.6), (3.10) and (3.12)

$$N_x = A \left( \frac{\partial \bar{u}}{\partial x} + \frac{v}{a} \bar{v} + \frac{v}{a} \bar{w} - \frac{c^2}{4a} \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{c^2}{4a} \frac{\partial \bar{y}_{xz}}{\partial x} \right) \cos \varphi = \bar{N}_x \cos \varphi , \qquad (6.23)$$

$$T_{x\varphi} = A\left(-\frac{1-\nu}{2}\frac{\bar{u}}{a} + \frac{1-\nu}{2}\frac{\partial\bar{v}}{\partial x} + \frac{1-\nu}{2}\frac{c^2}{4a^2}\frac{\partial\bar{w}}{\partial x} + \frac{1-\nu}{2}\frac{c^2}{4a}\frac{\partial\gamma_{\varphi z}}{\partial x}\right)\sin\varphi = \overline{T}_{x\varphi}\sin\varphi, \qquad (6.24)$$

$$Q_x = S_{xz}\bar{\gamma}_{xz}\cos\varphi = \bar{Q}_x\cos\varphi, \qquad (6.25)$$

$$M_x = B\left(-\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{v}{a^2} \bar{w} + \frac{\partial \bar{\gamma}_{xz}}{\partial x} + \frac{v}{a} \bar{\gamma}_{\varphi z} + \frac{1}{a} \frac{\partial \bar{u}}{\partial x} + \frac{v}{a^2} \bar{v}\right) \cos \varphi = \overline{M}_x \cos \varphi , \qquad (6.26)$$

$$M_{x\varphi} = B\left(\frac{1-\nu}{2}\frac{2}{a}\frac{\partial\overline{w}}{\partial x} - \frac{1-\nu}{2}\frac{\overline{\gamma}_{xz}}{a} + \frac{1-\nu}{2}\frac{\partial\overline{\gamma}_{\varphi z}}{\partial x} + \frac{1-\nu}{2}\frac{2}{a}\frac{\partial\overline{\nu}}{\partial x}\right)\sin\varphi = \overline{M}_{x\varphi}\sin\varphi.$$
(6.27)

The inertia loading, see  $(3.15) \dots (3.19)$  is rewritten into

$$X = \left\{ -m\vec{u} - \frac{I}{a} \left( \ddot{\gamma}_{xz} - \frac{\partial \vec{w}}{\partial x} \right) \right\} \cos \varphi = \overline{X} \cos \varphi , \qquad (6.28)$$

$$Y = \left\{ -m\vec{v} - \frac{I}{a} \left( \ddot{\vec{y}}_{\varphi z} - \frac{\vec{w}}{a} + \frac{\vec{v}}{a} \right) \right\} \sin \varphi = \overline{Y} \sin \varphi , \qquad (6.29)$$

$$Z = \{-m\overline{w}\}\cos\varphi = \overline{Z}\cos\varphi, \qquad (6.30)$$

$$\mathbf{m}_{x} = I\left(\bar{\gamma}_{\varphi z} + \frac{\bar{w}}{a} + 2\frac{\bar{v}}{a}\right)\sin\varphi = \overline{\mathbf{m}}_{x}\sin\varphi, \qquad (6.31)$$

$$\mathbf{m}_{\varphi} = -I\left(\ddot{\gamma}_{xz} - \frac{\partial \vec{w}}{\partial x} + \frac{\vec{u}}{a}\right)\cos\varphi = \overline{\mathbf{m}}_{\varphi}\cos\varphi .$$
(6.32)





From figs 5a, 5b and the expressions (6.23)-(6.32) the resultant loading for a cylinder element can now easily be obtained through integration. It appears that the loading as far as elastic effects are concerned is represented by the transverse shear force D and the bending moment M (all other components are zero), while the resultant inertia loading only consists of the translational load L and the rotatory inertia moment **m**. They read

$$D = \int_{0}^{2\pi} (Q_x \cos \varphi - T_{x\varphi} \sin \varphi) \, a \, d\varphi = \pi a (\overline{Q}_x - \overline{T}_{x\varphi}) \,, \tag{6.33}$$

$$M = \int_0^{2\pi} \left( N_x a \cos \varphi + M_x \cos \varphi \right) a \mathrm{d}\varphi = \pi a \left( a \overline{N}_x + \overline{M}_x \right), \tag{6.34}$$

$$L = \int_{0}^{2\pi} (Z \cos \varphi - Y \sin \varphi) a d\varphi = \pi a (\overline{Z} - \overline{Y}), \qquad (6.35)$$

$$\mathbf{m} = \int_{0}^{2\pi} \left( Xa \cos \varphi + M_{\varphi} \cos \varphi \right) a \mathrm{d}\varphi = \pi a \left( a \overline{X} + \overline{M}_{\varphi} \right), \tag{6.36}$$

or after substitution of (6.23)-(6.26) and (6.28), (6.29), (6.30), (6.32)

$$D = \pi a \left[ S_{xz} \bar{\gamma}_{xz} - A \left( -\frac{1-\nu}{2} \frac{\bar{u}}{a} + \frac{1-\nu}{2} \frac{\partial \bar{v}}{\partial x} + \frac{1-\nu}{2} \frac{c^2}{4a^2} \frac{\partial \bar{w}}{\partial x} + \frac{1-\nu}{2} \frac{c^2}{4a} \frac{\partial \bar{\gamma}_{\varphi z}}{\partial x} \right) \right], \tag{6.37}$$

$$M = \pi a \left[ A \left( a \frac{\partial \bar{u}}{\partial x} + v \bar{v} + v \bar{w} - \frac{c^2}{4} \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{c^2}{4} \frac{\partial \bar{\gamma}_{xz}}{\partial x} \right) +$$
(6.38)

$$+B\left(-\frac{\partial^2 \bar{w}}{\partial x^2}+\frac{v}{a^2}\bar{w}+\frac{\partial \bar{\gamma}_{xz}}{\partial x}+\frac{v}{a}\bar{\gamma}_{\varphi z}+\frac{1}{a}\frac{\partial \bar{u}}{\partial x}+\frac{v}{a^2}\bar{v}\right)\right].$$
(6.39)

$$L = \pi a \left[ -m \ddot{\bar{w}} + m \ddot{\bar{v}} + \frac{I}{a} \left( \ddot{\bar{\gamma}}_{\varphi z} + \frac{\ddot{\bar{w}}}{a} + \frac{\ddot{\bar{v}}}{a} \right) \right], \tag{6.40}$$

$$\mathbf{m} = \pi a \left[ -ma\ddot{\vec{u}} - I\left(\ddot{\vec{\gamma}}_{xz} - \frac{\partial \ddot{\vec{w}}}{\partial x}\right) - I\left(\ddot{\vec{\gamma}}_{xz} - \frac{\partial \ddot{\vec{w}}}{\partial x} + \frac{\ddot{\vec{u}}}{a}\right) \right].$$
(6.41)

Now when the cylinder acts as a uniform beam and undergoes a vertical deflection f(x) there may be put, see fig. 6.



Fig. 6. Relations between a vertical deflection f(x), the radial displacement w and the tangential displacement v.

$$v_z = v = \bar{v} \sin \phi = -f \sin \phi, \text{ or } \bar{v} = -f, \qquad (6.42)$$

$$w = \overline{w} \cos \varphi = f \cos \varphi$$
, or  $\overline{w} = f$ , (6.43)

which immediately yields the additional conclusion, see eq. (2.8),  $\gamma_{\varphi z} - \frac{\partial w}{a \partial \varphi} + \frac{v}{a} = 0$ , or with (6.22), (6.42) and (6.43)

$$\bar{\gamma}_{\varphi z} = 0 . \tag{6.44}$$

With these simplifications the number of five unknown distortions is now decreased to three, viz. u, f and  $\gamma_{xz}$ . Since the equilibrium of a cylinder-element requires satisfaction of only two conditions, which read from fig. 7

$$\frac{\partial D}{\partial x} + L = 0, \qquad (6.45)$$

$$\frac{\partial M}{\partial x} - D + \mathbf{m} = 0, \qquad (6.46)$$

there is one condition lacking. It may, however, be deduced in the following way.

If the assumptions (6.42), (6.43) on  $\bar{v}$  and  $\bar{w}$  are correct, any element of the shell is in equilibrium when the conditions (4.1)–(4.6) and hence the eqs (6.45), (6.46) are satisfied.

As it may be shown that the eqs (6.45), (6.46) are fulfilled through the conditions (4.1), (4.2), (4.3), (4.5) and (4.6), one could say reversely that if the assumptions on  $\bar{v}$  and  $\bar{w}$  are part of an exact solution, the conditions (6.45), (6.46) guarantee the satisfaction of all equilibrium conditions for a shell element, but for (4.4).



Fig. 7. Equilibrium of a cylinder element.

Therefore, in the present case, the third equilibrium condition must be (4.4), i.e.

$$\frac{\partial M_{\varphi}}{\partial \phi} + \frac{\partial M_{x\varphi}}{\partial x} - Q_{\varphi} - \mathbf{m}_{x} = 0.$$
(6.47)

With (6.22), the expressions for  $Q_{\varphi}$  and  $M_{\varphi}$  become from (3.7), (3.11)

$$Q_{\varphi} = S_{\varphi z} \bar{\gamma}_{\varphi z} \sin \varphi , \qquad (6.48)$$

$$M_{\varphi} = B \left\{ -\nu \frac{\partial^2 \bar{w}}{\partial x^2} + \nu \frac{\partial \bar{\gamma}_{xz}}{\partial x} + \frac{\bar{\gamma}_{\varphi z}}{a} \right\} \cos \varphi .$$
 (6.49)

When all quantities small of order  $c^2/a^2$  are neglected, evaluation of the conditions (6.45), (6.46), (6.47), by using the assumptions (6.42)–(6.44) and the special relation (see eqs (3.5), (3.14))

$$B=A\,\frac{c^2}{4}$$

yields after elimination of  $\bar{\gamma}_{xz}$  the two coupled differential equations

$$\begin{cases} \frac{1-\nu}{2}\frac{1}{a}\frac{\partial}{\partial x} \right\} \bar{u} + \left\{ \left(\frac{1-\nu}{2} + \frac{2\nu}{1+\nu}\frac{S_{xz}}{A}\right)\frac{\partial^2}{\partial x^2} - 2\left(\frac{m}{A} - \frac{1}{1+\nu}\frac{S_{xz}}{A}\frac{I}{B}\right)\frac{\partial^2}{\partial t^2} \right\} f = 0, \qquad (6.50)$$

$$\begin{cases} \frac{\partial^3}{\partial x^3} - \frac{1-\nu}{2}\frac{1}{a^2}\frac{\partial}{\partial x} - \frac{m}{A}\frac{\partial^3}{\partial x\partial t^2} \right\} \bar{u} + \\ + \left\{ -\frac{1-\nu}{1+\nu}\frac{2}{a}\frac{B}{A}\frac{\partial^4}{\partial x^4} - \left(\frac{1-\nu}{2} + \frac{2\nu}{1+\nu}\frac{S_{xz}}{A}\right)\frac{1}{a}\frac{\partial^2}{\partial x^2} + \frac{3-\nu}{1+\nu}\frac{2}{a}\frac{I}{A}\frac{\partial^4}{\partial x^2\partial t^2} - \frac{4}{1+\nu}\frac{1}{a}\frac{I}{A}\frac{B}{\partial t^4} - \frac{2}{1+\nu}\frac{1}{a}\frac{S_{xz}}{A}\frac{I}{B}\frac{\partial^2}{\partial t^2} \right\} f = 0. \qquad (6.50)$$

By taking

$$\tilde{u} = U \cos\left(\frac{\omega}{C_E} x - \omega t\right),$$
  

$$f = F \sin\left(\frac{\omega}{C_E} x - \omega t\right),$$
(6.52)

the characteristic equation becomes after again neglecting all terms that are small of order  $c^2/a^2$ 

$$\left(\frac{1-\nu}{2} + \frac{2\nu}{1+\nu}\frac{S_{xz}}{A}\right)\frac{1}{C_E^4} - \left\{2\left(\frac{m}{A} - \frac{1}{1+\nu}\frac{S_{xz}}{A}\frac{I}{B}\right) + \frac{m}{A}\left(\frac{1-\nu}{2} + \frac{2\nu}{1+\nu}\frac{S_{xz}}{A}\right)\right\}\frac{1}{C_E^2} + \left\{2\frac{m}{A}\left(\frac{m}{A} - \frac{1}{1+\nu}\frac{S_{xz}}{A}\frac{I}{B}\right) - \frac{1-\nu}{a^2}\frac{m}{A}\frac{1}{\omega^2}\right\} = 0$$
(6.53)

and the only propagation speed  $C_E$  for

$$\omega < \frac{1}{a} \sqrt{\frac{1}{2}} \frac{\beta}{\alpha} \frac{A}{m}$$

becomes

where

$$C_E = \left\{ \frac{m}{A} \left( \alpha + \frac{1}{2} \right) + \sqrt{\left( \frac{m}{A} \right)^2 \left( \alpha - \frac{1}{2} \right)^2 + \frac{\beta}{a^2} \frac{m}{A} \frac{1}{\omega^2}} \right\}^{-\frac{1}{2}},$$
(6.54)

$$\alpha = 2 \frac{1 + v - \frac{S_{xz}}{B} \frac{1}{m}}{1 - v^2 + 4v \frac{S_{xz}}{A}}$$
(6.55)

(6.56)

and

Of course, the expression (6.54) needs numerical verification.

# 6.2 High-frequency behaviour

In the first place the behaviour for extremely high frequencies will be considered. For that purpose the characteristic equation (6.8) will be reduced to the term of highest frequency power. Table 2 then shows that it is sufficient to account for the parts with highest frequency power of each  $a_{ij}$  in

 $\beta = 2 \frac{1 - v^2}{1 - v^2 + 4v \frac{S_{xx}}{4}}.$ 

$$a_{11}a_{33}a_{22}a_{44}a_{55} - a_{11}a_{33}a_{24}a_{42}a_{55} - a_{31}a_{13}a_{22}a_{44}a_{55} + a_{31}a_{13}a_{24}a_{42}a_{55} + a_{11}a_{35}a_{42}a_{53}a_{24} - a_{11}a_{35}a_{44}a_{53}a_{22} - a_{31}a_{15}a_{42}a_{53}a_{24} + a_{31}a_{15}a_{44}a_{53}a_{22} - a_{51}a_{22}a_{33}a_{44}a_{15} + a_{51}a_{22}a_{44}a_{35}a_{13} - a_{51}a_{24}a_{13}a_{35}a_{42} + a_{51}a_{24}a_{15}a_{33}a_{42}$$

and the approximate characteristic equation becomes, when all quantities small of order  $c^2/a^2$  are neglected,

$$-\left(\frac{1-\nu}{2}\right)^{2}\frac{S_{xz}}{A}\left(\frac{1}{C_{E}^{2}}-\frac{m}{A}\right)\left(\frac{1}{C_{E}^{2}}-\frac{2m}{A(1-\nu)}\right)\left(\frac{1}{C_{E}^{2}}-\frac{m}{S_{xz}}\right)\left(\frac{1}{C_{E}^{2}}-\frac{I}{B}\right)\left(\frac{1}{C_{E}^{2}}-\frac{2I}{B(1-\nu)}\right)\omega^{10}=0.$$
 (6.57)

The solutions for extremely high frequencies thus read

$$C_{E_4} = \bar{C}_L = \sqrt{\frac{A}{m}},\tag{6.58}$$

which again equals the propagation speed of longitudinal waves in a flat panel in case the contraction in the direction of the wave fronts is fully restrained and the motion of the shell is due to axial displacements u,

$$C_{E_2} = \sqrt{\frac{A(1-v)}{2m}},$$
 (6.59)

where shear of the faces (tangential displacement v) is decisive,

$$C_{E_1} = \sqrt{\frac{S_{xz}}{m}},\tag{6.60}$$

being primarily due to transverse shear  $\gamma_{xz}$ ,

$$C_{E_5} = \sqrt{\frac{B}{I}}, \qquad (6.61)$$

where the motion of the shell again mainly consists of transverse shear  $\gamma_{xz}$  and

$$C_{E_3} = \sqrt{\frac{B(1-\nu)}{2I}},$$
 (6.62)

where the shell motion is associated with transverse shear  $\gamma_{\varphi z}$  exclusively. That both  $C_{E_{1w-w}} = \sqrt{\frac{S_{xz}}{m}}$  and  $C_{E_{5w-w}} = \sqrt{\frac{B}{I}}$  are determined by the transverse shear angle  $\gamma_{xz}$  can be explained by the following arguments. In case of  $C_{E_1}$  the mass-term *m* indicates that deflections *w* must be present, while the absence of rotatory-inertia terms indicates that the shell elements do not rotate (significantly) about the  $\varphi$ -axis. The relation between  $\gamma_{xz}$  and *w* is then given by

$$\gamma_{xz} = \frac{\partial w}{\partial x}$$

The corresponding relation between their amplitudes, which reads

$$\Gamma_{xz} = \frac{\omega}{C_E} W = \frac{\pi W}{\lambda}$$

then explains why  $\gamma_{xz}$  can yet be the most important distortion (W is of the order of the wavelength and the shellmotion is as shown in fig. 8).



Fig. 8. Shell-motions for an infinitely high frequency when  $\gamma_{xz}$  is the most important distortion.

In case of  $C_{E_s}$  only the quantity I and the bending stiffness (B) are of importance. This is only possible if the motion of the shell is as shown in fig. 8.

It may be observed that the approximate characteristic equation (6.57) is no longer dependent on the number of waves in circumferential direction (i.e. the quantity *n*), nor on the cylinder radius "*a*". From the latter statement it may obviously be concluded that for extremely high frequencies the cylinder again behaves as a flat panel. It will be interesting to investigate in how far the "flat panel behaviour" remains valid when the frequencies are decreased from these extremely high to somewhat lower values.

For that purpose all terms in Table 1 that vanish when  $a \rightarrow \infty$  are left out and the characteristic equation becomes from (6.8)

$$-\left(\frac{\omega}{C_{E}}\right)^{2} - \frac{1-v}{2}\left(\frac{n}{a}\right)^{2} + \frac{m}{A}\omega^{2} \qquad \frac{1+v}{2}\left(\frac{\omega}{C_{E}}\right)\left(\frac{n}{a}\right) \qquad 0 \qquad 0 \qquad 0$$

$$\frac{1+v}{2}\left(\frac{\omega}{C_{E}}\right)\left(\frac{n}{a}\right) \qquad -\left(\frac{n}{a}\right)^{2} - \frac{1-v}{2}\left(\frac{\omega}{C_{E}}\right)^{2} + \frac{m}{A}\omega^{2} \qquad 0 \qquad 0 \qquad 0$$

$$0 \qquad 0 \qquad \frac{m}{A}\omega^{2} \qquad \frac{S_{\sigma z}}{A}\left(\frac{n}{a}\right) \qquad -\frac{S_{\sigma z}}{A}\left(\frac{\omega}{C_{E}}\right) = 0$$

$$0 \qquad 0 \qquad -\left(\frac{\omega}{C_{E}}\right)^{2}\left(\frac{n}{a}\right) - \left(\frac{n}{a}\right)^{3} + \frac{1}{B}\left(\frac{n}{a}\right)\omega^{2} \qquad -\left(\frac{n}{a}\right)^{2} - \frac{1-v}{2}\left(\frac{\omega}{C_{E}}\right)^{2} - \frac{S_{\sigma z}}{B} + \frac{1}{B}\omega^{2} \qquad \frac{1+v}{2}\left(\frac{\omega}{C_{E}}\right)\left(\frac{n}{a}\right) = 0$$

$$0 \qquad 0 \qquad \left(\frac{\omega}{C_{E}}\right)^{3} + \left(\frac{\omega}{C_{E}}\right)\left(\frac{n}{a}\right)^{2} - \frac{1}{B}\left(\frac{\omega}{C_{E}}\right)\omega^{2} \qquad \frac{1+v}{2}\left(\frac{\omega}{C_{E}}\right)\left(\frac{n}{a}\right) - \left(\frac{\omega}{C_{E}}\right)^{2} - \frac{1-v}{2}\left(\frac{n}{a}\right)^{2} - \frac{1-v}{2}\left(\frac{n}{$$

which can immediately be separated into the two requirements

$$\begin{vmatrix} -\left(\frac{\omega}{C_E}\right)^2 - \frac{1-\nu}{2}\left(\frac{n}{a}\right)^2 + \frac{m}{A}\omega^2 & \frac{1+\nu}{2}\left(\frac{\omega}{C_E}\right)\left(\frac{n}{a}\right) \\ \frac{1+\nu}{2}\left(\frac{\omega}{C_E}\right)\left(\frac{n}{a}\right) & -\left(\frac{n}{a}\right)^2 - \frac{1-\nu}{2}\left(\frac{\omega}{C_E}\right)^2 + \frac{m}{A}\omega^2 \end{vmatrix} = 0$$
(6.64)

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and

$$\begin{vmatrix} \frac{m}{A}\omega^{2} & \frac{S_{\varphi z}}{A}\left(\frac{n}{a}\right) & -\frac{S_{xz}}{A}\left(\frac{\omega}{C_{E}}\right) \\ -\left(\frac{\omega}{C_{E}}\right)^{2}\left(\frac{n}{a}\right) - \left(\frac{n}{a}\right)^{3} + \frac{I}{B}\left(\frac{n}{a}\right)\omega^{2} & -\left(\frac{n}{a}\right)^{2} - \frac{1-\nu}{2}\left(\frac{\omega}{C_{E}}\right)^{2} - \frac{S_{\varphi z}}{B} + \frac{I}{B}\omega^{2} & \frac{1+\nu}{2}\left(\frac{\omega}{C_{E}}\right)\left(\frac{n}{a}\right) \\ \left(\frac{\omega}{C_{E}}\right)^{3} + \left(\frac{\omega}{C_{E}}\right)\left(\frac{n}{a}\right)^{2} - \frac{I}{B}\left(\frac{\omega}{C_{E}}\right)\omega^{2} & \frac{1+\nu}{2}\left(\frac{\omega}{C_{E}}\right)\left(\frac{n}{a}\right) & -\left(\frac{\omega}{C_{E}}\right)^{2} - \frac{1-\nu}{2}\left(\frac{n}{a}\right)^{2} - \frac{S_{xz}}{B} + \frac{I}{B}\omega^{2} \end{vmatrix} = 0$$

$$(6.65)$$

Evaluation of (6.64), which delivers propagation speeds associated with axial- and circumferential displacements u and v in the neutral plane only, yields

$$C_{E_4} = \left\{ -\left(\frac{n}{a}\right)^2 \frac{1}{\omega^2} + \frac{m}{A} \right\}^{-\frac{1}{2}}$$
(6.66)

and

$$C_{E_2} = \left\{ -\left(\frac{n}{a}\right)^2 \frac{1}{\omega^2} + \frac{2m}{A(1-\nu)} \right\}^{-\frac{1}{2}}.$$
(6.67)

When  $\omega \rightarrow \infty$  the values that were already mentioned in (6.58) and (6.59) reappear.

Furthermore it may be observed that  $C_{E_4}$  and  $C_{E_2}$  have asymptotes at and are only valid for higher frequencies than

$$\omega_{n4} = \frac{n}{a} \sqrt{\frac{A}{m}} \tag{6.68}$$

and

$$\omega_{n2} = \frac{n}{a} \sqrt{\frac{A(1-\nu)}{2m}} \tag{6.69}$$

respectively.

At  $\omega_{n4}$  the motion of the shell is governed by the tangential displacements v and at  $\omega_{n2}$  by the axial displacements u. These asymptotes coincide with those already obtained in (6.17), (6.13) (the former only for  $n \ge 1$ ).

The requirement (6.65) yields propagation speeds where the motion of the shell is only determined by the radial deflection w and the transverse shear angles  $\gamma_{xz}$  and  $\gamma_{\varphi z}$  of the core.

For extremely high frequencies again only the term of highest frequency power is essential and it may easily be verified that in that case the values of (6.60), (6.61) and (6.62) reappear.

Any attempt, however, to write the total determinant of (6.65) in such form that conclusions concerning the influence of n for high frequencies could be drawn, did fail. Therefore, this matter will be investigated by numerical means.

#### 7 Numerical example

For the numerical calculations a configuration that is representative for an actual aircraft will be taken.

Consider a sandwich shell having two dural faces ( $E = 7000 \text{ kg mm}^{-2}$ , v = 0.3) of 0.5 mm thickness and a honeycomb core.

The total thickness of the shell be 25 mm. Hence

$$h = 0.5 \text{ mm}$$

and

$$c = 25 - 0.5 = 24.5 \text{ mm}$$
.

The weights per unit area of one face, one gluelayer between face and core, and the core are respectively

$$G_f = 1.4 \text{ kg m}^{-2}$$
,  
 $G_g = 0.5 \text{ kg m}^{-2}$ ,  
 $G_c = 0.92 \text{ kg m}^{-2}$ .

The transverse shear moduli of the core  $G_{xz}$  and  $G_{\omega z}$  read

The mean radius of the cylinder is

a = 1425 mm .

When the gravitational acceleration g amounts to

$$g = 9.81 \text{ m sec}^{-2}$$

all necessary quantities can now be computed.

They are

$$m_f = G_f/g = 1.427 \times 10^{-10} \text{ kg mm}^{-3} \text{ sec}^2 ,$$
  

$$m_g = G_g/g = 0.510 \times 10^{-10} \text{ kg mm}^{-3} \text{ sec}^2 ,$$
  

$$m_c = G_c/g = 0.938 \times 10^{-10} \text{ kg mm}^{-3} \text{ sec}^2 .$$

From the eqs (B.6), (B.7) it then follows that

 $m = m_c + 2(m_a + m_f) = 4.812 \times 10^{-10} \text{ kg mm}^{-3} \text{ sec}^2$ 

and

 $I = \frac{1}{12}m_c c^2 + \frac{1}{2}(m_f + m_g)c^2 = 6.253 \times 10^{-8} \text{ kg mm}^{-1} \text{ sec}^2.$ 

The eqs (A.26), (A.27), (A.32), (A.33) finally yield

$$A = \frac{2Eh}{1 - v^2} = 7.692 \times 10^3 \text{ kg mm}^{-1} ,$$
  

$$B = \frac{Ec^2 h}{2(1 - v^2)} = 1.154 \times 10^6 \text{ kg mm} ,$$
  

$$S_{xz} = cG_{xz} = 4.729 \times 10^2 \text{ kg mm}^{-1} ,$$
  

$$S_{az} = cG_{az} = 3.651 \times 10^2 \text{ kg mm}^{-1} .$$

In case of axisymmetric waves associated with u, w,  $\gamma_{xz}$  the determinant of the coefficient matrix of the system of linear equations (5.21)–(5.23) must be zero and the propagation speed  $C_E$  can be found from this requirement by substituting the above numerical data and the desired value of the angular frequency  $\omega$ .

For non-axisymmetric waves the propagation speeds follow in a similar way from the requirement (6.8) (see also table 1). Thereby the values n = 1, 2, 3, 10 will be chosen. The latter value is representative for "large values of n" and it will be interesting to observe whether or not the high-frequency behaviour is strongly influenced.

Results for the propagation speed  $C_E$  and the corresponding half wavelength  $\lambda = \pi C_E/\omega$  are plotted in figs 9–13 and tables 3–7.

Figs 9–12 and the said tables also show a comparison with the approximate formulas (5.33), (5.37), (5.42), (5.45) in case of axisymmetric waves and (6.54), (6.60), (6.61), (6.62), (6.65), (6.66), (6.67) in case of non-axisymmetric waves.

#### 8 Discussion of results

# 8.1 Axisymmetric waves associated with u, w and $\gamma_{xz}$

The results for the propagation speeds for this kind of waves are listed in table 3 and plotted in fig. 9. As was already discussed in the theory of section 5, three types of solutions may be distinguished. They will be discussed successively. The first solution,  $C_{E_1}$ , shows a nearly constant value of 3815 m/sec (see (5.33)) in the frequency range of 0–100 cps. This value equals the propagation speed of longitudinal waves in a flat panel that is free to expand in the direction of the wave fronts and the displacements are mainly axial (u).

Above 100 cps the propagation speed  $C_{E_1}$  begins to decrease and shows a sudden steep drop at approximately 426 cps, caused by the fact that the determining stiffness changes from axial to bending and shear of the shell itself.

It may be observed that the low frequency approximation (5.33), which is given in fig. 9 by the dotted line, shows good agreement up to approximately 420 cps.

The afore mentioned value of 426 cps is the frequency, where the approximation formula (5.33) gives a zero value for the propagation speed, see eq. (5.34). In literature (ref. 4) the environment of this frequency is called the transition region. Just above this region, at approximately 600 cps, the propagation speed has a minimum of 493 m/sec and thereafter shows a continuous increase up to approximately 20,000 cps. Above this frequency the value of the pro-

pagation speed is again nearly constant and amounts to 991.3 m/sec. As may be seen from the dotted line in fig. 9, the high-frequency approximations (5.42), (5.43) show good agreement above approximately 1200 cps. Since these formulas are correct for the flat panel, when only deflections w and transverse shear of the core  $\gamma_{xz}$  are taken into account, the motion of the shell may be said to be primarily due to deflections w and transverse shear  $\gamma_{xz}$ , however, in so far that for frequencies over approximately 20,000 cps, where the propagation speed becomes nearly constant, the transverse shear  $\gamma_{xz}$  is decisive.

Finally it is remarked that for extremely high frequencies the results of the given calculations must be handled with care, because in the analysis the shear angles of the core were assumed to be constant throughout the thickness. Since this assumption is only reliable when the half wavelengths are some times greater than the thickness of the shell, i.e. 0.025 m. it is suggested that the solution  $C_{E_1}$  for the propagation speed of axisymmetric waves is reliable up to 10,000 cps approximately (see the curve for  $\lambda_1$  in fig. 9). The second solution for the propagation speed,  $C_{E_4}$ , only exists for frequencies higher than  $f_{04}$ =447 cps, see eq. (5.26), where  $C_{E_4} \rightarrow \infty$ .

Because then all points in axial (x)-direction are in phase, due to the fact that the wavelength tends to infinity, this frequency represents a two-dimensional resonance frequency, viz. that corresponding to radial expansion (w). When the frequency increases the propagation speed steeply drops to reach a nearly constant value of 3998,1 m/sec at approximately 2000 cps. This latter value may be found from eq. (5.37) and is equivalent with the propagation speed of longitudinal waves in a flat panel in case the contraction in the direction of the wavefronts is fully restrained. The motion of the shell is thus primarily due to axial displacements u and therefore the validity of this solution is not restricted to a certain high frequency.

The third solution for the propagation speed  $C_{E_5}$  only exists above  $f_{05} = 13841$  cps, see eq. (5.27), where  $C_{E_5} \rightarrow \infty$ . Again all points in axial (x)-direction are in phase, so that this frequency corresponds to a two-dimensional resonance, viz. that associated with  $\gamma_{xz}$ . Above this frequency the propagation speed again strongly decreases to the limiting value of 4295,9 m/sec (see eq. (5.46)) where the motion of the shell is again determined by  $\gamma_{xz}$ . The approximation formula (5.45), which applies to the flat panel when only deflections w and shear  $\gamma_{xz}$  are taken into account, gives fair agreement as long as the half wavelength  $\lambda_5$  remains small with respect to the cylinder radius. From the curve for  $\lambda_5$  it is finally remarked that the results for  $C_{E_5}$  from the present theory are expected to be valid up to a frequency of approximately 45000 cps.

#### 8.2 Non-axisymmetric waves

The cases considered involve 1, 2, 3 and 10 wavelengths in circumferential direction (indicated by n = 1, 2, 3, 10). Results are shown in tables 4–7 and figs. 10–13, and compared, also with those for axisymmetric waves, in fig. 14.

In the first place the solutions  $C_{E_1}$  will be discussed. In case n = 1, literature states that the cylindrical shell behaves as a uniform beam in the low frequency range (ref. 4) and indeed the approximation formula (6.54), which was derived on such basis, proves to be a fairly good approximation up to a frequency of about 100 cps (see fig. 10). When, in the low frequency range, the frequency becomes higher, the propagation speed  $C_{E_1}$  increases too, at least up to a maximum value at approximately 240 cps. Thereafter a strong drop occurs, due to the fact that the leading stiffness changes from that of the uniform beam to the bending- and shear stiffness ( $S_{xz}$ ) of the shell itself, and at approximately 600 cps the propagation speed becomes equal to that of axisymmetric waves (see fig. 14). This latter phenomenon can also be observed in the cases n=2, n=3. In case n=10, however, it only occurs for much higher frequencies, viz. over 5000 cps approximately, because then the wavelength in circumferential direction becomes of the order of the wavelength in axial direction (or even smaller for large values of n) so that the transverse shear angle  $\gamma_{oxz}$  gains in importance. Figs 10–13 show that the flat panel, where only deflections w and shear angles  $\gamma_{oxz}$ ,  $\gamma_{xxz}$ are taken into account, is a good high frequency approximation (see eqs (6.65), (6.60)). The low frequency behaviour in cases n=2, 3, 10 is quite different from the cases n=0 and n=1. In fact, in these cases the solution for  $C_{E_1}$  only exists above a certain frequency, where the propagation speed tends to infinity (in the figs 11–13 this behaviour is indicated by a dotted line because the required data were not calculated. The asymptotes are indicated).

These frequencies, which read

$$f_{21} = 10.3 \text{ cps}, \quad n = 2 \quad ,$$
  

$$f_{31} = 28.9 \text{ cps}, \quad n = 3 \quad ,$$
  

$$f_{10.0} = 351 \text{ cps}, \quad n = 10 \quad ,$$

represent two-dimensional resonance frequencies, which are mainly governed by deflections w. When the frequency increases the propagation speed steeply drops down to a minimum value, followed (at least in the cases n=2, 3) by a maximum, just like in case n=1. With increasing n-value there is a tendency for this maximum to flatten and in the case n=10 it is completely vanished. For the cases n=2, 3, 10 no low-frequency approximation for  $C_{E_1}$  was derived. For the same reasons as were mentioned in case of axisymmetric waves, the validity of the solutions  $C_{E_1}$ 

again extends to a frequency of about 10,000 cps. Concerning the solutions  $C_{E_4}$  and  $C_{E_5}$  it can be remarked (see fig. 14) that they make one family with those found for axisymmetrical waves, because they approach the same value when  $\omega \to \infty$  (compare eqs (5.27), (6.58) and (5.46), (6.61)), while the motion of the shell is also governed by the same distortions, viz. u and  $\gamma_{\varphi z}$ .

Their vertical asymptotes (two-dimensional resonance frequencies), however, are different, viz. these frequencies increase with increasing *n*-value. In case of  $C_{E_5}$ , however, this tendency is very weak and the propagation speed is nearly independent of *n*. For axisymmetric waves the motion of the shell, when the propagation speed  $C_{E_4}$  tends to infinity, is governed by deflections *w*, whereas for non-axisymmetric waves the main distortions are tangential displacements *v*. This is understandable because in both cases the normal stiffness of the shell in tangential direction is decisive. At  $C_{E_4}$  the main shell-motion for infinite propagation speeds remains transverse shear  $\gamma_{xz}$ .

The approximation formulas (6.66), (6.58) for  $C_{E_4}$  and (6.65), (6.61) for  $C_{E_5}$  prove again to be fairly good approximations as long as the wavelengths in x-direction are some times smaller than the cylinder radius, especially for  $C_{E_5}$  where the difference was beyond drawing in the range of values calculated. The approximations (6.19) for  $f_{14}$ , (6.68) for  $f_{n4}$   $(n \ge 1)$ , (6.17) for  $f_{n4}$   $(n \ge 1)$  and (6.14) for  $f_{n5}$   $(n \ge 1)$  turned out to be fairly good (see tables 4–7), only with the exception that the approximation (6.68) for  $f_{n4}$  can indeed only be used for  $n \ge 1$  as was to be expected, because the eqs (6.17) and (6.68) are identical.

From the curves for the half wavelengths and the leading distortions for  $\omega \rightarrow 0$  it may be concluded that the validity of the present theory towards high frequencies is not restricted in case of  $C_{E_4}$ , but that for  $C_{E_5}$  a boundary of 45000 cps approximately is indicated.

Finally the solutions  $C_{E_2}$  and  $C_{E_3}$  will be discussed. The solution  $C_{E_2}$  only exists for frequencies higher than  $f_{n,2}$  (n=1, 2, 3, 10), see eqs (6.13), (6.69), where the propagation speed tends to infinity and the motion of the shell is determined by axial displacements u. The value of  $f_{n,2}$  may be seen to increase with increasing *n*-value.

For frequencies little higher than  $f_{n,2}$  the propagation speed drops down to a nearly constant value (2365,3 m/sec for  $\omega \rightarrow \infty$ , see eq. (6.59)) which is independent of *n*. The shell motion is then governed by tangential displacements *v*, in fact a torsion of the cylinder, and the validity of the present theory is not affected in the high frequency range. As long as the wavelength in x-direction is some times smaller than the cylinder radius the eq. (6.67) proves to be a good approximation.

The behaviour of the solutions  $C_{E_3}$  shows much conformity with that of  $C_{E_5}$  since they also are nearly independent of *n* (see fig. 14). On the lower-frequency side their existence is bounded by two-dimensional resonance frequencies (infinite propagation speeds) which are related to deflection w and transverse shear  $\gamma_{\varphi z}$ . According to table 4, the value of  $f_{1,3}$  is approximated by eq. (6.19) in a suitable manner. Their limiting value for  $\omega \rightarrow \infty$  is 2541,5 m/sec, see (6.62). Also here the equation (6.65) proves to be a good approximation (the difference between approximate and exact values could not clearly be drawn for the values calculated).

Concerning the validity of the present theory in this case it may be said that it is not decided by the wavelength in x-direction, but by the value of n (the number of wavelengths in circumferential direction).

#### 9 References

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# APPENDIX A

# Derivation of forces and moments caused by elastic effects

From eqs (2.7), (2.8), the axial- and circumferential displacements in the faces may be given as (see figs 2, 3)

$$u^{u,l} = u \pm \left(\gamma_{xz} - \frac{\partial w}{\partial x}\right) \frac{c}{2} \quad (A.1) \quad \text{and} \quad v^{u,l} = v \pm \left(\gamma_{\varphi z} - \frac{\partial w}{a \partial \varphi} + \frac{v}{a}\right) \frac{c}{2}, \tag{A.2}$$

where the superfices "u" and "l" respectively indicate the upper and lower face.

The corresponding normal- and shear strains then read

.

,

$$\varepsilon_{x}^{u,l} = \frac{\partial u}{\partial x} \pm \left(\frac{\partial \gamma_{xz}}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) \frac{c}{2},$$

$$\varepsilon_{\varphi}^{u,l} = \frac{1}{a \pm \frac{c}{2}} \left\{ \frac{\partial v}{\partial \varphi} \pm \left(\frac{\partial \gamma_{\varphi z}}{\partial \varphi} - \frac{\partial^2 w}{a \partial \varphi^2} + \frac{\partial v}{a \partial \varphi}\right) \frac{c}{2} + w \right\},$$

$$\gamma_{x\varphi}^{u,l} = \frac{1}{a \pm \frac{c}{2}} \left\{ \frac{\partial u}{\partial \varphi} \pm \left(\frac{\partial \gamma_{xz}}{\partial \varphi} - \frac{\partial^2 w}{\partial x \partial \varphi}\right) \frac{c}{2} \right\} + \frac{\partial v}{\partial x} \pm \left(\frac{\partial \gamma_{\varphi z}}{\partial x} - \frac{\partial^2 w}{a \partial x \partial \varphi} + \frac{\partial v}{a \partial x}\right) \frac{c}{2}.$$
(A.3)

Since  $a \ge c$  the latter two expressions may be approximated by

$$\varepsilon_{\varphi}^{u,l} = \left\{ \left(1 - \frac{c^2}{4a^2}\right) \frac{\partial v}{a \,\partial \varphi} + \frac{w}{a} + \frac{c^2}{4a} \frac{\partial^2 w}{a^2 \,\partial \varphi^2} - \frac{c^2}{4a} \frac{\partial \gamma_{\varphi z}}{a \,\partial \varphi} \right\} \pm \left\{ \frac{\partial \gamma_{\varphi z}}{a \,\partial \varphi} - \frac{\partial^2 w}{a^2 \,\partial \varphi^2} - \frac{w}{a^2} \right\} \frac{c}{2}, \tag{A.4}$$

$$\gamma_{x\varphi}^{u,l} = \left\{ \frac{\partial u}{a \partial \varphi} + \frac{\partial v}{\partial x} + \frac{c^2}{4a} \frac{\partial^2 w}{a \partial x \partial \varphi} - \frac{c^2}{4a} \frac{\partial \gamma_{xz}}{a \partial \varphi} \right\} \pm \left\{ -2 \frac{\partial^2 w}{a \partial x \partial \varphi} + \frac{\partial \gamma_{xz}}{a \partial \varphi} + \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} - \frac{1}{a} \frac{\partial u}{a \partial \varphi} + \frac{1}{a} \frac{\partial v}{\partial x} \right\} \frac{c}{2}.$$
 (A.5)

Using the strain-stress relations for isotropic plates the stresses in the face become from (A.3)-(A.5)

$$\sigma_{x}^{u,l} = \frac{E}{1-v^2} \left\{ \frac{\partial u}{\partial x} + v \left( 1 - \frac{c^2}{4a^2} \right) \frac{\partial v}{a \partial \varphi} + v \frac{w}{a} + v \frac{c^2}{4a} \frac{\partial^2 w}{a^2 \partial \varphi^2} - v \frac{c^2}{4a} \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} \right\} \pm \frac{Ec}{2(1-v^2)} \left\{ -\frac{\partial^2 w}{\partial x^2} - v \frac{\partial^2 w}{a^2 \partial \varphi^2} + \frac{\partial \gamma_{xz}}{\partial x} + \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} - v \frac{w}{a^2} \right\},$$

$$\sigma_{\varphi}^{u,l} = \frac{E}{1-v^2} \left\{ v \frac{\partial u}{\partial x} + \left( 1 - \frac{c^2}{4a^2} \right) \frac{\partial v}{a \partial \varphi} + \frac{w}{a} + \frac{c^2}{4a} \frac{\partial^2 w}{a^2 \partial \varphi^2} - \frac{c^2}{4a} \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} \right\} \pm \frac{Ec}{2(1-v^2)} \left\{ -v \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{a^2 \partial \varphi^2} + v \frac{\partial \gamma_{xz}}{\partial x} + \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} - \frac{w}{a^2} \right\},$$

$$(A.6)$$

$$\tau_{x\varphi}^{u,l} = \frac{E}{1-v^2} \left\{ \frac{1-v}{2} \frac{\partial u}{a \partial \varphi} + \frac{1-v}{2} \frac{\partial v}{\partial x} + \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial^2 w}{a \partial x \partial \varphi} - \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial \gamma_{xz}}{a \partial \varphi} \right\} \pm \frac{Ec}{2(1-v^2)} \left\{ -\frac{1-v}{2} \frac{2}{a \partial \varphi} + \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial^2 w}{a \partial x \partial \varphi} - \frac{1-v}{2} \frac{1-v}{2} \frac{1-v}{a \partial \varphi} + \frac{1-v}{2} \frac{1-v}{a \partial z} \frac{1-v}{\partial x} \right\}.$$

$$(A.6)$$

The resultant forces and moments (fig. 4) may now be obtained from the relations

$$N_{x} = \sigma_{x}^{u} \left(1 + \frac{c}{2a}\right) h + \sigma_{x}^{l} \left(1 - \frac{c}{2a}\right) h, \qquad T_{x\varphi} = \tau_{x\varphi}^{u} \left(1 + \frac{c}{2a}\right) h + \tau_{x\varphi}^{l} \left(1 - \frac{c}{2a}\right) h, \\ N_{\varphi} = \sigma_{\varphi}^{u} h + \sigma_{\varphi}^{l} h, \qquad T_{\varphi x} = \tau_{x\varphi}^{u} h + \tau_{x\varphi}^{l} h, \\ M_{x} = \sigma_{x}^{u} \left(1 + \frac{c}{2a}\right) h \frac{c}{2} - \sigma_{x}^{l} \left(1 - \frac{c}{2a}\right) h \frac{c}{2}, \qquad M_{x\varphi} = \tau_{x\varphi}^{u} \left(1 + \frac{c}{2a}\right) h \frac{c}{2} - \tau_{x\varphi}^{l} \left(1 - \frac{c}{2a}\right) h \frac{c}{2}, \\ M_{\varphi} = \sigma_{\varphi}^{u} h \frac{c}{2} - \sigma_{\varphi}^{l} h \frac{c}{2}, \qquad M_{\varphi x} = \tau_{x\varphi}^{u} h \frac{c}{2} - \tau_{x\varphi}^{l} h \frac{c}{2}, \qquad (A.9)$$

which deliver after substitution of (A.6)-(A.8)

.

.

$$N_x = \frac{2Eh}{1-v^2} \left\{ \frac{\partial u}{\partial x} + v \left( 1 - \frac{c^2}{4a^2} \right) \frac{\partial v}{a \partial \varphi} + v \left( 1 - \frac{c^2}{4a^2} \right) \frac{w}{a} - \frac{c^2}{4a} \frac{\partial^2 w}{\partial x^2} + \frac{c^2}{4a} \frac{\partial \gamma_{xz}}{\partial x} \right\},$$
(A.10)

$$N_{\varphi} = \frac{2Eh}{1-v^2} \left\{ v \frac{\partial u}{\partial x} + \left(1 - \frac{c^2}{4a^2}\right) \frac{\partial v}{a \partial \varphi} + \frac{w}{a} + \frac{c^2}{4a} \frac{\partial^2 w}{a^2 \partial \varphi^2} - \frac{c^2}{4a} \frac{\partial \dot{\gamma}_{\varphi z}}{a \partial \varphi} \right\},\tag{A.11}$$

$$T_{x\varphi} = \frac{2Eh}{1-v^2} \left\{ \frac{1-v}{2} \left( 1 - \frac{c^2}{4a^2} \right) \frac{\partial u}{a \partial \varphi} + \frac{1-v}{2} \left( 1 + \frac{c^2}{4a^2} \right) \frac{\partial v}{\partial x} - \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial^2 w}{a \partial x \partial \varphi} + \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial \gamma_{\varphi z}}{\partial x} \right\}, \quad (A.12)$$

$$T_{\varphi x} = \frac{2Eh}{1-v^2} \left\{ \frac{1-v}{2} \frac{\partial u}{a \partial \varphi} + \frac{1-v}{2} \frac{\partial v}{\partial x} + \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial^2 w}{a \partial x \partial \varphi} - \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial \gamma_{xz}}{a \partial \varphi} \right\},$$
(A.13)

$$M_{x} = \frac{Ec^{2}h}{2(1-v^{2})} \left\{ -\frac{\partial^{2}w}{\partial x^{2}} - v\left(1 - \frac{c^{2}}{4a^{2}}\right)\frac{\partial^{2}w}{a^{2}\partial\varphi^{2}} + \frac{\partial\gamma_{xz}}{\partial x} + v\left(1 - \frac{c^{2}}{4a^{2}}\right)\frac{\partial\gamma_{\varphi z}}{a\partial\varphi} + \frac{1}{a}\frac{\partial u}{\partial x} + v\left(1 - \frac{c^{2}}{4a^{2}}\right)\frac{1}{a}\frac{\partial v}{a\partial\varphi} \right\}, \quad (A.14)$$

$$M_{\varphi} = \frac{Ec^2 h}{2(1-v^2)} \left\{ -v \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{a^2 \partial \varphi^2} + v \frac{\partial \gamma_{xz}}{\partial x} + \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} - \frac{w}{a^2} \right\},$$
(A.15)

$$M_{x\varphi} = \frac{Ec^2 h}{2(1-v_{\perp}^2)} \left\{ -\frac{1-v}{2} \left( 2 - \frac{c^2}{4a^2} \right) \frac{\partial^2 w}{a \partial x \partial \varphi} + \frac{1-v}{2} \left( 1 - \frac{c^2}{4a^2} \right) \frac{\partial \gamma_{xz}}{a \partial \varphi} + \frac{1-v}{2} \frac{\partial \gamma_{\varphi z}}{\partial x} + \frac{1-v}{2} \frac{2}{a} \frac{\partial v}{\partial x} \right\}, \quad (A.16)$$

$$M_{\varphi x} = \frac{Ec^2h}{2(1-v^2)} \left\{ -\frac{1-v}{2} 2\frac{\partial^2 w}{a\partial x \partial \varphi} + \frac{1-v}{2} \frac{\partial \gamma_{xz}}{a\partial \varphi} + \frac{1-v}{2} \frac{\partial \gamma_{\varphi z}}{\partial x} - \frac{1-v}{2} \frac{1}{a} \frac{\partial u}{a\partial \varphi} + \frac{1-v}{2} \frac{1}{a} \frac{\partial v}{\partial x} \right\}.$$
 (A.17)

By inserting (A.12), (A.13) and (A.17) the equilibrium condition (4.6) proves to be always satisfied and thus needs no further attention. Because  $a \ge c$  the expression (A.10)–(A.17) may be simplified into

$$N_x = A \left\{ \frac{\partial u}{\partial x} + v \frac{\partial v}{a \partial \varphi} + v \frac{w}{a} - \frac{c^2}{4a} \frac{\partial^2 w}{\partial x^2} + \frac{c^2}{4a} \frac{\partial \gamma_{xz}}{\partial x} \right\},$$
(A.18)

$$N_{\varphi} = A \left\{ v \frac{\partial u}{\partial x} + \frac{\partial v}{a \partial \varphi} + \frac{w}{a} + \frac{c^2}{4a} \frac{\partial^2 w}{a^2 \partial \varphi^2} - \frac{c^2}{4a} \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} \right\},\tag{A.19}$$

$$T_{x\varphi} = A \left\{ \frac{1-\nu}{2} \frac{\partial u}{\partial \phi} + \frac{1-\nu}{2} \frac{\partial v}{\partial x} - \frac{1-\nu}{2} \frac{c^2}{4a} \frac{\partial^2 w}{\partial x \partial \phi} + \frac{1-\nu}{2} \frac{c^2}{4a} \frac{\partial \gamma_{\varphi z}}{\partial x} \right\},\tag{A.20}$$

$$T_{\varphi x} = A \left\{ \frac{1-v}{2} \frac{\partial u}{\partial \phi} + \frac{1-v}{2} \frac{\partial v}{\partial x} + \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial^2 w}{\partial x \partial \phi} - \frac{1-v}{2} \frac{c^2}{4a} \frac{\partial \gamma_{xz}}{\partial \phi \phi} \right\},\tag{A.21}$$

$$M_{x} = B \left\{ -\frac{\partial^{2} w}{\partial x^{2}} - v \frac{\partial^{2} w}{a^{2} \partial \varphi^{2}} + \frac{\partial \gamma_{xz}}{\partial x} + v \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} + \frac{1}{a} \frac{\partial u}{\partial x} + \frac{v}{a} \frac{\partial v}{a \partial \varphi} \right\},$$
(A.22)

$$M_{\varphi} = B \left\{ -\nu \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{a^2 \partial \varphi^2} + \nu \frac{\partial \gamma_{xz}}{\partial x} + \frac{\partial \gamma_{\varphi z}}{a \partial \varphi} - \frac{w}{a^2} \right\},$$
(A.23)

$$M_{x\varphi} = B \left\{ -\frac{1-\nu}{2} 2 \frac{\partial^2 w}{a \partial x \partial \varphi} + \frac{1-\nu}{2} \frac{\partial \gamma_{xz}}{a \partial \varphi} + \frac{1-\nu}{2} \frac{\partial \gamma_{\varphi z}}{\partial x} + \frac{1-\nu}{2} \frac{2}{a} \frac{\partial v}{\partial x} \right\},\tag{A.24}$$

$$M_{\varphi x} = B \left\{ -\frac{1-\nu}{2} 2 \frac{\partial^2 w}{a \partial x \partial \varphi} + \frac{1-\nu}{2} \frac{\partial \gamma_{xz}}{a \partial \varphi} + \frac{1-\nu}{2} \frac{\partial \gamma_{\varphi z}}{\partial x} - \frac{1-\nu}{2} \frac{1}{a} \frac{\partial u}{a \partial \varphi} + \frac{1-\nu}{2} \frac{1}{a} \frac{\partial v}{\partial x} \right\}$$
(A.25)

with

$$A = \frac{2Eh}{1 - v^2} \tag{A.26}$$

and

$$B = \frac{Ec^2 h}{2(1-v^2)}.$$
 (A.27)

Further simplifications will not be made, because these would depend on the wavelengths in x- and  $\varphi$ -direction. Within the limit of approximations used the above expressions are comparable with those derived by Naghdi and Cooper (ref. 2) for the isotropic shell.

When all terms that vanish for  $a \to \infty$  are neglected, except for the two with  $\frac{w}{a}$  in  $N_x$  and  $N_{\varphi}$ , the expressions are the same as those obtained by Stein and Mayers (ref. 3) for curved sandwich plates.

From section 2 the shear angles of the core read respectively

$$\gamma_{xz}(z) = \gamma_{xz} \tag{A.28}$$

and

$$\gamma_{\varphi z}(z) = \gamma_{\varphi z} \left( 1 - \frac{z}{a} \right). \tag{A.29}$$

With the introduction of the shear moduli  $G_{xz}$  and  $G_{\varphi z}$  of the core, the transverse shear forces thus become

$$Q_{x} = G_{xz} \int_{-c/2}^{c/2} \gamma_{xz} \left( 1 + \frac{z}{a} \right) dz = S_{xz} \gamma_{xz} , \qquad (A.30)$$

$$Q_{\varphi} = G_{\varphi z} \int_{-c/2}^{c/2} \gamma_{\varphi z} \left( 1 - \frac{z}{a} \right) \mathrm{d}z = S_{\varphi z} \gamma_{\varphi z} , \qquad (A.31)$$

where

$$S_{xz} = cG_{xz} \tag{A.32}$$

and

$$S_{\varphi z} = cG_{\varphi z} \tag{A.33}$$

are the shear stiffnesses of the core in the xz- and  $\varphi$ z-plane respectively.

#### APPENDIX B

#### Derivation of inertia forces and moments

When  $m_c$ ,  $m_f$  and  $m_g$  respectively represent the masses per unit area of the core, one face and one glue layer between core and face, the inertia loading (fig. 4) reads (see the expressions for the distortions in section 2)

$$X = -\frac{m_c}{c} \int_{-c/2}^{c/2} \left\{ \ddot{u} + \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right) z \right\} \left( 1 + \frac{z}{a} \right) dz + -(m_f + m_g) \left\{ \ddot{u} + \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right) \frac{c}{2} \right\} \left( 1 + \frac{c}{2a} \right) + -(m_f + m_g) \left\{ \ddot{u} - \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right) \frac{c}{2} \right\} \left( 1 - \frac{c}{2a} \right) \right\}$$
$$= - \left\{ m_c + 2(m_f + m_g) \right\} \ddot{u} - \frac{1}{a} \left\{ \frac{1}{12} m_c c^2 + \frac{1}{2} (m_f + m_g) c^2 \right\} \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right),$$
(B.1)

$$Y = -\frac{m_c}{c} \int_{-c/2}^{c/2} \left\{ \ddot{v} + \left( \ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{\partial \phi} + \frac{\ddot{v}}{a} \right) z \right\} \left( 1 + \frac{z}{a} \right) dz + \\ - (m_f + m_g) \left\{ \ddot{v} + \left( \ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{a \partial \phi} + \frac{\ddot{v}}{a} \right) \frac{c}{2} \right\} \left( 1 + \frac{c}{2a} \right) + \\ - (m_f + m_g) \left\{ \ddot{v} - \left( \ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{a \partial \phi} + \frac{\ddot{v}}{a} \right) \frac{c}{2} \right\} \left( 1 - \frac{c}{2a} \right) \right\} \\ = - \left\{ m_c + 2(m_f + m_g) \right\} \ddot{v} - \frac{1}{a} \left\{ \frac{1}{12} m_c c^2 + \frac{1}{2} (m_f + m_g) c^2 \right\} \left( \ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{a \partial \phi} + \frac{\ddot{v}}{a} \right),$$
(B.2)

$$Z = -\frac{m_c}{c} \int_{-c/2}^{c/2} \vec{w} \left(1 + \frac{z}{a}\right) dz + -(m_f + m_g) \vec{w} \left(1 + \frac{c}{2a}\right) + -(m_f + m_g) \vec{w} \left(1 - \frac{c}{2a}\right) = = -\{m_c + 2(m_f + m_g)\} \vec{w}, \qquad (B.3)$$

$$\mathbf{m}_{\mathbf{x}} = \frac{m_{c}}{c} \int_{-c/2}^{c/2} \left\{ \ddot{v} + \left( \ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{a \partial \varphi} + \frac{\ddot{v}}{a} \right) z \right\} \left( 1 + \frac{z}{a} \right) z \, \mathrm{d}z + \\ + \left( m_{f} + m_{g} \right) \left\{ \ddot{v} - \left( \ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{a \partial \varphi} + \frac{\ddot{v}}{a} \right) \frac{c}{2} \right\} \left( 1 + \frac{c}{2a} \right) \frac{c}{2} + \\ - \left( m_{f} + m_{g} \right) \left\{ \ddot{v} - \left( \ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{a \partial \varphi} + \frac{\ddot{v}}{a} \right) \frac{c}{2} \right\} \left( 1 - \frac{c}{2a} \right) \frac{c}{2} \\ = \left\{ \frac{1}{12} m_{c} c^{2} + \frac{1}{2} (m_{f} + m_{g}) c^{2} \right\} \left\{ \frac{\ddot{v}}{a} + \left( \ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{a \partial \varphi} + \frac{\ddot{v}}{a} \right) \right\},$$
(B.4)

$$\mathbf{m}_{\varphi} = -\frac{m_c}{c} \int_{-c/2}^{c/2} \left\{ \ddot{u} + \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right) z \right\} \left( 1 + \frac{z}{a} \right) z \, \mathrm{d}z$$

$$- (m_f + m_g) \left\{ \ddot{u} + \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right) \frac{c}{2} \right\} \left( 1 + \frac{c}{2a} \right) \frac{c}{2} + (m_f + m_g) \left\{ \ddot{u} - \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right) \frac{c}{2} \right\} \left( 1 - \frac{c}{2a} \right) \frac{c}{2}$$

$$= - \left\{ \frac{1}{12} m_c c^2 + \frac{1}{2} (m_f + m_g) c^2 \right\} \left\{ \frac{\ddot{u}}{a} + \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right) \right\}.$$
(B.5)

Since

$$\{m_{c}+2(m_{f}+m_{g})\}=m$$
(B.6)

equals the total mass and there is put

$$\frac{1}{12}m_c c^2 + \frac{1}{2}(m_f + m_g)c^2 = I \tag{B.7}$$

which equals the moment of inertia with respect to rotation, the expressions (B.1)-(B.5) may be rewritten in the form

$$X = -m\ddot{u} - \frac{1}{a} \left( \ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} \right), \tag{B.8}$$

$$Y = -m\ddot{v} - \frac{I}{a}\left(\ddot{\gamma}_{\varphi z} - \frac{\partial \ddot{w}}{a \partial \varphi} + \frac{\ddot{v}}{a}\right), \tag{B.9}$$

$$Z = -m\ddot{w}, \tag{B.10}$$

$$\mathbf{m}_{\mathbf{x}} = I\left(\ddot{\gamma}_{\varphi z} - \frac{\partial H}{\partial \phi} + 2\frac{\partial}{a}\right), \tag{B.11}$$

$$\mathbf{m}_{\varphi} = -I\left(\ddot{\gamma}_{xz} - \frac{\partial \ddot{w}}{\partial x} + \frac{\ddot{u}}{a}\right). \tag{B.12}$$

# APPENDIX C

# Axisymmetric waves associated with tangential displacements v and transverse shear of the core $\gamma_{\varphi z}$

In this case the only two equilibrium conditions that are to be satisfied read from (4.1)-(4.6)

$$\frac{\partial T_{x\varphi}}{\partial x} + \frac{Q_{\varphi}}{a} + Y = 0, \qquad (C.1)$$

$$\frac{\partial M_{x\phi}}{\partial x} - Q_{\phi} - \mathbf{m}_{x} = 0.$$
 (C.2)

The forces and moments involved become from sections 3.1, 3.2

$$T_{x\varphi} = A\left(\frac{1-\nu}{2}\frac{\partial\nu}{\partial x} + \frac{1-\nu}{2}\frac{c^2}{4a}\frac{\partial\gamma_{\varphi z}}{\partial x}\right),\tag{C.3}$$

$$Q_{\varphi} = S_{\varphi z} \gamma_{\varphi z} , \qquad (C.4)$$

$$M_{x\varphi} = B\left(\frac{1-v}{2}\frac{\partial\gamma_{\varphi z}}{\partial x} + \frac{1-v}{2}\frac{2}{a}\frac{\partial v}{\partial x}\right),\tag{C.5}$$

$$Y = -m\ddot{v} - \frac{I}{a}\left(\ddot{\gamma}_{\varphi z} + \frac{\ddot{v}}{a}\right), \qquad (C.6)$$

$$\mathbf{m}_{\mathbf{x}} = I\left(\ddot{\mathbf{y}}_{\varphi z} + 2\frac{\ddot{v}}{a}\right). \tag{C.7}$$

Substitution gives the two coupled differential equations

$$\left(\frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} - \frac{m+I/a^2}{A}\frac{\partial^2}{\partial t^2}\right)\nu + \left(\frac{1-\nu}{2}\frac{c^2}{4a}\frac{\partial^2}{\partial x^2} + \frac{S_{\varphi z}}{A}\frac{1}{a} - \frac{I}{A}\frac{1}{a}\frac{\partial^2}{\partial t^2}\right)\gamma_{\varphi z} = 0, \quad (C.8)$$

$$\left(\frac{1-v}{2}\frac{2}{a}\frac{\partial^2}{\partial x^2} - 2\frac{I}{B}\frac{1}{a}\frac{\partial^2}{\partial t^2}\right)v + \left(\frac{1-v}{2}\frac{\partial^2}{\partial x^2} - \frac{S_{\varphi z}}{B} - \frac{I}{B}\frac{\partial^2}{\partial t^2}\right)\gamma_{\varphi z} = 0,$$
(C.9)

which can be solved by putting

$$v = V \sin\left(\frac{\omega}{C_E}x - \omega t\right),$$

$$\gamma_{\varphi z} = \Gamma_{\varphi z} \sin\left(\frac{\omega}{C_E}x - \omega t\right).$$
(C.10)

This leads to two homogeneous linear equations

$$\left\| \left\{ -\frac{1-\nu}{2} \left( \frac{\omega}{C_E} \right)^2 + \frac{m+I/a^2}{A} \omega^2 \right\} \quad \left\{ -\frac{1-\nu}{2} \frac{c^2}{4a} \left( \frac{\omega}{C_E} \right)^2 + \frac{S_{\varphi z}}{A} \frac{1}{a} + \frac{I}{A} \frac{1}{a} \omega^2 \right\} \right\| \quad \left\| \nu \right\|_{\varphi z} = 0,$$

$$\left\{ -\frac{1-\nu}{2} \frac{2}{a} \left( \frac{\omega}{C_E} \right)^2 + 2\frac{I}{B} \frac{1}{a} \omega^2 \right\} \quad \left\{ -\frac{1-\nu}{2} \left( \frac{\omega}{C_E} \right)^2 - \frac{S_{\varphi z}}{B} + \frac{I}{B} \omega^2 \right\} \quad \left\| \Gamma_{\varphi z} \right\| = 0,$$

$$(C.11)$$

which only have a solution if the determinant of the coefficient matrix vanishes. The resulting characteristic equation is

$$\left(\frac{1-v}{2}\right)^{2} \left(1-\frac{c^{2}}{2a^{2}}\right) \left(\frac{\omega}{C_{E}}\right)^{4} + \left\{\frac{1-v}{2}\frac{S_{\varphi z}}{B}\left(1+\frac{2B}{a^{2}A}\right)-\frac{1-v}{2}\frac{I}{B}\omega^{2}\left(1-\frac{2B}{a^{2}A}-\frac{c^{2}}{2a^{2}}\right)-\frac{1-v}{2}\frac{m+I/a^{2}}{A}\omega^{2}\right\} \left(\frac{\omega}{C_{E}}\right)^{2} + \left\{-\frac{m+I/a^{2}}{A}\frac{S_{\varphi z}}{B}\omega^{2}-2\frac{I}{B}\frac{S_{\varphi z}}{B}\frac{B}{Aa^{2}}\omega^{2}+\frac{m+I/a^{2}}{A}\frac{I}{B}\omega^{4}-\frac{I}{B}\frac{I}{B}\frac{2B}{Aa^{2}}\omega^{4}\right\}=0.$$
(C.12)

Now  $\frac{B}{A}$  and  $\frac{1}{m}$  are both of order  $c^2$ . Hence, if  $c \ll a$  the eq. (C.12) reduces to

$$\left\{-\frac{1-\nu}{2}\left(\frac{\omega}{C_E}\right)^2 + \frac{m}{A}\omega^2\right\} \left\{-\frac{1-\nu}{2}\left(\frac{\omega}{C_E}\right)^2 - \frac{S_{\varphi z}}{B} + \frac{I}{B}\omega^2\right\} = 0, \qquad (C.13)$$

which is the characteristic equation for the flat panel where v and  $\gamma_{\varphi z}$  are uncoupled, see eq. (C.11). For shallow shells the coupling between v and  $\gamma_{\varphi z}$  is thus weak and the solutions for  $C_E$  read approximately

$$C_{\mathbf{E}_2} = \sqrt{\frac{1-\nu}{2}} \frac{A}{m},\tag{C.14}$$

$$C_{E_{3}} = \sqrt{\frac{\frac{1-\nu}{2}\frac{B}{I}}{1-\frac{S_{\varphi z}}{I}\frac{1}{\omega^{2}}}}.$$
 (C.15)

The solution  $C_{E_3}$  implies nearly pure shear  $\gamma_{\varphi z}$  and exists only for frequencies higher than

$$\omega_{03} = \sqrt{\frac{S_{\varphi z}}{l}}, \qquad (C.16)$$

which is the two-dimensional resonance frequency for the case that the inner and outer face of the shell possess an opposite rotation.

For very high frequencies the solution reduces to the constant value

$$C_{E_{3_{m-\infty}}} = \sqrt{\frac{1-\nu}{2}} \frac{B}{I}.$$
(C.17)

With the solution  $C_{E_2}$  the displacements are almost pure tangential (v), i.e. the waves are of torsion type. Of course the value of  $C_{E_2}$  equals that of shear waves in the plane of a flat sandwich panel.

In fig. 15 the solutions  $C_{E_2}$  and  $C_{E_3}$  are illustrated.
f	$C_{E_1}$ appr.	C _E , exact	$\lambda_1$ exact	$C_{E_4}$ appr.	C _{E4} exact	λ ₄ exact	C _E , appr.	C _{Es} exact	ر exact
0	3815	3815							· <u> </u>
18,75	3813.6	3813.9	101,71						
37,50	3812,6	3812,9	50,84						
75	3808,5	3808,8	25.39						
150	3789.9	3790,2	12,63					•	
225	3749,2	3749,6	8.33						
300	3655,5	3655,8	6.09		6		341	341	
350	3500,7	3501,0	5.00	447	44		138	138	
400	2950,3	2951,4	3,69	<u> </u>	1		=		
410	2611,5	2614.1	3,19	at	at		at	at	
420	1874.1	1899,1	2,26	8	8		8	8	
430		903,81	1.05	11	1		<u>"</u>		
440		712,00	0.809	$C_E$	Č		$C_{E}$	CE	
450		638,51	0.709		10461	11,62			
500	376,5	527,41	0_527		4654.2	4,054			
550	393,3	500,64	0.455		4332,6	3,939			
600	409,1	493,00	0.411		4216.0	3,513			
700	438,4	496,72	0,355		4120.1	2,943			
1200	551,6	573,43	0.239		4027,3	1.678			
2400	711,9	719,37	0,150		4004.9	0.834			
4800	855.4	857,95	0,089		4000,0	0.417			
9600	943,0	943,79	0,049		3998.9	0,208			
19200	977,6	977,80	0.025		3998,6	0,104	6285,8	6256,6	0,163
38400	987.8	987,83	0.013		3998,5	0.052	4622,2	4610,6	0,060
80	991.3	991.3		3998,1	3998,1		4295.9	4295.9	

For approximations see, for  $C_{E_1}$  eqs (5.31), (5.33), (5.42), (5.43) , for  $C_{E_4}$  eqs (5.26), (5.37)

, for  $C_{E_5}$  eqs (5.27), (5.45), (5.46)

## TABLE 4

Propagation speed and half wavelength of non-axisymmetric elastic waves (one wave in circumferential direction. n = 1) in an infinitely long, sandwich-type. cylinder of circular cross-section.

f	$C_{E_1}$ appr.	$C_{E_1}$ exact	41 exact	$C_{E_2}$ appr.	C _{E2} exact	λ ₂ exact	C _{E4} appr.	C _{E4} exact	λ ₄ exact	C _{E3} appr.	$\overline{C}_{E_3}$ exact	λ ₃ exact	$C_{E_s}$ appr.	$C_{E}$ , exact	λ ₅ exact
18.75	671.1	644.69	17,19												
37.50	908,7	875.22	11.67	12	5										
75	1184,9	1144.7	7,63	8 8	568										
150	1453.5	1395,6	4.65	j "	j _ ∎		5)								
225	1578	1467.4	3,26	ີສີບ	at C		631								
300		1402.6	2,34	4991.5	7757.1	12,93	5)	.46							
350		1248,1	1,78	3605,9	5365.1	7,66	446	631				ч <b>-</b> .			
400		909.21	1,14	3150,1	4478,0	5.60	Ű.	_					E		
450		595.53	0.662	2921,8	3942,1	4,38	atj	at			,		844		
500	377.8	521.30	0,521	2785,9	3538.1	3.54	8	. 8				£	(13		
550	394,5	499,06	0,454	2696.8	3216,4	2.92	1	١		11	4		4	11	
600	410,3	492.73	0,411	2634,4	2979.7	2,48	Č,	C ^r		12]	121		138	138	
700	439,4	497.21	0,355	2554,2	2712,5	1.94	5191.7	6924.8	4,946	Ĩ	1		Ľ	"	
1200	552,3	574.06	0,239	2424,8	2438.0	1.016	4307,5	4372,6	1.822	at	at )		ੰਬ	, at	
2400	712.3	719.72	0.150	2379,8	2380.7	0.496	4069.2	4077,5	0.849	8	8		8	8	
4800	855,5	858.09	0.089	2368,9	2369.1	0,247	4015,5	4017,5	0,418	 	ľ		1	"	
9600	943.0	943.83	0.049	2366,2	2366,4	0.123	4002.5	4003,2	0,209	ີບ	CE		$C_E$	$C_E$	
19200	977.6	977,81	0.025	2365.5	2365.7	0.0616	3999.2	3999.6	0,104	3285,1	3273.0	0,085	6289.2	6259.9	0.163
38400	987,8			2365,4			3998,4	3998,7	0.052	2679.5			4622.6	4611.0	0.060
~	991.3	991.3		2365,3	2365.3		3998.1	3998.1		2541.5	2541.5		4295.9	4295.9	

For approximations see, for  $C_{E_1}$  eqs (6.54), (6.65), (6.60)

, for  $C_{E_2}$  eqs (6.67), (6.59), (6.13), (6.69)

. for  $C_{E_3}$  eqs (6.65), (6.62), (6.19)

. for  $C_{E_4}$  eqs (6.66), (6.58), (6.19), (6.68)

. for  $C_{E_5}$  eqs (6.65). (6.61), (6.14)

TABLE 3

## Propagation speed and half wavelength of axisymmetric elastic waves in an infinitely long, sandwich-type, cylinder of circular cross-section.

	sandwich-type, cylinder of circular cross-section.														
f	$C_{E_1}$ appr.	$C_{E_1}$ exact	λ ₁ exact	$C_{E_2}$ appr.	$C_{E_2}$ exact	λ ₂ exact	С _{Е4} аррг.	$\overline{C_{E_4}}$ exact	$\lambda_4$ exact	$C_{E_3}$ appr.	$C_{E_3}$ exact	$\lambda_3$ exact	• C _{Es} appr.	$C_{E_3}$ exact	λ ₅ exact
		$f_{E_1} = \infty$ t $f = 10.3$													
18 75		407.51	10.97												
37.5		518.83	6018												
75		679.84	4 532												
150		842.81	7,809	4	4		<u>-</u>	4,		001	13		52	50	
225		886.33	1.970	528	528		893	866		122	121		138	138	
300		834.12	1,390	j j	Ĩ		Îl.	<u> </u>		<u> </u>	Ľ		j.	ji –	
350		743,00	1.061	at	atj		atj	atj		at J	atj		atj	at J	
400		625,21	0,782	8	8		8	8		8	8		. 8	8	
450		544,16	0,605	1	.e		1	1		н,	1		١Ļ	л,	
500	381,8	508,88	0,509	C.	C"		C.	$C_{E}$		C.	ů.		C.	CE	
550	398,3	495,61	0.451	8515,2	11028	10,03	•								
600	413,8	492,37	0,410	4991,5	6249,5	5,208									
700	442,6	498,78	0,356	3605,9	4233,4	3,024				21					
1200	554,4	575,97	0,240	2634,4	2690,9	1,121	5986,0	6597,8	2,749						
2400	713.3	720,76	0,150	2424,8	2427,9	0,506	4307,5	4322,0	0,900						
4800	855,9	858,52	0,089	2379,8	2380,1	0,248	4069,2	4070,5	0,424						
9600	943,2	943,97	0,049	2368,9	2369,1	0,123	4015,5	4016,3	0,209						
19200	977.6	977,84	0,025	2366,2	2366,5	0,0616	4002,5	4002,9	0,104	3287.2	3275,1	0.085	6299.5	6270.1	0,163
38400	987,8	987,84	0,013	2365,5	2365,7	0,0308	3999.2	3999,6	0,052	2679.8	2673.8	0.035	4623.8	4612.2	0.060
76800		990.46	0.0064		2365.5	0.0154		3998,7	0.026		2568.3	0.017		4362.0	0,028
	991.3	991.3		2365,3	2365.3		3998.1	3998.1		2541.5	2541.5		4295.9	4295,9	

For approximations see, for  $C_{E_1}$  eqs (6.65), (6.60); for  $C_{E_2}$  eqs (6.67), (6.59), (6.13), (6.69); for  $C_{E_3}$  eqs (6.65), (6.62); for  $C_{E_4}$  eqs (6.66), (6.58), (6.68); for  $C_{E_5}$  eqs (6.65), (6.61), (6.14).

TABLE 6 Propagation speed and half wavelength of non-axisymmetric elastic waves (three waves in circumferential direction, n=3) in an infinitely long, sandwich-type, cylinder of circular cross-section.

f	С _{Е1} аррг.	$C_{E_1}$ exact	$\lambda_1$ exact	$C_{E_2}$ appr.	$C_{E_2}$ exact	$\lambda_2$ exact	C _{E₄} appr.	$C_{E_4}$ exact	$\hat{\lambda}_4$ exact	$C_{E_3}$ appr.	C _{E3} exact	λ ₃ exact	$\overline{C_{E_s}}$ appr.	$C_{E_s}$ exact	λ ₅ exact
		28.9													
•		. <u> </u>												-	
18,75		at _													
37,5		467,85	6,238												
75		502,76	3,352												
150		606,41	2,021												
225		639.43	1,421												
300		616,54	1,028												
350		579,54	0,828	\$	<b>,</b>		9,6	6.1		50	22			67	35
400		540,43	0,676	792	792		133	141		122	122			138	138
450		512.68	0,570	J	<u>)</u> r		_	<u> </u>		, I	Į.			, II	ji ji
500	388,8	498,18	0,498	atj	at /		at j	at J		at	at J			at )	at J
550	404,8	492,83	0,448	8.	8		8	8		8	8			8	8
600	419,9	492.98	0,411	1	n		n,	n		พ	ĩ			Ĩ	) j
700	448,0	501,77	0,358	C_	Č.		$C_E$	C.		C.	C.			$C_{\vec{E}}$	ي ت
1200	557,9	579,20	0,241	-3150,1	3295,5	1,137				•					
2400	715,1	722,50	0,151	2505,9	2513,0	0,524	4818,6	4853,1	1.011						
4800	856,7	859,23	0,090	2398.2	2398.8	0,250	4163,6	4166,4	0,434						
9600	943.4	944,19	0,049	2373,4	2373,6	0,124	4037,6	4038,4	0,210						
19200	977.7	977,91	0,025	2367.3	2367.5	0,062	4007.9	4008,3	0,104	3290,8	3278,6	0,085	6316,7	6287,1	0.164
38400	987,8	987.86	0.013	2365.8	2365.9	0.031	4000.6	4000.9	0.052	2680.2	2673,7	0.035	4625.8	4614.2	0.060
	991.3	991.3		2365.3	2365,3		3998.1	3998.1		2541.5	2541.5		4295,9	4295.9	

For approximations see, for  $C_{E_1}$  eqs (6.65), (6.60); for  $C_{E_2}$  eqs (6.67), (6.59), (6.13), (6.69); for  $C_{E_3}$  eqs (6.65), (6.62); for  $C_{E_4}$  eqs (6.66), (6.58), (6.68); for  $C_{E_5}$  eqs (6.65), (6.61), (6.14).

 TABLE 5

 Propagation speed and half wavelength of non-axisymmetric elastic waves (two waves in circumferential direction. n = 2) in an infinitely long.

## TABLE 7

Propagation speed of non-axisymmetric elastic waves (ten waves in circumferential direction, n = 10) in an infinitely long, sandwich-type, cylinder of circular cross-section.

f	C _{E1} appr.	$C_{E_1}$ exact	$\lambda_1$ exact	C _{E2} appr.	C _{E2} exact	$\lambda_2$ exact	C _{E4} appr.	$C_{E_4}$ exact	$\lambda_4$ exact	C _{E3} appr.	$C_{E_3}$ exact	$\lambda_3$ - exact	C _E , appr	C _{Es} exact	λ ₅ exact
		=351													
		it C													
		8													
		Ű			1										
		C.													
400		961.56	1,202												
450		770,83	0,856	42	4		65	87		105	770		129	<u>0</u> 960	
500	677,8	700.45	0,700	:26	: 264		44(	448		13.	-13(		14	14	
550	639,8	665,82	0,605	<u>_</u>	<u> </u>		<u>ل</u>	<u>ب</u>		<u>, 1</u>	<del>ر</del> س		1	<u>س</u>	
600	619,9	646,88	0,539	at	at		at	at		at	31		at	at	
700	604,4	630,48	0,450	8	8		8	8		8	8		8	8	
1200	636,9	652,96	0,272	, <u>,</u> ,	<u>,</u>		, <u> </u>	 		, <u>e</u>	j ŝ			ц. Ц	
2400	750,2	/20,80	0,158	2822.0	10107	0.204	10900 7	11102.0	1 166	0	0		0	0	
4600	0/0,1	049.25	0,091	2833,0	2838,7	0,290	10699,7	4518.0	1,100						
10200	947,5	946,33	0,045	2400,3	2400.0	0,120	4110.0	4010,0	0,235	3367.6	3344.7	0.0871	66564	6622.6	0 172
38400	988,1	988,14	0,023	2370,9	2371.1	0,031	4025,4	4025.8	0,052	2687,9	2681,3	0.0349	4663,0	4651.1	0,061
œ	991,3	991,3		2365,3	2365,3		3998.1	3998,1		2541,5	2541,5		4295,9	4295,9	

For approximations see, for  $C_{E_1}$  eqs (6.65), (6.60), (6.16)

, for  $C_{E_1}$  eqs (6.67), (6.59), (6.13), (6.69) , for  $C_{E_2}$  eqs (6.67), (6.59), (6.13), (6.69) , for  $C_{E_3}$  eqs (6.65), (6.62), (6.16) , for  $C_{E_4}$  eqs (6.66), (6.58), (6.17), (6.68) , for  $C_{E_5}$  eqs (6.65), (6.61), (6.14)

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Fig. 9. Propagation speed and half wavelength of axisymmetric elastic waves in an infinitely long, sandwich-type, cylinder of circular cross-section. Governing distortions are given within brackets.



Fig. 10. Propagation speed and half wavelength of non-axisymmetric elastic waves (one wave in circumferential direction, n=1) in an infinitely long, sandwich-type, cylinder of circular cross-section. Governing distortions are given within brackets.



Fig. 11. Propagation speed and half wavelength of non-axisymmetric elastic waves (two waves in circumferential direction, n=2) in an infinitely long, sandwich-type, cylinder of circular cross-section. Governing distortions are given within brackets.



Fig. 12. Propagation speed and half wave length of non-axisymmetric elastic waves (three waves in circumferential direction, n=3) in an infinitely long, sandwich-type, cylinder of circular cross-section. Governing distortions are given within brackets.

Fig. 13. Propagation speed and half wavelength of non-axisymmetric elastic waves (ten waves in circumferential direction, n=10) in an infinitely long sandwich-type, cylinder of circular cross-section. Governing distortions are given within brackets.

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Fig. 15. Propagation speed (approximate solution) of axisymmetric elastic waves in a shallow, infinitely long, sandwich-type, cylinder of circular cross-section. Governing distortions are given within brackets.



